Topological aspects of SU(2) Weyl fermion and global anomaly

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(Received 22 June 1992; accepted for publication 4 February 1994)

It is argued here that the global anomaly of SU(2) Weyl fermions is related to the residual topological properties of massive Dirac fermions leading to the topological index corresponding to the fermion number. As in the case of the chiral anomaly with Dirac fermions, the SU(2) anomaly with a Weyl fermion vanishes when the effect of this topological property is taken into account in the Lagrangian formulation.

I. INTRODUCTION

Witten[1] discusses the mathematical inconsistency in an SU(2) gauge theory involving an odd number of Weyl doublets. In the Euclidean functional integral approach, as $\pi^4(\mathrm{SU}(2))=Z_2$, nontrivial gauge transformations U(x) exist so that the Euclidean path integral in the presence of a single doublet of the left handed (Weyl) fermion, viz.,

$$Z=\int [dA_\mu]\exp\left(-\frac{1}{2g^2}\int \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})d^4x\right)\int [d\psi\, d\bar{\psi}]_{\mathrm{Weyl}} \exp\left(-\int \bar{\psi}i\not{D}\psi\, d^4x\right)$$

vanishes identically when A_μ, the SU(2) gauge potential, is allowed to have all possible configurations. This is because corresponding to any A_μ, we have an A_μ^U given by

$$A_\mu^U=U^{-1}A_\mu U-iU^{-1}\partial_\mu U$$

and $U(x)$ being a nontrivial gauge transformation in four-dimensional Euclidean space cannot be deformed to identity, if U(x) "wraps" around the gauge group an odd number of times—a consequence of $\pi^4(\mathrm{SU}(2))=Z_2$. To see how this actually happens, one has just to perform the fermionic part integration in Z. This comes out to be $[\det i\not{D}(A_\mu)]^{1/2}$, where $\not{D}(A_\mu)$ is the full four-dimensional Euclidean Dirac operator. Under $A_\mu \to A_\mu^U$; $U(x)$, the implementing nontrivial gauge transformation, we get

$$[\det i\not{D}(A_\mu^U)]^{1/2}=-[\det i\not{D}(A_\mu)]^{1/2}.$$

Since both A_μ, A_μ^v configurations are admissible while evaluating Z, we get $Z=0$ identically. For any physical quantity to be represented by a gauge-invariant operator x (say), a similar argument leads to Z_x, the path integral for x also equals zero. Hence the expectation $\langle x\rangle=Zx/Z$ becomes indeterminate. This leads to an inconsistency. Clearly, it arises due to the change in signature of the Weyl fermionic determinant when we consider all possible A_μ configurations. This is the reason for recognizing it as a global anomaly. As a matter of fact, there is no local anomaly in any SU(2) gauge theory.

However, subsequently a few authors[2] have pointed out that the inconsistency which cropped up in Witten's analysis is due to the fact that he disregarded the zero mode of the Dirac operator altogether. Indeed, it is the zero modes which play a decisive role in resolving the crisis of the global anomaly associated with Weyl fermions. The zero mode contribution is nonperturbative in origin and is connected with a number of normalizable positive chirality zero modes (n_+) and a number of normalizable negative chirality zero modes (n_-) of the Dirac operator in a compactified space. The quantity $n_+ - n_- = \nu$ is an invariant (topological) index of the Dirac operator. For any SU(2) gauge field configuration having nontrivial ν ($\nu \neq 0$), the Dirac operator will have zero modes. ν is also called the Pontryagin index and is given by

$$\nu = n_+ - n_- = \frac{1}{16\pi^2} \int \mathrm{Tr}({}^*F^{\mu\nu} F_{\mu\nu}) d^4x.$$

$F_{\mu\nu}$ is the SU(2) gauge field strength tensor; ${}^*F^{\mu\nu}$ is its dual. This zero mode contribution is reminiscient of the connection of the *ABJ* anomaly with chiral fermions. Indeed, the anomaly shows up in this case because we have a Weyl (left) fermion doublet (helicity and chirality concepts are equivalent) and the Jacobian of the Weyl fermion measure for U(1) transformations changes if the Dirac operator has zero modes.

In this article, we want to emphasize that the global anomaly as mentioned above, is related to the problem of how the topological properties of a massive fermion leading to the fermion number as a topological index survives in case of massless Weyl fermions. In Sec. II, we recapitulate briefly how the topological index arises in the case of a massive fermion and is associated with the the origin of the fermion number. In Sec. III, we discuss how the incorporation of this topological property helps us to understand the chiral anomaly of a Dirac fermion and how this anomaly is avoided when we take into account the proper topological features.[3] In Sec. IV, we shall show how the residual effect of this topological property of a Dirac fermion manifests itself in the case of a Weyl fermion and helps us to understand the global anomaly. In fact, the anomaly is avoided when this residual property for massless Weyl fermions is taken into account. Thereafter, a brief discussion follows in Sec. V.

II. TOPOLOGICAL PROPERTIES OF A DIRAC FERMION AND THE ORIGIN OF FERMION NUMBER

The conventional stochastic quantization scheme of Nelson[4] can be generalized to the relativistic domain when we consider Brownian motion processes both in the external and internal spaces.[5] The nonrelativistic case is then given by the sharp point limit. To have the quantization of a fermion we have to introduce an anisotropy in the internal space so that this gives rise to two internal helicities depicting particle and antiparticle states. Thus the internal helicities may be taken to have a geometrical realization of the fermion number. These features relating to the quantization procedure of a fermion and the geometrical interpretation of fermion number effectively leads to the fact that fermions in general correspond to Skyrme solitons and the Skyrme term which has been introduced to have the stability of the soliton appears here just as an effect of quantization. From this relativistic generalization procedure, where we have incorporated the Brownian motion process in the internal space also apart from that in the external space, *after quantization*, for an observational procedure, we can think of the mean position of the particle $\bar{x}_\mu$ in the external (observable) space with a stochastic extension determined by the internal stochastic variable ξ_μ.[6] In the nonrelativistic case, the sharp point limit helps us to realize the quantum mechanical correlations from the stochastic variables.[7] So for the expectation values of a stochastic variable, we can assign a quantum mechanical observable with the same expectation value. Indeed, to have quantization in Minkowski space–time, we have to take the space–time coordinate in complexified space–time as $Z_\mu = x_\mu + i\xi_\mu$. Now the observation of Heisenberg's uncertainty relation from stochastic mechanics along with the customary time-energy uncertainty relation helps us to formulate the commutation relations[8]

$$[Q_\mu, P_\nu] = i\hbar g_{\mu\nu}; \quad [Q_\mu, Q_\nu] = 0 = [P_\mu, P_\nu], \tag{1}$$

where Q_μ and P_ν are defined as

$$Q_\mu = q_\mu + i\hat{Q}_\mu; \quad P_\nu = p_\nu + i\hat{P}_\nu, \quad g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \tag{2}$$

and $q_\mu(p_\mu)$ denotes the mean position (momentum) in the external (observable) space and $\hat{Q}_\mu(\hat{P}_\mu)$ is given by the internal space variable $\xi_\mu(i(\partial/\partial\xi_\mu))$ denoting the stochastic extension. We introduce a new constant $\omega=\hbar/lmc$, where m is the mass of the particle and l is the fundamental length, with the following representation of Q_μ/ω and P_μ/ω where the latter are considered as acting on functions defined on phase space[9]

$$\frac{Q_\mu}{\omega}=-i(\partial/\partial p_\mu+\phi_\mu),\quad \frac{P_\mu}{\omega}=i(\partial/\partial q_\mu+\psi_\mu), \tag{3}$$

where ϕ_μ and ψ_μ are some complex-valued functions. Now when we introduce an anisotropy in the internal space giving rise to the internal helicity to quantize a fermion, ϕ_μ and ψ_μ become matrix-valued functions due to the noncommutativity character of the components of $Q_\mu(P_\mu)$. The anisotropic feature of the ξ space helps us to consider ξ_μ as an attached "direction vector" to the space–time point x_μ. The two opposite orientations of the "direction vector" give rise to two opposite internal helicities corresponding to fermion and antifermion. This internal helicity can easily be formulated in terms of the extended space–time metric $g_{\mu\nu}(x,\theta,\bar{\theta})$ where $\theta(\bar{\theta})$ are two-component spinorial variables.[10] In fact, for a massive spinor, we can choose the chiral coordinates in this extended space as

$$Z^\mu=x^\mu+\frac{i}{2}\lambda_\alpha^\mu\theta^\alpha \quad (\alpha=1,2), \tag{4}$$

where we identify the coordinate in the complex manifold $Z^\mu=x^\mu+i\xi^\mu$ with $\xi^\mu=\frac{1}{2}\lambda_\alpha^\mu\theta^\alpha$. We can now replace the chiral coordinates by their matrix representatives

$$Z^{AA'}=X^{AA'}+\frac{i}{2}\lambda_\alpha^{AA'}\theta^\alpha, \tag{5}$$

where

$$X^{AA'}=\frac{1}{\sqrt{2}}\begin{bmatrix} x^0-x^1 & x^2+ix^3 \\ x^2-ix^3 & x^0+x^1 \end{bmatrix} \quad \text{and} \quad \lambda_\alpha^{AA'}\in \mathrm{SL}(2,c).$$

With these relations, the twistor equation is now modified as

$$\bar{Z}_a Z^a+\lambda_\alpha^{AA'}\theta^\alpha\bar{\pi}_A\pi_{A'}=0, \tag{6}$$

where $\pi_A(\pi_{A'})$ denote the spinorial variable (complex conjugate spinor) corresponding to the four-momentum p_μ (the canonical conjugate of x^μ) and is given by the usual matrix representation for p_μ, viz.,

$$p_{AA'}=\bar{\pi}_A\pi_{A'},\quad Z^a=(\omega^A,\pi_{A'}),\quad \bar{Z}_a=(\bar{\pi}_A,\omega^{A'}), \tag{7}$$

with

$$\omega^A=i\left(X^{AA'}+\frac{i}{2}\lambda_\alpha^{AA'}\theta^\alpha\right)\pi_{A'}.$$

Eq. (6) now involves the helicity operator

$$S=-\lambda_\alpha^{AA'}\theta^\alpha\bar{\pi}_A\pi_{A'}, \tag{8}$$

which we identify as the internal helicity of the particle and relate it to the fermion number. It may be noted that we have taken the matrix representation of p_μ (the canonical conjugate of x^μ occurring in the complex coordinate expression $Z^\mu=x^\mu+i\xi^\mu$) as $p_{AA'}=\bar{\pi}_A\pi_{A'}$ necessarily implying $p_\mu^2=0$. So, the particle will have mass due to the nonvanishing character of the quantity ξ_μ^2. It is observed that the complex conjugate of the chiral coordinate given by Eq. (4) will give rise to a massive particle with opposite internal helicity corresponding to an antifermion. In the null plane where $\xi_\mu^2=0$, we can write the chiral coordinate as

$$Z^{AA'}=X^{AA'}+\frac{i}{2}\bar{\theta}^A\theta^{A'}, \tag{9}$$

where the coordinate ξ^μ is replaced by $\xi^{AA'}=\frac{1}{2}\bar{\theta}^A\theta^{A'}$. In this case, the helicity operator is given by

$$\mathscr{S}=-\bar{\theta}^A\theta^{A'}\bar{\pi}_A\pi_{A'}=-\bar{\mathscr{E}}\mathscr{E} \tag{10}$$

where $\mathscr{E}=i\theta^{A'}\pi_{A'}$; $\bar{\mathscr{E}}=-i\bar{\theta}^A\bar{\pi}_A$. Shirafuji[11] noted that the state with the helicity $+1/2$ is the vacuum state of the fermion operator

$$\mathscr{E}|S=+\tfrac{1}{2}\rangle=0. \tag{11}$$

Similarly, the state with the internal helicity $-1/2$ is the vacuum state of the fermion operator

$$\bar{\mathscr{E}}|S=-\tfrac{1}{2}\rangle=0. \tag{12}$$

In the case of a massive spinor, we can define a negative-definite plane D^- where for the coordinate $Z^\mu=x^\mu+i\xi^\mu$, ξ^μ belongs to the interior of the forward light cone $\xi \gg 0$ (in ξ space) and as such, represents the upper half plane with the condition det $\xi>0$ and $\frac{1}{2}$ Tr $\xi>0$. The positive-definite plane D^+ is given by the set of all coordinates Z^μ with ξ^μ in the interior of the backward light cone (in ξ space). The map $z\mapsto z^*$ sends a negative-definite plane to a positive-definite plane. The space M of the null plane (det $\xi=0$) is the Shilov boundary so that a function holomorphic in D^- (D^+) is determined by its boundary values. Thus if we consider any function $\phi(z)=\phi(x)+i\phi(\xi)$ that is holomorphic in the whole domain, we note that the helicity $+1/2$ ($-1/2$) given by the operator $i\theta^{A'}\pi_{A'}$ ($-i\bar{\theta}^A\bar{\pi}_A$) in the null plane may be taken to be the limiting value of the internal helicity in the upper (lower) half plane.

In this complexified space–time exhibiting the internal helicity states, we can write the metric

$$g_{\mu\nu}(x,\theta,\bar{\theta})=g_{\mu\nu}^{AA'}(x)\bar{\theta}^A\theta^{A'}. \tag{13}$$

It has been shown elsewhere[12] that this metric structure gives rise to the SL(2,C) gauge theory of gravitation and generates the field-strength tensor $F_{\mu\nu}$ given in terms of the gauge fields B_μ which are matrix-valued having the SL(2,C) group structure and is given by

$$F_{\mu\nu}=-\partial_\mu B_\nu+\partial_\nu B_\mu+[B_\mu,B_\nu]. \tag{14}$$

Since $\theta(\bar{\theta})$ is the spinorial variable which represents the "direction vector" attached to the space–time point, this effectively represents the stochastic extension of a relativistic quantum particle representing a fermion. So from the relations we can now identify ϕ_μ with B_μ and we can associate another gauge field C_μ with ψ_μ satisfying the relation (14). This suggests that for a relativistic quantum particle which is taken as a stochastically extended one, the fermionic character of a particle associates the functions defined on stochastic phase space with matrix-valued non-Abelian gauge fields having the SL(2,C) group structure. The asymptotic zero curvature condition then implies that we can write the non-Abelian gauge field as

$$B_\mu = U^{-1}\partial_\mu U, \quad \text{for} \quad x_\mu \to \infty; \quad U \in \mathrm{SL}(2,C). \tag{15}$$

With this substitution, we note that the corresponding Lagrangian is given by

$$L = M^2 \, \mathrm{Tr}(\partial_\mu U^\dagger \partial_\mu U) + \mathrm{Tr}[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2, \tag{16}$$

where M is a suitable constant having the dimension of mass. The first term arises here from a gauge noninvariant term $M^2 B_\mu B^\mu$ and the second term clearly arises from the quantity $F_{\mu\nu} F^{\mu\nu}$ where $F_{\mu\nu}$ is given by Eq. (14).

Thus we find that the quantization of a Fermi field considering an anisotropy in the internal space leading to an internal helicity description corresponds to the realization of a nonlinear sigma model where the Skyrme term in the Lagrangian $L_{\mathrm{Skyrme}} = \mathrm{Tr}[\partial_\mu U U^\dagger, \, \partial_\nu U U^\dagger]^2$, necessary for the stability of the soliton, arises here as an effect of quantization. Thus in this picture fermions appear as solitons and the fermion number is found to have a topological origin. Indeed, for the Hermitian representation, we can take the group manifold as SU(2) and this leads to a mapping from the space three-sphere $\mathscr{S}^3$ to the group space $\mathscr{S}^3$ ($\mathrm{SU}(2) \cong \mathscr{S}^3$), and the corresponding winding number is given by

$$q = \frac{1}{24\pi^2} \int_{\mathscr{S}^3} dS_\mu \, \epsilon^{\mu\nu\alpha\beta} \, \mathrm{Tr}[(U^{-1}\partial_\nu U)(U^{-1}\partial_\alpha U)(U^{-1}\partial_\beta U)]. \tag{17}$$

Evidently, q is a topological index and represents the fermion number.

III. TOPOLOGICAL PROPERTIES OF A DIRAC FERMION AND CHIRAL ANOMALY

However, if we demand SL(2,C) invariance of the Lagrangian, following Malin and Carmeli,[13] we can choose the simplest Lagrangian density in spinor affine space

$$L = -\tfrac{1}{4} \, \mathrm{Tr}[\epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}]. \tag{18}$$

Applying the usual procedure of variational calculus, we find the field equations

$$\partial_\delta(\epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta}) - [B_\delta, \, \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta}] = 0. \tag{19}$$

Now one can write

$$B_\mu = \mathbf{b}_\mu \cdot \mathbf{g}, \quad F_{\mu\nu} = \mathbf{f}_{\mu\nu} \cdot \mathbf{g}, \tag{20}$$

where $\mathbf{g} = (g_1, g_2, g_3)$ are tangent vectors to the generators of the SL(2,C) group

$$g_1(Z) = \begin{bmatrix} 1 & 0 \\ Z & 1 \end{bmatrix}, \quad g_2(Z) = \begin{bmatrix} e^Z & 0 \\ 0 & e^{-Z} \end{bmatrix}, \quad g_3(Z) = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}, \tag{21}$$

where Z is complex. With the definition

$$g_m = \left[\frac{dg_m(Z)}{dZ} \right]_{Z=0} \tag{22}$$

implying $g_m(Z) = \exp(Z g_m)$, we find

$$g_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{23}$$

Evidently in this space, these SL(2,C) gauge fields will appear as background fields.

Now, to describe a matter field in this geometry, the Lagrangian will be modified by the introduction of this SL(2,C) invariant Lagrangian density.[18] Hence for a massless Dirac field, we write for the Lagrangian

$$L = -\bar{\psi}\gamma^{\mu}D_{\mu}\psi - \tfrac{1}{4}\,\mathrm{Tr}(\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}), \tag{24}$$

where D_{μ} is the SL(2,C) gauge-covariant derivative

$$D_{\mu} = \partial_{\mu} - igB_{\mu} \tag{25}$$

and g is some coupling strength. Now, from the properties of the SL(2,C) generators, we find

$$\mathbf{j}_{\theta}^{\mu} = \epsilon^{\mu\nu\alpha\beta}\mathbf{b}_{\nu}\times\mathbf{f}_{\alpha\beta} = \epsilon^{\mu\nu\alpha\beta}\partial_{\nu}\mathbf{f}_{\alpha\beta}, \tag{26}$$

with

$$\partial_{\mu}\mathbf{j}_{\theta}^{\mu} = 0. \tag{27}$$

However, in the Lagrangian (24), if we split the Dirac massless spinor in chiral forms and identify the internal helicity $+1/2$ $(-1/2)$ with left (right) chirality corresponding to θ and $\bar{\theta}$, we can write

$$\begin{aligned}\bar{\psi}\gamma_{\mu}D_{\mu}\psi &= \bar{\psi}\gamma_{\mu}\partial_{\mu}\psi - ig\bar{\psi}\gamma_{\mu}B_{\mu}^{a}g^{a}\psi \\ &= \bar{\psi}\gamma_{\mu}\partial_{\mu}\psi - \frac{ig}{2}\{\bar{\psi}_R\gamma_{\mu}B_{\mu}^{1}\psi_R - \bar{\psi}_R\gamma_{\mu}B_{\mu}^{2}\psi_R + \bar{\psi}_L\gamma_{\mu}B_{\mu}^{2}\psi_L + \bar{\psi}_L\gamma_{\mu}B_{\mu}^{3}\psi_L\}.\end{aligned} \tag{28}$$

Then the three SL(2,C) gauge field equations yield the following three conservation laws:

$$\begin{gathered}\partial_{\mu}[\tfrac{1}{2}(-ig\bar{\psi}_R\gamma_{\mu}\psi_R) + j_{\mu}^{1}] = 0, \\ \partial_{\mu}[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_{\mu}\psi_L + ig\bar{\psi}_R\gamma_{\mu}\psi_R) + j_{\mu}^{2}] = 0, \\ \partial_{\mu}[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_{\mu}\psi_L) + j_{\mu}^{3}] = 0.\end{gathered} \tag{29}$$

These three equations represent a consistent set of equations if we choose

$$j_{\mu}^{1} = -\tfrac{1}{2}j_{\mu}^{2}; \quad j_{\mu}^{3} = \tfrac{1}{2}j_{\mu}^{2}, \tag{30}$$

which evidently guarantees the vector current conservation. Then we can write

$$\partial_{\mu}(\bar{\psi}_R\gamma_{\mu}\psi_R + j_{\mu}^{2}) = 0, \quad \partial_{\mu}(\bar{\psi}_L\gamma_{\mu}\psi_L - j_{\mu}^{2}) = 0. \tag{31}$$

From these, we find

$$\partial_{\mu}(\bar{\psi}\gamma_{\mu}\gamma_5\psi) = \partial_{\mu}J_{\mu}^{5} = -2\partial_{\mu}j_{\mu}^{2}. \tag{32}$$

Thus the anomaly is expressed here in terms of the second SL(2,C) component of the gauge field current j_{μ}^{2}. However, since in this formalism, the chiral currents are modified by the introduction of j_{μ}^{2}, we note that the anomaly vanishes.

This current j_{μ}^{2} is related to the topological origin of fermion number, as we have

$$q = \int j_0^2(x)\,d^3x. \tag{33}$$

Indeed, the term $\mathrm{Tr}(\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta})$ in the Lagrangian can be expressed as a four-divergence of the form $\partial_\mu\Omega^\mu$ where

$$\Omega^\mu = -\frac{1}{16\pi^2}\,\epsilon^{\mu\alpha\beta\gamma}\,\mathrm{Tr}\left[B_\alpha F_{\beta\gamma} - \frac{2}{3}\,B_\alpha B_\beta B_\gamma\right]. \tag{34}$$

The Pontryagin density is

$$P = -\frac{1}{16\pi^2}\,\mathrm{Tr}({}^*F^{\mu\nu}F_{\mu\nu}) = \partial_\mu\Omega^\mu, \tag{35}$$

where Ω^μ is the Chern–Simons secondary characteristic class. The Pontryagin index

$$q = \int P\, d^4x \tag{36}$$

is a topological invariant and is related to the current j_μ^2 through the relation (33) which suggests that

$$\tilde{J}_\mu^5 = J_\mu^5 + 2\hbar\Omega_\mu = J_\mu^5 + 2j_\mu^2 \tag{37}$$

implying $\partial_\mu\tilde{J}_\mu^5 = 0$. That means, when the topological properties of a fermion related to the origin of fermion number is taken into account, we are not confronted with the chiral anomaly.

IV. TOPOLOGICAL PROPERTY OF WEYL FERMIONS AND GLOBAL ANOMALY

This topological property will have its residual effect in Weyl fermions also. This is due to the fact that as discussed earlier, the SU(2) doublet of massless Weyl fermions are now represented by the null plane $\xi_\mu^2 = 0$ which is the Shilov boundary of the negative- and positive-definite planes D^- and D^+ corresponding to the forward and backward light cones in the ξ space depicting the massive Dirac fermion and antifermion. Indeed, for any function holomorphic in the whole domain, the internal helicities in the upper and lower half planes will have their limiting values in the boundary in this complexified Minkowski space–time. So the topological index denoting the fermion number will have its residual effect in this boundary. Thus the conserved current in this case will also be given by

$$\partial_\mu\tilde{J}_\mu^5 = \partial_\mu J_\mu^5 + 2\partial_\mu j_\mu^2 = 0, \tag{38}$$

where j_μ^2 will be denoted here by boundary values. Indeed, with the identity of j_μ^2 with Ω_μ, the Chern–Simons term also suggests that in the case of Weyl fermions, Ω_μ will have its residual effect in the boundary. Thus for a left Weyl fermion, the conserved current $\tilde{J}_\mu^L$ is given by

$$\tilde{J}_\mu^L = \tfrac{1}{2}\{J_\mu^V - (J_\mu^5 + 2\Omega_\mu)\} = \tfrac{1}{2}\{J_\mu^V - (J_\mu^5 + 2j_\mu^2)\}. \tag{39}$$

The corresponding conserved charge is then expressed as

$$\tilde{Q}_{5L} = \int \psi_L^\dagger\psi_L\, d^3x - \int j_0^2\, d^3x. \tag{40}$$

It is the Chern–Simons term $\Omega_\mu(j_\mu^2)$ which defines the fermion number through the associated charge $\int j_0^2\, d^3x$. Generally, we get two equal and opposite charges in the upper and lower half planes for a massive fermion and antifermion. The boundary values of these two charges can then be taken to represent massless Weyl fermions ψ_L and ψ_R. $\tilde{Q}_{5L}$ thus is responsible for generating U(1) rotation in the ψ_L sector. The term $\int j_0^2\, d^3x = q$ can be identified with the topological index corresponding to the fermion number and is actually an integer.

Now, in the conventional Hamiltonian formulation of the gauge theory of the SU(2) Weyl fermion the states are labeled by an index n, the winding number of SU(2) gauge potential A_μ, and eigenvalues of the generator T_3. Besides these, we also have the conserved left charge. To define it, we introduce the SU(2) Pontryagin index density $(i/16\pi^2)\,\mathrm{Tr}(^*F^{\mu\nu}F_{\mu\nu})$; $F_{\mu\nu}$ being the SU(2) gauge field strength tensor, $^*F^{\mu\nu}$ being its dual. This term can be written as $\partial_\mu C^\mu$; C_μ being the Chern–Simons current given by

$$C^\mu = \frac{ie}{16\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[A_\nu F_{\rho\sigma} - \frac{2}{3}\, ie^2 A_\nu A_\rho A_\sigma\right]. \tag{41}$$

Then, from $\partial_\mu J_\mu^L = i/16\pi^2\,\mathrm{Tr}(^*F^{\mu\nu}F_{\mu\nu})$ where $J_\mu^L \equiv \bar{\psi}_L \gamma_\mu \psi_L$ is the usual anomalous left current, one constructs a modified conserved left current $\tilde{J}_\mu^L = (J_\mu^L - C_\mu)$ and defines the chiral charge $\tilde{Q}_5 = \int \tilde{J}_0^L(x)\, d^3x$. $\tilde{J}_\mu^L$ is gauge noninvariant due to C_μ. $\tilde{Q}_5$ is conserved in time and is taken as the generator of chiral (left) transformations. The inadequacy with $\tilde{Q}_5$ is that it is not invariant under the time independent SU(2) transformation Ω_n with winding number n. One actually gets the relation

$$\Omega_n^{-1}\tilde{Q}_5\Omega_n = \tilde{Q}_5 + n. \tag{42}$$

Now, T_3, the diagonal SU(2) generator, is not supposedly affected by n because it is associated with the physical charge and not the axial charge. Hence, if one considers a transformation on the physical states by the composite operator $G = V^{-1}\Omega_n^{-1} V \Omega_n$ where $V = e^{2\pi i T_3}$, one finds that $G = I$. Thus G implements a trivial transformation.

However, considering any particular SU(2) left member, V can be realized also through U(1) chiral rotation through the generator $\tilde{Q}_5$. Then one gets[14,15]

$$G = e^{-i\pi\tilde{Q}_5}\Omega_n^{-1} e^{i\pi\tilde{Q}_5}\Omega_n = e^{i\pi n}. \tag{43}$$

For n odd, $G \neq I$. This is considered as the inconsistency of the SU(2) Weyl fermions.

In our approach, we have the conserved left current as $\bar{\psi}_L\gamma_\mu\psi_L - j_\mu^2$; j_μ^2 being the second component of the SL(2,C) gauge field current. This current defines a time-invariant left charge given by Eq. (40). For left fields, $\int j_0^2\, d^3x$ could be set at some even or odd integer value to begin with—this being a convention. Then in presence of SU(2) gauge fields we must have

$$G = e^{-i\pi\tilde{Q}_{5L}}\Omega_n^{-1} e^{i\pi\tilde{Q}_{5L}}\Omega_n = e^{-i\pi\tilde{Q}_{5L}} e^{i\pi(\tilde{Q}_5 - q' + n)} = e^{i\pi}(q - q' + n) = I \quad \text{as } q = q' - n. \tag{44}$$

The explanation is quite simple—when SU(2) gauge fields wrap n times, the SL(2,C) gauge fields already associated with the left matter field space–time region unwrap that many times so that the left charge $\tilde{Q}_{5L}$ never gets offset with time. So in Ω_n whether n is even or odd we are not confronted with any real inconsistency if we focus our attention to the whole $\psi_L - B_\mu - A_\mu$ field system.

The origin[1] of the global SU(2) anomaly lies in the change of sign of the partition function for a disconnected gauge transformation, i.e.,

$$[\det i\not{D}(A^U)]^{1/2} = -[\det i\not{D}(A)]^{1/2}, \tag{45}$$

where

$$i\not{D}(A)\psi \equiv i\gamma_\mu(\partial_\mu - ieA_\mu)\psi = 0. \tag{46}$$

ψ is an SU(2) doublet and A_μ is a matrix-valued gauge potential. This determinant is not invariant under large gauge transformations U, but changes sign when U belongs to an odd homotopy group. When A_μ is labeled by a homotopy parameter τ such that $A_\mu(x,\tau)$ vanishes at $\tau=-\infty$ and is a pure gauge $U^{-1}\partial_\mu U$ at $\tau=+\infty$ with U belonging to the first homotopy class, the determinant at $\tau=+\infty$ has opposite sign from that at $\tau=-\infty$. Indeed defining

$$\Delta(A)\equiv\Delta_{(4)}^{1/2}(A)=[\det(\partial\!\!\!/+A\!\!\!/)_{(4)}]^{1/2}, \tag{47}$$

where the subscript (4) indicates that we are dealing with the usual 4×4 γ matrices, we can take $\Delta_{(4)}^{1/2}(A)$ to be the product of positive eigenvalues (which may be odd or even in number) at $\tau=-\infty$ and follow these eigenvalues as τ passes to $+\infty$. If an odd number crosses the axis, the determinant changes sign. It is assumed that there is no zero mode, so the determinant is nonvanishing. This suggests that the eigenvalue $\lambda(\tau)$ of $(\partial\!\!\!/+A\!\!\!/)_{(4)}$ must have the kink shape changing sign as τ passes from $-\infty$ to $+\infty$. This is avoided in our formalism as follows.

The SL(2,C) gauge fields B_μ makes it obligatory for us to write the partition function for a Dirac doublet as

$$Z=\int dA_\mu\int dB_\mu\int(d\bar\psi\, d\psi)_{\rm Dirac}\exp\left[-\int\bar\psi(iD\!\!\!\!/+gB\!\!\!/)\psi\, d^4x\right]$$
$$\times\exp\left[-\int\frac{1}{2g^2}\,{\rm Tr}(\epsilon^{\alpha\beta\gamma\delta}\tilde F_{\alpha\beta}\tilde F_{\gamma\delta})d^4x\right]\exp\left[-\int\frac{1}{2e^2}\,{\rm Tr}(F^{\mu\nu}F_{\mu\nu})d^4x\right]. \tag{48}$$

Here,

$$A_\mu=SU(2)\ \text{gauge potentials (one-form)},$$

$$B_\mu=SL(2,C)\ \text{gauge potentials (one-form)},$$

$$D\!\!\!\!/\equiv\gamma_\mu(\partial_\mu-ieA_\mu);\quad g=\psi_L-B_\mu\ \text{coupling strength},$$

$$\tilde F_{\mu\nu}=SL(2,C)\ \text{gauge field strength tensor (two-form)},$$

$$F_{\mu\nu}=SU(2)\ \text{gauge field strength tensor (two-form)}.$$

Now, we will surely have

$$\int(d\bar\psi\, d\psi)_{\rm Dirac}\exp\left[-\int\bar\psi(iD\!\!\!\!/+gB\!\!\!/)\psi\, d^4x\right]=\det(iD\!\!\!\!/+gB\!\!\!/). \tag{49}$$

However, in the SU(2) Weyl left doublet case, asymmetry in point of contributions received by the distinct Weyl species when one breaks up the term $\bar\psi B\!\!\!/\psi$ [shown in Eq. (28)] does not permit us to write in general

$$\int(d\bar\psi\, d\psi)_{\rm Weyl}\exp\left[-\int\bar\psi(iD\!\!\!\!/+gB\!\!\!/)\psi\, d^4x\right]=[\det(iD\!\!\!\!/+gB\!\!\!/)]^{1/2} \tag{50}$$

unless we set B_μ identically to zero. But then, left fermions cannot be consistently described. Thus the crisis of a global ambiguity regarding the signature of the square root of det $(iD\!\!\!\!/+gB\!\!\!/)$ never arises; whereas if $B_\mu=0$ one gets $[\det iD\!\!\!\!/(A)]^{1/2}=-[\det.\ iD\!\!\!\!/(A^U)]^{1/2}$ as stated in Eq. (45) resulting in a global anomaly.

In Refs. 1 and 2, the term $-\int i\bar\psi D\!\!\!\!/\psi\, d^4x$ has been employed as an exponent and integrated over Dirac path measure $d\bar\psi\, d\psi$. The result obtained is $[\det iD\!\!\!\!/(A)]$. As the operator $iD\!\!\!\!/(A)$ is

Hermitian and anticommutes with γ_5, it is clear that we will in general have a large number of real positive as well as negative eigenvalues in the spectrum of $i\not{D}(A)$. Corresponding to any eigenvalue λ_n realized in one eigenspace $\{\phi_n\}$ there will be another eigenvalue $-\lambda_n$ realized in the γ_5 transformed eigenspace $\{\gamma_5\phi_n\}$ and vice versa. This argument effectively brought us to the front door of the crisis when we integrated over Weyl path measures and as a result got the factor $[\det i\not{D}(A)]^{1/2}$. It may be observed that no such room exists in the present formalism where the geometry (topology) of the Weyl fermions have an SL(2,C) gauge field (B_μ) description to start with. This leads us to consider the operator $i\not{D}(A)+g\not{B}$ instead of $i\not{D}(A)$. But then $i\not{D}+g\not{B}$ is not necessarily Hermitian. Thus while considering the factor $\det[i\not{D}(A)+g\not{B}]$, we cannot make simple conclusions in the same line as mentioned above in the context of the factor $\det[i\not{D}(A)]$. The extra additive factor $g\not{B}$ associated with the Hermitian differential operator $i\not{D}(A)$ leads to an inhomogeneous eigenvalue equation. It seems that the eigenvalue network actually is planar if one considers the B_μ fields. Hence, one cannot easily conclude about rearrangement of eigenvalues within the eigenvalue network without choosing some specific B_μ configuration.

V. DISCUSSIONS

Very recently, it has been argued by some authors[2] that the global SU(2) anomaly is due to the contribution of the zero modes of the Dirac operator and hence the inconsistencies are basically of nonperturbative origin. However, since the zero mode contribution is just the $m \rightarrow 0$ limit of the conventional mass term, the present formalism suggests that this limiting value is just determined by the topological properties on the boundary of the upper and lower half planes corresponding to massive fermions and antifermions. Thus the zero mode contribution is actually absorbed in the quantities $\tilde{J}^L_\mu$ and $\tilde{J}^R_\mu$ when we take the boundary values. In Ref. 2 it is stated that the zero modes of the Dirac operator $i\not{D}(A)$ actually save the SU(2) global anomaly crisis. In our formalism zero modes do not enter unless we put $B_\mu=0$. But physically this will mean that we are admitting ambiguities in the very definition of a Weyl fermion. Eventually, it is this shortcoming which manifests itself in the form of the ambiguities discussed in Refs. 1 and 2. However, from the above analysis, we note that as in case of Dirac fermions, massless SU(2) Weyl fermions can also be thought of as free from anomaly when the residual topological property giving rise to the fermion number is taken into account.

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