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SOME ASYMPTOTIC PROPERTIES OF RISK FUNCTIONS WHEN THE LIMIT OF THE EXPERIMENT IS MIXED NORMAL

By P. JEGANATHAN

Indian Statistical Institute

SUMMARY. In recent times there occur situations, especially in estimation problems in certain stochastic processes, in which the limit of the suitably normalised log-likelihood ratios is a family of mixed or weighted normal-type distributions. When the limit is of normal-type, several basic results of LeCam and Hájek concerning asymptotic properties of risk functions are well-known; some of these results are extended in the present paper to the forementioned more general situation. It also appears that the proofs of the present paper, when they are specialised to the normal case, are simpler than the existing direct proofs. The present paper might also be useful in the sense that several closely related results are presented at a single place.

1. Introduction

In a fundamental paper LeCam (1953) obtained several basic results concerning the asymptotic properties of risk functions. In another fundamental paper LeCam (1960) introduced what is now called *locally asymptotically normal* (LAN) families of distributions and obtained several basic results regarding the asymptotic theory of estimation and testing and thereby demonstrating that a very large part of the asymptotic theory depends only on the approximating form of the suitably normalised log—likelihood ratios. Based on LeCam (1960), Hájek (1972) then improved and extended the local results of LeCam (1953) and presented them in a rather completely general form, i.e., the only assumption assumed by Hájek (1972) is the very weak LAN condition; the results presented in this paper of Hájek are known as the locally asymptotically minimax and admissibility results. Recently, Strasser (1978) has shown that the other global results of LeCam (1953) can also be improved using the technical essence of LeCam (1960), Hájek (1970) and certain "invariance" results of LeCam (1973 and 1979); in this connection see also the remark 2 in Section 2 below.

In two important papers LeCam (1972 and 1974b) (see LeCam (1979) for a more detailed presentation) further obtained certain extremely general results in a general framework concerning, among several other things, the local asymptotic minimaxity and admissibility and in particular showed that the results given in Hájek (1972) can be viewed as special case of these general results; in fact LeCam's results are applicable even to the situations which are remote from the usual LAN case, see Section 5 of Chapter 10 of LeCam (1979) and Miller (1979).

In recent times, there occur situations (c.f. Jeganathan, 1979 and the references therein) where the LAN condition is not satisfied but a more general condition, which may be called the *locally asymptotically mixed normal* (LAMN) condition is satisfied, see Definition 1 below for the precise definition of the LAMN condition. The main purpose of this paper is to extend the forementioned results of LeCam, Hájek and Strasser to the more general LAMN case and to present explicitly some further results, which are implicit in the various works of LeCam and Hájek, regarding the asymptotic properties of risk functions and posterior approximation. It also appears that the proofs of the present paper, when they are specialised to the LAN case, are simpler than the existing direct proofs given in the forementioned papers. The present paper might also be useful in the sense that several closely related results are presented at a single place.

More specifically, in Theorem 1 we present a result concerning the asymptotic lower bound for risk functions; a more familiar result, Theorem 2, follows from this result under the usual invariance restriction. Sequences of estimators which attain the lower bound of Theorem 1 are characterised in Theorem 3.

In Theorem 4 we present a result, for the LAMN case, which is an extension of the local asymptotic minimax results of Hájek (1972) and LeCam (1972 and 1974b) ; this result turns out to be an immediate consequence of Theorem 1, and one more simple proof is also indicated. In Theorem 5 we present, when $\dim \Theta \leqslant 2$, an extension of the uniqueness results of Hájek (1972) and LeCam (1974b); it may be noted here that this uniqueness result does not hold when $\dim \Theta > 2$ for the reasons explained in LeCam (1972). It may be further noted here that the uniqueness result of Theorem 3 is weaker than the uniqueness result of Theorem 5, as is easily seen by considering James-Stein type estimators.

In Theorems 7 and 8 we present analogous global asymptotic properties of risk functions; actually, we deduce these results from a general result (Theorem 6) concerning a certain kind of posterior approximation.

In connection with Theorems 4 and 5 of this paper, the following remarks should be made. As we have already remarked, LeCam (1972, 1974b) has obtained some very deep and general results which amount to the following. If one is interested in proving asymptotic properties such as local asymptotic minimaxity and admissibility for the given sequence experiments, it is just enough to prove the statements for the limit of the experiments and then the corresponding limiting statements for the sequence of experiments can be concluded from his results. Thus once we have proved minimax and admissibility results for the limit of the LAMN experiment, the conclusions of Theorems 4 and 5 are the consequences of LeCam's results, since LeCam's results are not restricted to any particular form of the limit of the experiments. The main reason for presenting a rather complete proof of Theorems 4 and 5 is the following. Once the powerful Lemma 2 below of LeCam is given, it turns out that the proof of the unique admissibility and minimaxity results for the limit of the LAMN experiments and the proof of the local asymptotic admissibility and minimaxity results for the LAMN experiments are almost identical.

This paper constitutes Chapter 6 of our Ph.D. thesis (1980). A referee of an earlier version of this paper has remarked that some of the results of this paper might be already known to some of the workers working in this field, e.g., R. B. Davies. During the final stage of prepararation of our Ph.D. thesis we received a copy of Ph.D. thesis from Swensen (1980, September), where he has independently obtained our Theorems 4 and 5. His proof consists of first proving minimax and admissibility results for the limit of the LAMN experiment and then using the above mentioned results of LeCam to get the desired conclusion, whereas our proofs are based directly on Lemma 2 below.

In Section 2 we present the results and in Section 3 we present some preliminary lemmas. In Section 4 we present the proofs of the results.

2. Notations and definitions

Let $\boldsymbol{E}_n = \{\boldsymbol{X}_n, \mathcal{A}_n, P_{\theta,n} ; \theta \in \Theta\}$, $n \geq 1$, be a sequence of experiments; throughout what follows it will be assumed, without any further mentioning, that Θ is an open subset of $\boldsymbol{R}^k$.

We use the following notations. If P and Q are probability measures on a measurable space $(\boldsymbol{X}, \boldsymbol{A})$, then dP/dQ denotes the Radon-Nikodym derivative of the Q-continuous part of P with respect to Q. If p and q are densities of P and Q with respect to some σ-finite measure λ, then

$$\|P-Q\| = \int |p-q|\, d\lambda$$

is the L_1-norm. If $\boldsymbol{Y}$ is a random vector its distribution will be denoted by $\mathcal{L}(\boldsymbol{Y})$ or by $\mathcal{L}(\boldsymbol{Y}|P)$ when $\boldsymbol{Y} : (\boldsymbol{X}, \boldsymbol{A}) \to (\boldsymbol{R}^q, \boldsymbol{B}^q)$, $q \geq 1$, $\boldsymbol{B}^q$ being the σ-field of Borel subsets of $\boldsymbol{R}^q$. For a vector $\boldsymbol{h} \in \boldsymbol{R}^k$, $\boldsymbol{h}'$ denotes the transpose of $\boldsymbol{h}$ and $|\boldsymbol{h}|$ denotes the euclidean norm; for a square matrix $\boldsymbol{D}$, $\|\boldsymbol{D}\|$ denotes the norm defined by the square root of the sum of squares of its elements. '$\Longrightarrow$' denotes the convergence in distribution. $\text{Log}\,\dfrac{dP_{s,n}}{dP_{\theta,n}}$, $\theta, s \in \Theta$, $n \geqslant 1$, will be denoted by $\Lambda_n(s, \theta)$.

We now introduce the LAMN condition.

Definition 1 : The sequence of experiments $\{\boldsymbol{E}_n\}$ satisfies the LAMN condition at $\theta = \theta_0 \in \Theta$ if the following two conditions are satisfied.

(A.1). There exists a sequence $\{\boldsymbol{W}_n(\theta_0)\}$ of $\boldsymbol{A}_n$-measurable k-vectors and a sequence $\{\boldsymbol{T}_n(\theta_0)\}$ of $\boldsymbol{A}_n$-measurable $k \times k$ symmetric matrices such that $P_{\theta_0, n}[\boldsymbol{T}_n(\theta_0)$ is p.d.$] = 1$ for every $n \geqslant 1$ and the difference

$$\Lambda_n(\theta_0+\boldsymbol{\delta}_n\boldsymbol{h}, \theta_0)-[\boldsymbol{h}'\boldsymbol{T}_n^{1/2}(\theta_0)\boldsymbol{W}_n(\theta_0)-\tfrac{1}{2}\boldsymbol{h}'\boldsymbol{T}_n(\theta_0)\boldsymbol{h}]$$

converges to zero in $P_{\theta_0,n}$-probability for every $\boldsymbol{h} \in \boldsymbol{R}^k$, where $\{\boldsymbol{\delta}_n\}$ is a sequence of p.d. matrices such that $\|\boldsymbol{\delta}_n\| \to 0$ as $n \to \infty$.

(A.2). There exists an almost surely (a.s.) p.d. random matrix $\boldsymbol{T}(\theta_0)$ such that

$$\mathcal{L}(\boldsymbol{W}_n(\theta_0), \boldsymbol{T}_n(\theta_0) \mid P_{\theta_0,n}) \Longrightarrow \mathcal{L}(\boldsymbol{W}, \boldsymbol{T}(\theta_0))$$

where $\boldsymbol{W}$ is a copy of the standard k-variate normal distribution independent of $\boldsymbol{T}(\theta_0)$.

In the special case when $\boldsymbol{T}_n(\theta_0)$ is equal to a non-random matrix $\boldsymbol{T}(\theta_0)$ for all $n \geqslant 1$, $\{\boldsymbol{E}_n\}$ is said to satisfy the LAN-condition at $\theta = \theta_0 \in \Theta$.

Remark 1 : A detailed study of the LAMN-condition can be found in Davies (1978), Jeganathan (1982, 1980) and Swensen (1980).

Let L be the class of all loss functions $l : \boldsymbol{R}^k \to [0, \infty)$ of the form $l(x) = l(|x|)$, $l(0) = 0$ and $l(x) \leqslant l(y)$ if $|x| \leqslant |y|$.

In what follows $E_{\theta+\boldsymbol{\delta}_n\boldsymbol{h}}$ denotes the expectation with respect to the measure $P_{\theta+\boldsymbol{\delta}_n\boldsymbol{h},n}$, $n \geqslant 1$, $\boldsymbol{h} \in \boldsymbol{R}^k$. We set $\boldsymbol{D}_\alpha = \{\boldsymbol{h} \in \boldsymbol{R}^k ; |\boldsymbol{h}| \leqslant \alpha\}$, $\alpha > 0$.

Theorem 1 : *Assume that the sequence $\{\boldsymbol{E}_n\}$ of experiments satisfies the LAMN condition at $\theta = \theta_0 \in \Theta$. Further assume that the functions $h \to P_{\theta_0+\boldsymbol{\delta}_n\boldsymbol{h},n}(A)$, $\boldsymbol{A} \in \boldsymbol{A}_n$, $n \geqslant 1$ are Borel measurable. Then for every sequence $\{V_n\}$ of estimators and for eevery $l \in L$*

$$\lim_{\alpha\to\infty}\ \liminf_{n\to\infty} \frac{1}{\mu^k(\boldsymbol{D}_\alpha)} \int_{\boldsymbol{D}_\alpha} E_{\theta_0+\boldsymbol{\delta}_n\boldsymbol{h}}[l(\boldsymbol{\delta}_n^{-1}(V_n-\theta_0-\boldsymbol{\delta}_n\boldsymbol{h}))]d\boldsymbol{h}$$

$$\geqslant E[l(\boldsymbol{T}^{-1/2}(\theta_0)\boldsymbol{W})]. \quad \ldots \ (1)$$

Theorem 2 : *Suppose that the sequence* $\{\boldsymbol{E}_n\}$ *satisfies the LAMN condition at* $\theta = \theta_0 \in \Theta$. *Let* $\{V_n\}$ *be a sequence of estimators such that the difference*

$$E(f(\boldsymbol{\delta}_n^{-1}(V_n-\theta_0-\boldsymbol{\delta}_n\boldsymbol{h}))\,|\,P_{\theta_0+\boldsymbol{\delta}_n\boldsymbol{h},n})-E(f(\boldsymbol{\delta}_n^{-1}(V_n-\theta_0))\,|\,P_{\theta_0,n})$$

converges to zero for every $\boldsymbol{h} \in \boldsymbol{R}^k$ *and for every continuous functions* $f : \boldsymbol{R}^k \to R$ *vanishing outside compacts. Then for all* $l \in L$

$$\liminf_{n\to\infty} E_{\theta_0}[l(\boldsymbol{\delta}_n^{-1}(V_n-\theta_0))] \geqslant E[l(\boldsymbol{T}^{-1/2}(\theta_0)\boldsymbol{W})].$$

Theorem 3 : *Suppose that the assumptions of Theorem* 1 *are satisfied. Further suppose that for a sequence* $\{V_n\}$ *of estimators and for a non-constant* $l \in \boldsymbol{L}$

$$\lim_{a\to\infty}\lim_{n\to\infty}\frac{1}{\mu^k(\boldsymbol{D}_a)}\int_{\boldsymbol{D}_a} E_{\theta_0+\boldsymbol{\delta}_n\boldsymbol{h}}[l(\boldsymbol{\delta}_n^{-1}(V_n-\theta_0-\boldsymbol{\delta}_n\boldsymbol{h}))]\,d\boldsymbol{h}$$

$$= E[l(\boldsymbol{T}^{-1/2}(\theta_0)\boldsymbol{W})] < \infty. \quad \ldots \quad (2)$$

Then for every $\varepsilon > 0$

$$\lim_{a\to\infty}\lim_{n\to\infty}\frac{1}{\mu^k(\boldsymbol{D}_a)}\int_{\boldsymbol{D}_a} P_{\theta_0+\boldsymbol{\delta}_n\boldsymbol{h},n}[\,|\boldsymbol{\delta}_n^{-1}(V_n-\theta_0)-\boldsymbol{T}_n^{-1/2}(\theta_0)\boldsymbol{W}_n(\theta_0)| > \varepsilon]\,d\boldsymbol{h} = 0.$$

$$\ldots \quad (3)$$

Theorem 4 : *Suppose that the sequence* $\{\boldsymbol{E}_n\}$ *satisfies the LAMN condition at* $\theta = \theta_0 \in \Theta$. *Then for every sequence* $\{V_n\}$ *of estimators and for every* $l \in \boldsymbol{L}$

$$\lim_{a\to\infty}\liminf_{n\to\infty}\sup_{|\boldsymbol{h}|\leqslant n} E_{\theta_0+\boldsymbol{\delta}_n,\boldsymbol{h}}[l(\boldsymbol{\delta}_n^{-1}(V_n-\theta_0-\boldsymbol{\delta}_n\boldsymbol{h}))] \geqslant E[l(\boldsymbol{T}^{-1/2}(\theta_0)].$$

Theorem 5 : (i) *Suppose that dim* $\Theta = 1$ *and that the assumption of Theorem* 4 *is satisfied. Further assume that* $E(\boldsymbol{T}^{-1/2}(\theta_0)) < \infty$ *if* l *is bounded and* $E[\boldsymbol{T}^{-1}\boldsymbol{W}^2 l(\boldsymbol{T}^{-1/2}\boldsymbol{W})] < \infty$ *if* l *is unbounded. Let* $\{V_n\}$ *be a sequence of estimators such that for a non-constant* $l \in \boldsymbol{L}$ *and for every* $\boldsymbol{h} \in R$ *and* $b > 0$, *setting* $l_b = min(b, l)$,

$$\limsup_{n\to\infty} E_{\theta_0+\boldsymbol{\delta}_n\boldsymbol{h}}[l_b(\boldsymbol{\delta}_n^{-1}(V_n-\theta_0-\boldsymbol{\delta}_n\boldsymbol{h}))] \leqslant E[l(\boldsymbol{T}^{-1/2}(\theta_0)\boldsymbol{W})]. \quad \ldots \quad (4)$$

Then the difference

$$\boldsymbol{\delta}_n^{-1}(V_n-\theta_0)-\boldsymbol{T}_n^{-1/2}(\theta_0)\boldsymbol{W}_n(\theta_0) \quad \ldots \quad (5)$$

converges to zero in $P_{\theta_0,n}$-*probability.*

(ii) *Suppose that* $\dim \Theta = 2$ *and that* $E(|\boldsymbol{T}^{-1/2}(\theta_0)\boldsymbol{W}|^5) < \infty$. *Further suppose that* (4) *holds, for all* $b > 0$, *for the quadratic loss* $l(x) = |x|^2$ *and for all* $\boldsymbol{h} \in \boldsymbol{R}^2$. *Then the difference* (5) *converges to zero in* $P_{\theta_0, n}$*-probability.*

Theorem 6 : *Let* $\lambda | \boldsymbol{B}^k$ *be a measure such that* $\lambda \ll \mu^k$ *and* $\lambda(\Theta) < \infty$. *Assume that the sequence* $\{\boldsymbol{E}_n\}$ *of experiments satisfies the LAMN condition for* μ^k*-almost all* $\theta \in \Theta$. *Further assume that the functions* $(\theta, h) \to P_{\theta+\delta_n h, n}(A)$, $A \in \boldsymbol{A}_n$, $n \geqslant 1$, *are* $\boldsymbol{B}^{2k}$*-measurable and that the functions* $\Lambda_n(\theta+\delta_n \boldsymbol{h}, \theta)$, $\boldsymbol{h} \in \boldsymbol{R}^k$, $n \geqslant 1$, *and* $\boldsymbol{W}_n(\theta)$, $\boldsymbol{T}_n(\theta)$, $n \geqslant 1$, *are* $\boldsymbol{A} \times \boldsymbol{B}^k$*-measurable. Set*

$$S_n^*(\theta, \boldsymbol{h}) = \frac{|det \boldsymbol{T}_n(\theta)|^{1/2}}{(2\pi)^{1/2}} \, exp[-\tfrac{1}{2}(\boldsymbol{h}-\boldsymbol{T}_n^{-1/2}(\theta)\boldsymbol{W}_n(\theta))'\boldsymbol{T}_n(\theta)(\boldsymbol{h}-\boldsymbol{T}_n^{-1/2}(\theta)\boldsymbol{W}_n(\theta))].$$

Let $\boldsymbol{H}$ *be a class of uniformly bounded Borel measurable functions of* $\boldsymbol{R}^k$. *Then the difference*

$$\int_{\Theta} E_\theta[f(\delta_n^{-1}(V_n-\theta))] \, \lambda(d\theta)$$

$$-\int_{\Theta} \int_{\boldsymbol{X}_n} \{ \int_{\boldsymbol{R}^k} f(\delta_n^{-1}(V_n-\theta)-\boldsymbol{h})) S_n^*(\theta, \boldsymbol{h}) \, d\boldsymbol{h}] dP_{\theta,n} \, \lambda(d\theta)$$

converges to zero uniformly for all $f \in \boldsymbol{H}$ *and for all sequences* $\{V_n\}$ *of estimators.*

Theorem 7 : *Let the measure* λ *be as in Theorem* 6. *Assume that the assumptions of Theorem* 6 *are satisfied. Then for every sequence* $\{V_n\}$ *of estimators and for every* $l \in \boldsymbol{L}$

$$\liminf_{n\to\infty} \int_{\Theta} E_\theta[l(\delta_n^{-1}(V_n-\theta))] \, \lambda(d\theta) \geqslant \int_{\Theta} E[l(\boldsymbol{T}^{-1/2}(\theta)\boldsymbol{W})] \, \lambda(d\theta). \quad \ldots \quad (6)$$

Theorem 8 : *Let the measure* λ *be as in Theorem* 6. *Assume that the assumptions of Theorem* 6 *are satisfied. Further assume that for a sequence* $\{V_n\}$ *of estimators and for a non-constant* $l \in \boldsymbol{L}$

$$\lim_{n\to\infty} \int_{\Theta} E_\theta[l(\delta_n^{-1}(V_n-\theta))] \, \lambda(d\theta) = \int_{\Theta} E[l(\boldsymbol{T}^{-1/2}(\theta)\boldsymbol{W})] \, \lambda(d\theta) < \infty. \quad \ldots \quad (7)$$

Then for every $\varepsilon > 0$

$$\lim_{n\to\infty} \int_{\Theta} P_{\theta,n}[|\delta_n^{-1}(V_n-\theta)-\boldsymbol{T}_n^{-1/2}(\theta)\boldsymbol{W}_n(\theta)| > \varepsilon] \, \lambda(d\theta) = 0.$$

Remark 2 : In connection with Theorem 2, one should see Heyde (1978) where one can also find important examples satisfying the LAMN condition.

The restriction imposed in Theorem 2 is known as the "invariance" restriction; in Jeganathan (1982) it is shown that when this invariance restriction is satisfied 'jointly' with the sequence $\{T_n(\theta_0)\}$ the limit distribution can be conditionally decomposed as a convolution, extending a famous result of Hájek (1970), and Theorem 2 and several other related inequalities can also be obtained from this convolution result; see Jeganathan (1982) for details.

Note that this invariance restriction cannot be in general relaxed, see LeCam (1953), but it can be shown (see LeCam, 1973 and 1979 Ch. 8 and Jeganathan, 1981) that these convolution results hold for almost all [Lebesgue] points of the parameter space, and Theorem 5, which does not involve any such restriction, can be deduced from these results also.

3. Some preliminary results

Lemma 1 : *Suppose that the sequence of experiments* $\{E_n\}$ *satisfies the LAMN-condition at* $\theta = \theta_0 \in \Theta$. *Then the sequence* $\{P_{\theta_0,n}\}$ *and* $\{P_{\theta_0+\delta_n h,n}\}$ *are continguous for all* $h \in R^k$. *Furthermore*

$$\mathcal{L}(T_n(\theta_0), T^{-1/2}(\theta_0)W_n(\theta_0) \,|\, P_{\theta_0+\delta_n h,n}) \Longrightarrow \mathcal{L}(T(\theta_0), T^{-1/2}(\theta_0)W+h).$$

Proof: The proof is a simple application of Theorem (2.1) of LeCam (1960); see Jeganathan (1982) for details.

Lemma 2 : *Assume that the sequence of experiments* $\{E_n\}$ *satisfies the LAMN-condition at* $\theta = \theta_0 \in \Theta$. *Then there exist*

(i) *an increasing sequence* $\{k_n\}$ *tending to infinity as* $n \to \infty$,

(ii) *functions* $C_n : \Theta \times R^k \to R$ *such that*

$$\sup_{|h|<a} |C_n(\theta_0, h)-1| \to 0 \;\; as \;\; n \to \infty$$

for all $a > 0$, *such that the measures* $Q_n(\theta_0, h) \,|\, A_n$ *defined by*

$$\frac{dQ_n(\theta_0, h)}{dP_{\theta_0,n}} = C_n(\theta_0, h) \, exp[h' T_n^{-1/2}(\theta_0) W_n^*(\theta_0) - \tfrac{1}{2} h' T_n(\theta_0) h]$$

with $W_n^*(\theta_0) = W_n(\theta_0) I(|T_n^{-1/2}(\theta_0)W_n(\theta_0)| \leqslant k_n)$, *are probability measures and satisfy*

$$\|P_{\theta_0+\delta_n h,n} - Q_n(\theta_0, h)\| \to 0 \;\; as \;\; n \to \infty$$

for all $h \in R^k$.

Proof: The proof is similar to the proof of Theorem (3.1) of LeCam (1960), see Jeganathan (1982) for details.

Lemma 3 : *Suppose that the sequence* $\{\boldsymbol{E}_n\}$ *satisfies the LAMN-condition at all* $\theta \in \Theta$. *Then a sequence* $\{\boldsymbol{\Delta}_n(\theta)\}$ *of k-vectors and a sequence* $\{\boldsymbol{T}_n^*(\theta)\}$ *of almost surely* $[P_{\theta,n}]$ *positive definite* $k \times k$ *matrices can be constructed from the log-likelihood ratios* Λ_n *in such a way that the differences* $\boldsymbol{T}_n^*(\theta)-\boldsymbol{T}_n(\theta)$, $\boldsymbol{T}_n^*(\theta+\delta_n\boldsymbol{h})-\boldsymbol{T}_n^*(\theta)$, $\boldsymbol{h} \in \boldsymbol{R}^k$, $\boldsymbol{\Delta}_n(\theta)-\boldsymbol{T}_n^{-1/2}(\theta)\boldsymbol{W}_n(\theta)$ *and* $\boldsymbol{\Delta}_n(\theta+\boldsymbol{\delta}_n\boldsymbol{h})-[\boldsymbol{\Delta}_n(\theta)-\boldsymbol{h}]$, $\boldsymbol{h} \in \boldsymbol{R}^k$, *converges to zero in* $P_{\theta,n}$*-probability for all* $\theta \in \Theta$.

Proof: The proof is contained in Chapter 12 of LeCam (1974).

Throughout what follows we set

$$S_n(\theta, \boldsymbol{h}) = \exp[\boldsymbol{h}'\boldsymbol{T}_n^{1/2}(\theta)\boldsymbol{W}_n^*(\theta) - \tfrac{1}{2}\boldsymbol{h}'\boldsymbol{T}_n(\theta)\boldsymbol{h}],$$

$$S_n^*(\theta, \boldsymbol{h}) = \frac{|\det \boldsymbol{T}_n(\theta)|^{1/2}}{(2\pi)^{k/2}} \exp(-\tfrac{1}{2}(\boldsymbol{h}-\boldsymbol{T}_n^{-1/2}(\theta)\boldsymbol{W}_n^*(\theta))'\boldsymbol{T}_n(\theta)(\boldsymbol{h}-\boldsymbol{T}_n^{-1/2}(\theta)\boldsymbol{W}_n^*(\theta))]$$

and

$$S(\theta, \boldsymbol{h}) = \exp[(\boldsymbol{h}'\boldsymbol{T}^{1/2}(\theta)\boldsymbol{W} - \tfrac{1}{2}\boldsymbol{h}'\boldsymbol{T}(\theta)\boldsymbol{h}]$$

where the sequence $\{\boldsymbol{W}_n^*(\theta)\}$ is the one constructed in Lemma 2. Further let $\boldsymbol{C}$ be the class of all sequences of $\boldsymbol{A}_n$-measurable k-vectors and let $\boldsymbol{H}$ be a class of uniformly bounded Borel measurable functions of $\boldsymbol{R}^k$. Without further mentioning, we will use the sequence $\{Q_n(\theta_0, \boldsymbol{h})\}$ of probability measures that was constructed in Lemma 2.

Lemma 4 : *Suppose that the assumptions of Theorem* 1 *are satisfied. Then for every* $\alpha > 0$, *the difference*

$$\frac{1}{\mu^k(\boldsymbol{D}_\alpha)} \int_{\boldsymbol{D}_\alpha} E_{\theta_0+\boldsymbol{\delta}_n\boldsymbol{h}}[f(\boldsymbol{Z}_n-\boldsymbol{h})]\, d\boldsymbol{h}$$

$$-\frac{1}{\mu^k(\boldsymbol{D}_\alpha)} \int_{\boldsymbol{D}_\alpha} \int_{\boldsymbol{X}_n} \left\{ \frac{\int_{\boldsymbol{D}_\alpha} f(\boldsymbol{Z}_n-\boldsymbol{u})\, S_n^*(\theta_0, \boldsymbol{u})d\boldsymbol{u}}{\int_{\boldsymbol{D}_\alpha} S_n^*(\theta_0, \boldsymbol{u})\, d\boldsymbol{u}} \right\} dQ_n(\theta_0, \boldsymbol{h})\, d\boldsymbol{h} \qquad \dots \quad (8)$$

converges to zero uniformly for all $f \in \boldsymbol{H}$ *and* $\boldsymbol{Z}_n \in \boldsymbol{C}$.

Proof: First note that the difference between the r.h.s. of the above difference (8) and the quantity

$$\frac{1}{\mu^k(\boldsymbol{D}_\alpha)} \int_{\boldsymbol{D}_\alpha} \int_{\boldsymbol{X}_n} \left\{ \frac{\int_{\boldsymbol{D}_\alpha} f(\boldsymbol{Z}_n-\boldsymbol{u})S_n^*(\theta_0, \boldsymbol{u})d\boldsymbol{u}}{\int_{\boldsymbol{D}_\alpha} S_n^*(\theta_0, \boldsymbol{u})\, d\boldsymbol{u}} \right\} S_n(\theta_0, \boldsymbol{h})\, dP_{\theta_0,n}\, d\boldsymbol{h}$$

$$= \frac{1}{\mu^k(\boldsymbol{D}_\alpha)} \int_{\boldsymbol{D}_\alpha} \int_{\boldsymbol{X}_n} f(\boldsymbol{Z}_n-\boldsymbol{h})S_n(\theta_0, \boldsymbol{h})\, dP_{\theta_0,n}\, d\boldsymbol{h}$$

converges to zero uniformly for all $f \in \boldsymbol{H}$ and $\boldsymbol{Z}_n \in \boldsymbol{C}$ by the statement (ii) of Lemma 2. Hence the result follows again from Lemma 2.

Lemma 5 : *Suppose that the assumptions of Theorem 1 are satisfied. Then the difference*

$$\frac{1}{\mu^k(\boldsymbol{D}_\alpha)} \int_{\boldsymbol{D}_\alpha} E_{\theta_0+\delta_n \boldsymbol{h}}[f(\boldsymbol{Z}_n-\boldsymbol{h})]d\boldsymbol{h}$$

$$-\frac{1}{\mu^k(\boldsymbol{D}_\alpha)} \int_{\boldsymbol{D}_\alpha} \int_{\boldsymbol{X}_n} \{\int_{\boldsymbol{R}^k} f(\boldsymbol{Z}_n-\boldsymbol{u})S_n^*(\theta_0, \boldsymbol{u})d\boldsymbol{u}\}dQ_n(\theta_0, \boldsymbol{h})d\boldsymbol{h} \quad \ldots \quad (9)$$

converges to zero uniformly for all $f \in \boldsymbol{H}$ and $\boldsymbol{Z}_n \in \boldsymbol{C}$ by first letting $n \to \infty$ and then letting $\alpha \to \infty$.

Proof: In view of Lemma 4 it is enough to show that the difference between the r.h.s. of (8) and the r.h.s. of (9) converges to zero uniformly for all $f \in \boldsymbol{H}$ and $\boldsymbol{Z}_n \in \boldsymbol{C}$ by first letting $n \to \infty$ and then $\alpha \to \infty$. It is easily checked that this difference is absolutely bounded by

$$\frac{2}{\mu^k(\boldsymbol{D}_\alpha)} \int_{\boldsymbol{D}_\alpha} \int_{\boldsymbol{X}_n} \{\int_{\boldsymbol{D}_\alpha^c} S_n^*(\theta_0, \boldsymbol{u})d\boldsymbol{u}\}dQ_n(\theta_0, \boldsymbol{h})d\boldsymbol{h}, \quad \ldots \quad (10)$$

where $\boldsymbol{D}_\alpha^c$ denotes the complement of the set $\boldsymbol{D}_\alpha$. Let, for each fixed $\boldsymbol{T}(\theta_0)$, $N_{(\boldsymbol{T}\theta_0)}$ denotes the k-variate normal distribution with mean vector $\theta \in \boldsymbol{R}^k$ and co-variance matrix $\boldsymbol{T}^{-1}(\theta_0)$. Then first letting $n \to \infty$ and using the statement (6) of Theorem 2.1 of LeCam (1960) it is easily seen that (10) converges to

$$\frac{2}{\mu^k(\boldsymbol{D}_\alpha)} \int_{\boldsymbol{D}_\alpha} \int N_{\boldsymbol{T}(\theta_0)}(\boldsymbol{D}_\alpha^c - \boldsymbol{T}^{-1/2}(\theta_0)\boldsymbol{W})S(\theta_0, \boldsymbol{h})d\mathcal{L}(\boldsymbol{W}, \boldsymbol{T}(\theta_0))d\boldsymbol{h}$$

$$= E_{\boldsymbol{T}}\left\{\frac{2}{\mu^k(\boldsymbol{D}_\alpha)} \int_{\boldsymbol{D}_\alpha} \int N_{\boldsymbol{T}(\theta_0)}(\boldsymbol{D}_\alpha^c - \boldsymbol{T}^{-1/2}(\theta_0)\boldsymbol{W})S(\theta_0, \boldsymbol{h})d\mathcal{L}(\boldsymbol{W})d\boldsymbol{h}\right\}$$

(using the independence of $\boldsymbol{T}(\theta_0)$ and $\boldsymbol{W}$)

$$= E_{\boldsymbol{T}}\left\{\frac{2}{\mu^k(\boldsymbol{D}_\alpha)} \int_{\boldsymbol{D}_\alpha} N_{\boldsymbol{T}(\theta_0)} * N_{\boldsymbol{T}(\theta_0)}(\boldsymbol{D}_\alpha^c - \boldsymbol{h})d\boldsymbol{h}\right\}$$

where $E_{\boldsymbol{T}}$ denotes the expectation w.r.t. the law of $\boldsymbol{T}(\theta_0)$. The lemma now follows from the following lemma whose ideas are contained in Hájek (1970), and stated and proved separately in Strasser (1978, Lemma 5).

Lemma 6 : *Let $P|\boldsymbol{B}^k$ be a probability measure. Then*

$$\lim_{a\to\infty} \frac{1}{\mu^k(\boldsymbol{D}_a)} \int_{\boldsymbol{D}_a} P(\boldsymbol{D}_a^c-\boldsymbol{h})d\boldsymbol{h} = 0.$$

The following simple lemma will be used in the proof of Theorem 5; in the present context it serves the purpose of Lemmas 3.1, 3.2 and 3.3 of Hájek (1972).

Lemma 7 : *Let* $\pi(\boldsymbol{h}) = \frac{1}{\sigma(2\pi)^{1/2}} exp\left(\frac{-h^2}{2\sigma^2}\right), \sigma > 0.$ *Assume that* $dim\, \Theta = 1$ *and that the assumptions of Theorem 1 are satisfied. Then the difference*

$$\int_R E_{\theta_0+\delta_n h}[l(Z_n-h)]\pi(h)dh$$

$$-\int_R \int_{X_n} \{\int_R l(Z_n-t)\psi_\sigma(t, W_n^*, T_n)dt\}dP_{\theta_0+\delta_n,h}\pi(h)dh \qquad \dots (11)$$

tends to zero as $n\to\infty$ for every sequence $\{Z_n\}$ of $\boldsymbol{A}_n$-measureable k-vectors and for $l \in L$, where

$$\psi_\sigma(h, w, t) = \frac{(1+t\sigma^2)^{1/2}}{\sigma(2\pi)^{1/2}} exp\left[-\frac{1}{2}\left(\frac{1+t\sigma^2}{\sigma^2}\right)\left(h-\frac{t^{1/2}\sigma^2 w}{1+t\sigma^2}\right)^2\right].$$

Proof: First note that the difference between the l.h.s. of the above difference (11) and the quantity

$$\int_{|h|\leq a} E_{\theta_0+\delta_n h}[l(Z_n-h)]\pi(h)dh, \qquad \dots (12)$$

is absolutely bounded by

$$\int_{|h|>a} \pi(h)dh. \qquad \dots (13)$$

Further, since

$$\int_{|h|\leq a} \int_{X_n} l(Z_n-h)S_n(\theta_0, h)dP_{\theta_0,n}\, \pi(h)dh$$

$$= \int_{|h|\leq a} \int_{X_n} \left\{ \frac{\int_{|t|\leq a} l(Z_n-t)S_n^*(\theta_0, t)\pi(t)dt}{\int_{|t|\leq a} S_n^*(\theta_0, t)\pi(t)dt} \right\} S_n(\theta_0, h)dP_{\theta_0,h}\pi(h)dh,$$

the difference between (12) and the quantity

$$\int_{|h|\leq\alpha}\int_{X_n}\left\{\frac{\int_{|t|\leq\alpha} l(Z_n-t)S_n^*(\theta_0,t)\pi(t)dt}{\int_{|t|\leq\alpha} S_n^*(\theta_0,t)\pi(t)dt}\right\}dP_{\theta_0+\delta_n h,n}\,\pi(h)dh \quad \dots \quad (14)$$

converges to zero as $n\to\infty$ for every $\alpha > 0$ by Lemma 2. Now the difference between (14) and the quantity

$$\int_R\int_{X_n}\left\{\frac{\int_{|t|\leq\alpha} l(Z_n-t)S_n^*(\theta_0,t)\pi(t)dt}{\int_{|t|\leq\alpha} S_n^*(\theta_0,t)\pi(t)dt}\right\}dP_{\theta_0+\delta_n h,n}\,\pi(h)\,dh \quad \dots \quad (15)$$

is absolutely bounded by (13). Moreover it is easy to see that the difference between (15) and the r.h.s. of the difference (11) is absolutely bounded by

$$2\int_R\int_{X_n}\left\{\frac{\int_{|t|>\alpha} S_n^*(\theta_0,t)\pi(t)dt}{\int_R S_n^*(\theta_0,t)\pi(t)dt}\right\}dP_{\theta_0+\delta_n h,n}\,\pi(h)\,dh. \quad \dots \quad (16)$$

(Note that $\dfrac{S_n^*(\theta_0,h)\pi(h)}{\int S_n^*(\theta_0,h)\pi(h)dh} = \psi_*(h, W_n^*, T_n)$.)

Obviously (13) tends to zero as $\alpha\to\infty$. Using the statement (6) of Theorem 2.1 of LeCam (1960) it is easy to see that, for every $\alpha > 0$, (16) converges to a limit as $n\to\infty$, and it is clear that this limit tends to zero as $\alpha\to\infty$. Hence the proof of the lemma is complete.

Remark 3 : Note that Lemma 7 holds for more than one-dimension also and for any arbitrary finite prior; the proof is same as above. It will be indicated that Theorem 4 can be obtained as a simple consequence of this lemma also.

4. Proofs of the results

Proof of Theorem 1 : It is enough to prove the result for bounded loss functions. Then we have

$$\lim_{\alpha\to\infty}\liminf_{n\to\infty}\frac{1}{\mu^k(\boldsymbol{D}_\alpha)}\int_{D_\alpha} E_{\theta_0+\delta_n h}\,[l(\boldsymbol{\delta}_n^{-1}(V_n-\theta_0)-\boldsymbol{h})]\,d\boldsymbol{h}$$

$$=\lim_{\alpha\to\infty}\liminf_{n\to\infty}\frac{1}{\mu^k(\boldsymbol{D}_\alpha)}\int_{D_\alpha}\int_{X_n}\{\int_{R^k} l(\boldsymbol{\delta}_n^{-1}(V_n-\theta_0)-\boldsymbol{u})S_n^*(\theta_0,\boldsymbol{u})d\boldsymbol{u}\}dQ_n(\theta_0,\boldsymbol{h})d\boldsymbol{h}$$

(by Lemma 5)

$$\geqslant \lim_{a\to\infty} \liminf_{n\to\infty} \frac{1}{\mu^k(D_a)} \int_{D_a} \int_{X_n} \{ \int_{R^k} l(T_n^{-1/2}(\theta_0)W_n^*(\theta_0)-u) S_n^*(\theta_0, u) du\} dQ(\theta_0, h) dh$$

(by Anderson, 1955)

$$= \lim_{a\to\infty} \liminf_{n\to\infty} \frac{1}{\mu^k(D_a)} \int_{D_a} E_{\theta_0+\delta_n h_n}[l(T_n^{-1/2}(\theta_0)W_n^*(\theta_0)-h)]\, dh$$

(by Lemma 5)

$$= \lim_{a\to\infty} \liminf_{n\to\infty} \frac{1}{\mu^k(D_a)} \int_{D_a} \int P_{\theta_0+\delta_n h_n}[I(\,T_n^{-1/2}(\theta_0)W_n^*(\theta_0)-h\,| > a]dl(a)\, dh$$

$$\geqslant \int l(T^{-1/2}(\theta_0)W) d\mathcal{L}(W, T(\theta_0)). \qquad \text{(by Lemma 1)}$$

Hence the proof is complete.

Proof of Theorem 3 : Let $\lambda_{n1} < \lambda_{n2} < \ldots < \lambda_{nk}$ be the eigen values of $T_n(\theta_0)$. An application of a result of Anderson (1955) shows that whenever $|\delta_n^{-1}(V_n-\theta_0)-T_n^{-1/2}(\theta_0)W_n^*(\theta_0)| > \varepsilon > 0$, $0 < \delta \leqslant \lambda_{n1}$, $\lambda_{nk} \leqslant M$ and $l \in L$ is non-constant there exists a continuous function $\eta(\varepsilon, \lambda_{n1}, \ldots, \lambda_{nk})$ of $(\lambda_{n1}, \ldots, \lambda_{nk})$ such that $\eta(\epsilon, \lambda_{n1}, \ldots, \lambda_{nk}) > 0$ and that the difference

$$\int_{R^k} l(\delta_n^{-1}(V_n-\theta_0)-T_n^{-1/2}(\theta_0)W_n^*(\theta_0)-u) \exp(-\tfrac{1}{2}u'T_n(\theta_0)u)\, du$$

$$- \int_{R^k} l(u) \exp(-\tfrac{1}{2}u'T_n(\theta_0)u)\, du$$

is greater than or equal to $\eta(\epsilon, \lambda_{n1}, \ldots, \lambda_{nk})$. Let

$$\eta'(\varepsilon, \delta, M) = \inf\{\eta(\varepsilon, \lambda_{n1}, \ldots, \lambda_{nk}) : \delta \leqslant \lambda_{n1} < \ldots < \lambda_{nk} \leqslant M\},$$

In view of continuity $\eta'(\varepsilon, \delta, M) > 0$. Let

$$A_n = \{|\delta_n^{-1}(V_n-\theta_0)-T_n^{-1/2}(\theta_0)W_n^*(\theta_0)| > \epsilon, \delta \leqslant \lambda_{n1}, \lambda_{nk} \leqslant M\}.$$

Let $l_a = \min(a, l)$. Since the given l is non-constant there exists an $a_0 > 0$ such that la_0 is non-constant. Let η' above be the one corresponding to la_0. Also it is easy to see that $\eta'(a)$ corresponding to l_a, $a \geqslant a_0$, can be chosen in such a way that $\eta'(a)$ is increasing with a. Then it follows from the above arguments that, for $a \geqslant a_0$,

$$\frac{1}{\mu^k(D_a)} \int_{D_a} \int_{X_n} \{ \int_{R_k} l_a(\delta_n^{-1}(V_n-\theta_0)-u) S_n^*(\theta_0, u) du\} dQ_n(\theta_0, h) dh$$

$$\geqslant \frac{\eta'(\epsilon, \delta, M)}{\mu^k(D_a)} \int_{D_a} \int_{X_n} I(A_n) dQ_n(\theta_0, h) dh$$

$$+ \frac{1}{\mu^k(D_a)} \int_{D_a} \int_{X_n} \{ \int_{R^k} l_a(T_n^{-1/2}(\theta_0)W_n^*(\theta_0)-u) S_n^*(\theta_0, u) du\}$$

$$dQ_n(\theta_0, h) dh. \qquad \ldots (17)$$

In view of the given condition (2) and Lemma 5 we have

$$\lim_{a\to\infty}\lim_{n\to\infty}\frac{1}{\mu^k(D_a)}\int_{D_a}\int_{X_n}\int_{R^k} l_a(\delta_n^{-1}(V_n-\theta_0)-u)S_n^*(\theta_0, u)du\, dQ_n(\theta_0, h)dh$$

$$\leqslant E[l(T^{-1/2}(\theta_0)W)]. \qquad \ldots \quad (18)$$

Furthermore, in view of the arguments of the proof of Theorem 1 we see that

$$\lim_{a\to\infty}\liminf_{n\to\infty}\frac{1}{\mu^k(D_a)}\int_{D_a}\int_{X_n}\int_{R^k} l_a(T_n^{-1/2}(\theta_0)W_n^*(\theta_0)-u)S_n^*(\theta_0, u)du\, dQ_n(\theta_0, n)dh$$

$$\geqslant E[l_a(T^{-1/2}(\theta_0)W)]. \qquad \ldots \quad (19)$$

From (17), (18) and (19) it now follows that by letting $a\to\infty$

$$\lim_{a\to\infty}\lim_{n\to\infty}\frac{1}{\mu^k(D_a)}\int_{D_a}\int_{X_n} I(A_n)dQ_n(\theta_0, h)dh = 0. \qquad \ldots \quad (20)$$

Further, in view of the invariance relation (Lemma 3)

$$\mathcal{L}(T_n(\theta_0)\,|\,P_{\theta_0+\delta_n, h}) \Longrightarrow \mathcal{L}(T(\theta_0)$$

and since $T(\theta_0)$ is p.d. almost surely we see that for every $\varepsilon > 0$ there exist positive constants δ and M such that

$$\lim_{a\to\infty}\limsup_{n\to\infty}\frac{1}{\mu^k(D_a)}\int_{D_a}\int_{X_n}[I(\delta>\lambda_{n1})+I(\lambda_{nk}>M)]dQ_n(\theta_0, h)dh \leqslant \varepsilon \quad \ldots \quad (21)$$

From (20) and (21) it follows that for every $\varepsilon > 0$

$$\lim_{a\to\infty}\lim_{n\to\infty}\frac{1}{\mu^k(D_a)}\int_{D_a}\int_{X_n} I\,|(\delta_n^{-1}(V_n-\theta_0)-T_n^{-1/2}(\theta_0)W_n^*(\theta_0)| > \varepsilon)$$

$$dQ_n(\theta_0, h)dh = 0.$$

Hence the proof follows from Lemma 2.

Proof of Theorem 4 : Note that when the measurability condition of Theorem 1 is assumed, the proof is immediate from Theorem 1. To prove the general case, first note that the sequence $\{Q_n(\theta_0, h)\}$ satisfies the measurability condition of Theorem 1 and that Theorem 1 is valid when the sequence $\{P_{\theta_0+\delta_n h, n}\}$ is replaced by $\{Q_n(\theta_0, h)\}$. Partition D_a into blocks $C_{1m}, \ldots, C_{mm}$ such that $\sup_{1\leqslant j\leqslant m} \mu^k(C_{jm}) \to 0$ as $m\to\infty$. Let h be a fixed point in

C_{jm}, $j = 1, \ldots, m$. It is enough to consider bounded and continuous $l \in \boldsymbol{L}$. Then it is easy to see that the difference

$$\frac{1}{\mu^k(\boldsymbol{D}_a)} \sum_{j=1}^{m} \left[\int_{\boldsymbol{X}_n} l(\delta_n^{-1}(V_n-\theta_0)-h_j)dQ_n(\theta_0, h_j)\right]\mu^k(C_{jm})$$

$$-\frac{1}{\mu^k(D_a)} \int_{D_a} \int_{\boldsymbol{X}_n} l(\delta_n^{-1}(V_n-\theta_0)-\boldsymbol{h})dQ_n(\theta_0, \boldsymbol{h})d\boldsymbol{h} \qquad \ldots \quad (22)$$

converges to zero by first letting $n \to \infty$ and then $m \to \infty$. Further, it is clear that the difference between the l.h.s. of (22) and the quantity

$$\frac{1}{\mu^k(\boldsymbol{D}_a)} \sum_{j=1}^{m} E_{\theta_0+\delta_n h_j}[l(\delta_n^{-1}(V_n-\theta_0)-\boldsymbol{h}_j)]\mu^k(C_{jm})$$

converges to zero as $n \to \infty$ for every m and $\alpha > 0$. Hence

$$\lim_{a\to\infty} \liminf_{n\to\infty} \sup_{|h|\leqslant a} E_{\theta_0+\delta_n h}[l(\delta_n^{-1}(V_n-\theta_0)-\boldsymbol{h})]$$

$$\geqslant \lim_{a\to\infty} \lim_{m\to\infty} \liminf_{n\to\infty} \frac{1}{\mu^k(\boldsymbol{D}_a)} \sum_{j=1}^{m} E_{\theta_0-\delta_n h_j}[l(\delta_n^{-1}(V_n-\theta_0)-\boldsymbol{h}_j)]\mu^k(C_{jm})$$

$$= \lim_{a\to\infty} \liminf_{n\to\infty} \frac{1}{\mu^k(\boldsymbol{D}_a)} \int_{\boldsymbol{D}_a} \int_{\boldsymbol{X}_n} l(\delta_n^{-1}(V_n-\theta_0)-\boldsymbol{h})dQ_n(\theta_0, \boldsymbol{h})d\boldsymbol{h}.$$

Hence the proof.

Remark 4 : As was mentioned (see Remark 3 of Section 3) Lemma 7 holds for more than one-dimention also and for any arbitrary finite prior, and hence Theorem 4 can be obtained from this lemma also in the standard way by taking normal priors on $\boldsymbol{R}^k$ with co-variance matrix $m\boldsymbol{I}$, using Anderson's lemma (1955), and letting $m \to \infty$.

Proof of Theorem 5 *for dim* $\Theta = 1$: First note that the condition (4) and the corresponding conclusion of Theorem 5 hold for the sequence $\{P_{\theta_0+\delta_n h, n}\}$ if and only if they hold for the sequence $\{Q_n(\theta_0, h)\}$. Further note that Lemma 7 is valid when the sequence $\{P_{\theta_0+\delta_n h, n}\}$ is replaced by $\{Q_n(\theta_0, h)\}$. Hence it is enough to prove the theorem as it stands with the additional assumption that the functions $h \to P_{\theta_0+\delta_n h, n}(A)$, $A \in \boldsymbol{A}_n$, $n \geqslant 1$, are Borel measurable, since this measurability assumption is satisfied for the sequence

$\{Q_n(\theta_0, h)\}$. We then have, (in what follows we suppress θ_0 and write $\boldsymbol{W}_n$, $\boldsymbol{T}_n$ and $\boldsymbol{T}$ instead of $\boldsymbol{W}_n(\theta_0)$, $\boldsymbol{T}_n(\theta_0)$ and $\boldsymbol{T}(\theta_0)$), setting $l_b = \min(b, l)$,

$$E[l(T^{-1/2}W)] \geqslant \limsup_{n\to\infty} \int_R E_{\theta_0+\delta_n h}[l_b(\delta_n^{-1}(V_n-\theta_0-\delta_n h))]\pi(h)dh$$

(by (4))

$$= \limsup_{n\to\infty} \int_R \int_{X_n} \int_R l_b(\delta_n^{-1}(V_n-\theta_0-\delta_n t))\psi_\sigma(t, W_n, T_n)dtdP_{\theta_0+\delta_n h, n}\pi(h)dh$$

(by Lemma 7)

$$\geqslant \liminf_{n\to\infty} \int_R \int_{X_n} \int_R l_b\left[\frac{T_n^{1/2} W_n\sigma^2}{1+T_n\sigma^2} - t\right]\psi_\sigma(t, W_n, T_n)dtdP_{\theta_0+\delta_n h, n}\pi(h)dh$$

$$\geqslant \int_R \int_{R^2} \int_R l_b\left[\frac{T^{1/2}W\sigma^2}{1+T\sigma^2} - t\right]\psi_\sigma(t, W, T)dtS(\theta_0, h)d\mathcal{L}(T, W)\pi(h)dh$$

(by statement (6) of Theorem (2.1) of LeCam (1960))

$$= \int_R \int_{R^2} \int_R \frac{(1+T\sigma^2)^{1/2}}{\sigma(2\pi)^{1/2}} l_b(t)\exp\left(-\frac{(1+\sigma^2 T)}{2\sigma^2}t^2\right)dtS(\theta_0, h)d\mathcal{L}(T, W)\pi(h)dh$$

$$= E\left[\frac{(1+T\sigma^2)^{1/2}}{\sigma(2\pi)^{1/2}}\int_R l_b(t)\exp\left(-\frac{1+\sigma^2 T)}{2\sigma^2}t^2\right)dt\right]$$

[since $\mathcal{L}(T) = \mathcal{L}(T \mid R_{\theta_0, h})$ where $dP_{\theta_0, h} = S(\theta_0, h)d\mathcal{L}(T,)W$]

$$> E\left[\frac{T^{1/2}}{(2\pi)^{1/2}}\int_R l_b(t)\exp\left(-\frac{(1+\sigma^2 T)}{2\sigma^2}t^2\right)dt\right]$$

$$\geqslant E[l_b(T^{-1/2}W)] - E\left\{\frac{T^{1/2}}{2(\pi)^{1/2}}\int l(t)\left[\exp\left(-\frac{Tt^2}{2}\right) - \exp\left(-\frac{1+\sigma^2 T}{2\sigma^2}t^2\right]dt\right\}.$$

Let, for some $\epsilon > 0$, $M > \delta > 0$,

$$A_n = \left\{\left|\delta_n^{-1}(V_n-\theta_0) - \frac{T_n^{1/2}W_n\sigma^2}{1+T_n\sigma^2}\right| > \epsilon, \delta < T_n < M, |W_n| \leqslant M\right\}.$$

Now whenever $\sigma > a > 0$ and the event A_n is true the inequality $\delta < \left(\frac{1+T_n\sigma^2}{\sigma^2}\right) < M + a^{-2}$ holds. Also note that since l is non-constant there exists a b_0 such that l_b is non-constant for all $b \geqslant b_0$. Hence it is easily seen that there exists a positive constant K depending only on ϵ, δ, a, b and M such that

$$\int_R \int_{X_n} \int_R l_b(\delta_n^{-1}(V_n - \theta_0 - \delta_n t))\psi_\sigma(t, W_n, T_n)dt dP_{\theta_0+\delta_n h, n}\, \pi(h)dh$$

$$\geqslant K \int_R \int_{X_n} I(A_n)dP_{\theta_0+\delta_n h, n}\, \pi(h)dh$$

$$+ \int_R \int_{X_n} \int_R l_b\left[\frac{T_n^{1/2}W_n\sigma^2}{1+T_n\sigma^2} - t\right]\psi_\sigma(t, W_n, T_n)dt dP_{\theta_0+\delta_n h, n}\, \pi(h)dh$$

for every $n \geqslant 1$ whenever $\sigma > a$. Also it is easy to see that K can be chosen in such a way that it, as a function of b, increases with b. Hence it follows from the inequalities presented in the beginning of the proof and by letting $b \to \infty$, that

$$\limsup_{n\to\infty} \int_R \int_{X_n} I(A_n)P_{\theta_0+\delta_n h, n}\, \pi(h)dh$$

$$\leqslant CE\left\{\int_U \frac{T^{1/2}}{(2\pi)^{1/2}}\, l(t)\left[\exp\left(-\tfrac{1}{2}Tt^2\right) - \exp\left(-\frac{1+\sigma^2 T}{2\sigma^2}\, t^2\right)\right] dt\right\}$$

$$\leqslant CE(g(T, \sigma)), \quad \sigma > a, \qquad \ldots \quad (23)$$

where C is a positive constant depending only on ϵ, δ, a, b_0 and M, and

$$E(g(T,\sigma)) = \begin{cases} \alpha(1+T\sigma^2)^{1/2}[(1+T\sigma^2)^{1/2} + T^{1/2}\sigma]^{-1} & \text{if } l \text{ is bounded by } \alpha, \\ E[T^{-1}W^2\, l(T^{-1/2}W)]/2\sigma^2 & \text{if } l \text{ is unbounded.} \end{cases}$$

[When l is unbounded, we have used the inequality

$$\exp\left(-\frac{(1+\sigma^2 T)}{2\sigma^2}\, t^2\right) \geqslant \left(1 - \frac{t^2}{2\sigma^2}\right)\exp\left(-\frac{Tt^2}{2}\right)]$$

Now note that the difference

$$\int_{|h|\leqslant a} \int_{X_n} I(A_n)S_n(\theta_0, h)dP_{\theta_0, n}\, \pi(h)dh$$

$$- \int_{|h|\leqslant a} \int_{X_n} I(A_n)dP_{\theta_0+\delta_n, h n}\, \pi(h)dh$$

tends to zero for every $\alpha > 0$ by Lemma 2. Further

$$\int_{|h|>\alpha} \int_{X_n} I(A_n) S_n(\theta_0, h) dP_{\theta_0, n} \pi(h) dh$$

$$\leqslant \int_{|h|>\alpha} \int_{X_n} I(\delta < T_n < M, |W_n| \leqslant M) S_n(\theta_0, h) dP_{\theta_0, n} \pi(h) dh$$

$$\leqslant \int_{|h|>\alpha} \pi(h) dh \text{ for all } \alpha > \frac{2M^{3/2}}{\delta}$$

$$\to 0 \text{ as } \alpha \to \infty.$$

Also

$$\int_{|h|>\alpha} \int_{X_n} I(A_n) dP_{\theta_0+\delta_n h, n} \pi(h) dh \leqslant \int_{|h|>\alpha} \pi(h) dh \to 0 \text{ as } \alpha \to \infty.$$

Therefore from (23) we have

$$\limsup_{n\to\infty} \int_{B} \int_{X_n} I(A_n) S_n(\theta_0, h) dP_{\theta_0, n} \pi(h) dh \leqslant CE(g(T, \sigma)) \quad \dots \quad (24)$$

whenever $\sigma > a$. Now note that

$$\int_{B} \int_{X_n} I(A_n) S_n(\theta_0, h) dP_{\theta_0, n} \pi(h) dh$$

$$= \int_{X_n} \frac{1}{(1+\sigma^2 T_n)^{1/2}} I(A_n) \exp\left(\frac{T_n \sigma^2 W_n^2}{2(1+T_n \sigma^2)} \right) dP_{\theta_0, n}$$

$$\geqslant \exp\left(-\frac{\delta M^2 \sigma^2}{1+M\sigma^2} \right) (1+\sigma^2 M)^{-1/2} \int_{X_n} I(A_n) dP_{\theta_0, n},$$

since $\delta < T_n < M$ and $|W_n| \leqslant M$ whenever the event A_n occurs. Hence from (24) we have

$$\limsup_{n\to\infty} P_{\theta_0, n}[|\delta_n^{-1}(V_n - \theta_0) - T_n^{-1/2} W_n| > 2\varepsilon, \delta < T_n < M, |W_n| < M]$$

$$\leqslant \limsup_{n\to\infty} P_{\theta_0, n}(A_n) + \limsup_{n\to\infty} P_{\theta_0, n}\left[\left| T_n^{-1/2} W_n - \frac{T_n^{1/2} W_n \sigma^2}{1+T_n \sigma^2} \right| > \varepsilon \right]$$

$$\leqslant C\sigma M \exp\left(\frac{\delta M^2 \sigma^2}{1+M\sigma^2} \right) E(g(T, \sigma))$$

$$+ \limsup_{n\to\infty} P_{\theta_0, n}\left[\left| T_n^{-1/2} W_n - \frac{T_n^{1/2} W_n \sigma^2}{1+T_n \sigma^2} \right| > \varepsilon \right] \quad \dots \quad (25)$$

whenever $\sigma > a$. Now note that when l is bounded by α, $g(\boldsymbol{T}, \sigma)$ is bounded by both $\alpha \boldsymbol{T}^{-1}\sigma^{-2}$ and $\alpha \boldsymbol{T}^{-1/2}\sigma^{-1}$. Hence applying the dominated convergence theorem we see that the last term of the inequality (25) tends to zero as $\sigma \to \infty$. Hence the proof of the theorem is complete by choosing δ and M in such a way that

$$\limsup_{n\to\infty} [P_{\theta_0, n}(\boldsymbol{T}_n < \delta)+P_{\theta_0, n}(\boldsymbol{T}_n > M)+P_{\theta n}(|\boldsymbol{W}_n| > M)] \leqslant \varepsilon.$$

We now present the proof of Theorem 4 for $\dim\Theta = 2$. The proof of this case is given separately since in this case we are dealing with an unbounded loss function and this fact makes the proof very simple; note that the arguments of the proof of this case are applicable without any change to the case $\dim \Theta = 1$ also when the loss functions are unbounded.

The proof of Theorem 4 for the case $\dim \Theta = 2$. By Chebyshev's inequality we have

$$P_{\theta_0, n}(|\boldsymbol{\delta}_n^{-1}(V_n-\theta_0)| > \alpha) = [l(\alpha)]^{-1}E[l_{\varepsilon+1}(\boldsymbol{\delta}_n^{-1}(V_n-\theta_0))].$$

Hence setting $l(x) = |x|^2$ it follows from the given condition that the sequence $\{\boldsymbol{\delta}_n^{-1}(V_n-\theta_0)\}$ is relatively compact for $\{P_{\theta_0, n}\}$. Hence for every subsequence $\{r\} \subsetneq \{n\}$ there exists a further subsequence $\{m\} \subset \{r\}$ such that $\{\mathcal{L}(\boldsymbol{\delta}_m^{-1}(V_m-\theta_0), \boldsymbol{W}_m, \boldsymbol{T}_m | P_{\theta_0, m})\}$ converges in distribution to a distribution $F_{(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{T})}(v, w, t)$ of a random vector $(\boldsymbol{V}, \boldsymbol{W}, \boldsymbol{T})$. Hence using the statement (6) of Theorem (2.1) of LeCam (1960) we have

$$\liminf_{n\to\infty} P_{\theta_0+\boldsymbol{\delta}_m \boldsymbol{h}, m}(l_b(\boldsymbol{\delta}_m^{-1}(V_m-\theta_0-\boldsymbol{\delta}_m \boldsymbol{h})))$$

$$\geqslant \int l_b(v-h)Q(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{t})dF(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{t})$$

for all $\boldsymbol{h} \in R^2$ and $b > 0$, where we set

$$Q(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{t}) = \exp(\boldsymbol{h}'\boldsymbol{t}^{1/2}\boldsymbol{w}-\tfrac{1}{2}\boldsymbol{h}'\boldsymbol{t}\boldsymbol{h}).$$

Hence it follows from the given condition by letting $b \to \infty$ that

$$\int |v-h|^2 Q(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{t})dF(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{t}) \leqslant \int |\boldsymbol{t}^{-1/2}\boldsymbol{w}|^2 dF(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{t})$$
$$= \int |\boldsymbol{t}^{-1/2}\boldsymbol{w}-\boldsymbol{h}|^2 Q(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{t})dF(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{t})$$

for all $\boldsymbol{h} \in R^2$. We now invoke a result of Brown and Fox (1974) (see Remark 5 below) to conclude that the strict inequality in the above inequality cannot hold, i.e.,

$$\int |\boldsymbol{v}-\boldsymbol{h}|^2 Q(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{t})dF(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{t}) = \int |\boldsymbol{t}^{-1/2}\boldsymbol{w}|^2 dF(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{t})$$

for all $\boldsymbol{h} \in \boldsymbol{R}^2$. Similarly, using the identity

$$|\boldsymbol{v}-\boldsymbol{h}|^2+|\boldsymbol{t}^{-1/2}\boldsymbol{w}-\boldsymbol{h}|^2-2\left|\frac{\boldsymbol{v}+\boldsymbol{t}^{-1/2}\boldsymbol{w}}{2}-\boldsymbol{h}\right|^2 = |\boldsymbol{v}-\boldsymbol{t}^{-1/2}\boldsymbol{w}|^2,$$

we have

$$\int \left|\frac{\boldsymbol{v}+\boldsymbol{t}^{-1/2}\boldsymbol{w}}{2}-h\right|^2 Q(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{t})dF(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{t}) = \int |\boldsymbol{t}^{-1/2}\boldsymbol{w}|^2 dF(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{t})$$

for all $\boldsymbol{h} \in \boldsymbol{R}^2$. Hence it follows from the above identities that

$$\int |\boldsymbol{v}-\boldsymbol{t}^{-1/2}\boldsymbol{w}|^2 Q(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{t})dF(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{t}) = 0.$$

From this it follows easily that

$$\boldsymbol{V} = \boldsymbol{T}^{-1/2}\boldsymbol{W} \text{ a.s. } [F].$$

Now note that the difference $\boldsymbol{\delta}_m^{-1}(\boldsymbol{V}_m-\theta_0)-\boldsymbol{T}^{-1/2}\boldsymbol{W}_m$ converges weakly, under $P_{\theta_0, m}$, to the r.v. $\boldsymbol{V}-\boldsymbol{T}^{-1/2}W = 0$ a.s. Hence

$$\boldsymbol{\delta}_m^{-1}(\boldsymbol{V}_m-\theta_0)-\boldsymbol{T}^{-1/2}\boldsymbol{W}_m \to 0$$

in $P_{\theta_0, m}$ probability. Since this is true for every convergent subsequences $\{m\} \subseteq \{n\}$, the desired conclusion follows.

Remark 5 : In order to apply the results of Brown and Fox (1974), take $\boldsymbol{T}^{-1/2}\boldsymbol{W} = \boldsymbol{X}$ and $\boldsymbol{T} = \boldsymbol{Y}$ where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are as in their paper with mutually absolutely continuous probability measures $\{G_{\boldsymbol{h}}; \boldsymbol{h} \in \boldsymbol{R}^2\}$, such that

$$\frac{dG_{\boldsymbol{h}}}{dG_0} = Q(\boldsymbol{h}, \boldsymbol{W}, \boldsymbol{T})$$

where the function Q is as defined in the above proof and the $\boldsymbol{A}$-mble $\boldsymbol{T}$ and $\boldsymbol{W}$ are such that $\boldsymbol{W}$ is 2-variate $N(0, I)$ independent of the 2×2 random matrix $\boldsymbol{T}$. $\boldsymbol{T}^{-1/2}\boldsymbol{W}$ and $\boldsymbol{T}$ are together sufficient and since the loss function is quadratic we need only consider estimates which are functions of these. For the quadratic loss they have given some simple conditions and have verified that these specific conditions imply their more general conditions needed for the validity of the their result : for this the existence of the fifth moment of $|\boldsymbol{T}^{-1/2}\boldsymbol{W}|$ assumed by us will suffice.

Before going into the details of the proof of Theorem 6, let us first observe that, when the measurability condition of Theorem 6 is satisfied, the random functions $\boldsymbol{T}_n^*(\theta)$, $\boldsymbol{\Delta}_n(\theta)$, $n \geqslant 1$, constructed in Lemma 3 will be $\boldsymbol{A}_n \times \boldsymbol{B}^k$-measurable; furthermore, it is easy to see that it is enough to prove the statement of Theorem 6 with the sequences $\{\boldsymbol{W}_n(\theta)\}$ and $\{\boldsymbol{T}_n(\theta)\}$ replaced by $\{\boldsymbol{\Delta}_n(\theta)\}$ and $\{\boldsymbol{T}_n^*(\theta)\}$. Therefore in what follows we will assume, without loss of generality, that the random functions $\boldsymbol{W}_n(\theta_0)$, $\boldsymbol{T}_n(\theta)$, $n \geqslant 1$, satisfy the regularity properties of Lemma 3 and that they are jointly measurable.

Let us also observe that Lemma 5 is valid when the sequence $\{\boldsymbol{W}_n^*(\theta_0)\}$ is replaced by $\{\boldsymbol{W}_n(\theta_0)\}$ since the difference $\boldsymbol{W}_n(\theta)-\boldsymbol{W}_n^*(\theta_0)$ converges to zero in $P_{\theta_0, n}$ probability, and that the functions $\boldsymbol{C}_n(\theta, \boldsymbol{h})$ will be jointly measurable whenever $\boldsymbol{W}_n(\theta)$ and $\boldsymbol{T}_n(\theta)$ are jointly measurable.

The following well-known result of Lebesgue will be used in the proof; for the sake of convenience we state it separately.

Lemma 8 : *For any function* $f \in L_p(\mu^k)$, $p \geqslant 1$, *the function*

$$\int |f(x+h)-f(x)|^p dx$$

is uniformly continuous in h.

Proof: See, e.g., Corollary 39.2 of Parthasarathy (1977).

Proof of Proposition 1 : For simplicity assume that $\Theta = R^k$. Now consider

$$\int_{R^k} E_{\theta+\delta_n h}[l(\delta_n^{-1}(V_n-\theta-\delta_n \boldsymbol{h}))]g(\theta)d\theta$$

$$= \int_{R^k} E_\theta[l(\delta_n^{-1}(V_n-\theta))]g(\theta-\delta_n h)d\theta$$

where $g(\theta)$ is the density of λ with respect to μ^k. Hence it follows from Lemma 8 that the difference

$$\int_{R^k} \frac{1}{\mu^k(D_a)} \int_{D_a} E_{\theta+\delta_n h}[l(\delta_n^{-1}(V_n-\theta-\delta_n \boldsymbol{h}))]d\boldsymbol{h}g(\theta)d\theta$$

$$- \int_{R^k} E_\theta[l(\delta_n^{-1}(V_n-\theta))]g(\theta)d\theta \qquad \ldots (26)$$

converges to zero for every $\alpha > 0$. Now note that, for $\boldsymbol{u} \in \boldsymbol{R}^k$,

$$\int_{R^k} l(\delta_n^{-1}(V_n-\theta)-\boldsymbol{h})S_n^*(\theta, \boldsymbol{h})d\boldsymbol{h}$$

$$= \int_{R^k} l(\delta_n^{-1}(V_n-\theta)-\boldsymbol{h}-\boldsymbol{u})S_n^*(\theta, \boldsymbol{h}+\boldsymbol{u})d\boldsymbol{h}.$$

Hence in view of Lemma 3 and the preceeding remarks, it is easily seen that the difference

$$\int_{R^k} l(\delta_n^{-1}(V_n-\theta)-\boldsymbol{h})S_n^*(\theta, \boldsymbol{h})d\boldsymbol{h}$$

$$- \int_{R^k} l(\delta_n^{-1}(V_n-\theta)-\boldsymbol{h}-\boldsymbol{u})S_n^*(\theta+\delta_n \boldsymbol{u}, \boldsymbol{h})d\boldsymbol{h}$$

converges to zero in $P_{\theta, n}$ probability and hence, by contiguity, in $P_{\theta_0+\delta_n u, n}$ probability also for every $u \in R^k$. In particular the difference

$$\int_{X_n} \int_{R^k} l(\delta_n^{-1}(V_n-\theta)h) S_n^*(\theta, h) dh dP_{\theta+\delta_n u, n}$$

$$-\int_{X_n} \int_{R^k} l(\delta_n^{-1}(V_n-\theta)-h-u) S_n^*(\theta+\delta_n u, h) dh dP_{\theta+\delta_n u, n}$$

converges to zero as $n \to \infty$. Hence by an application of Lemma 8 it follows that the difference

$$\int_{R^k} \int_{X_n} \int_{R^k} l(\delta_n^{-1}(V_n-\theta)-h) S_n^*(\theta, h) dh dP_{\theta+\delta_n u, n}\, g(\theta) d\theta$$

$$-\int_{R^k} \int_{X_n} \int_{R^k} l(\delta_n^{-1}(V_n-\theta)-h) S_n^*(\theta, h) dh dP_{\theta, n}\, g(\theta) d\theta$$

converges to zero for every $u \in R^k$. This in turn implies, in view of Lemma 2, that the difference between the r.h.s. of this expression and the quantity

$$\frac{1}{\mu^k(D_a)} \int_{D_a} \int_{R^k} \int_{X_n} \int_{R^k} l(\delta_n^{-1}(V_n-\theta)-h) S_n^*(\theta, h) dh dQ_n(\theta, u) g(\theta) d\theta du \qquad \ldots \quad (27)$$

converges to zero for every $a > 0$. Hence the result follows since, by Lemma 5, the difference between the l.h.s. of (26) and (27) converges to zero by first letting $n \to \infty$ and then $a \to \infty$.

Proof of Theorems 7 *and* 8 : Using the arguments similar to the proof of Theorems 1 and 3, the proof easily follows from Theorem 6.

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