Analysis of Censored Data
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REMARKS ON CRAMER-RAO TYPE INTEGRAL INEQUALITIES FOR RANDOMLY CENSORED DATA

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Abstract

Cramer-Rao type integral inequalities for the integrated risk, for estimators for parameters based on ramdomly censored data, are derived. As applications, lower bounds for the locally asymptotic minimax risk for estimators of parameters in the exponential and Weibull case for the proportional hazard model, are obtained and locally minimax estimators of the relevant parameters are identified.

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1 Introduction

Suppose that on a certain probability space, ξ and η are random variables with distribution functions $F(x,\theta)$ and $G(x,\theta)$ respectiveldy where $\theta \in \Theta \subset R^1$. Further suppose that $\xi_1, \xi_2, ..., \xi_n$ are independent and identically distributed (i.i.d.) as ξ and $\eta_1, \eta_2, \cdots, \eta_n$ are i.i.d. as η. Define $\zeta_i = \min(\xi_i, \eta_i)$ and $\delta_i = I(\xi_i \leq \eta_i)$, $1 \leq i \leq n$, where $I(A)$ denotes the indicator function of the set A. It is easy to see that ζ_i, $1 \leq i \leq n$ are independent and δ_i, $1 \leq i \leq n$ are also independent random variables. We assume that ξ_i and η_i are not observable but (ζ_i, δ_i) is observable for $1 \leq i \leq n$. As is well known, the set of data (ζ_i, δ_i), $1 \leq i \leq n$ so obtained is termed in the literature as randomly censored data and is common in studies in survival analysis and reliability. It is assumed that ξ_i's are independent of censoring random variables η_i's.

Abdushukurov and Kim (1987) obtained lower bounds for the variance of unbiased estimators of the parameter θ based on the data (ζ_i, δ_i), $1 \leq i \leq n$. They have obtained analogues of Cramer-Rao and Bhattacharya bounds in the uncensored case and discussed the conditions under which the bounds are attained.

Wyckoff and Engelhardt (1980) derived Cramer-Rao lower bounds for estimators based on data obtained from type II censoring and presented a method for numerically evaluating these bounds. Among others, Eubank and La Riccia (1982) and Crow and Shimi (1982) studied the problem for type I censoring.

Here we consider a Bayesian version of the problem. Suppose $\lambda(\cdot)$ is a prior density for θ and $\hat{\theta}$ is an estimator of θ based on $\boldsymbol{Z}_i = (\zeta_i, \delta_i)$, $1 \leq i \leq n$. Then $R(\hat{\theta}, \theta) = E_\theta \; E_{\boldsymbol{Z}|\theta}(\hat{\theta} - \theta)^2$ is the risk of the estimator $\hat{\theta}$ under the squared error loss function. The problem of interest is to obtain lower bounds for the risk $R(\hat{\theta}, \theta)$ analogous to the Cramer-Rao lower bound obtained by Abdushukurov and Kim (1987) and other lower bounds over various classes of estimators.

For recent work on the Cramer-Rao type integral inequalities in the Bayesian frame work, see Prakasa Rao (1991, 92) and Bhattacharya and Prakasa Rao (1995).

2 Preliminaries

In the sequel, we will assume that F and G are absolutely continuous with densities f and g respectively and that $\lambda(\cdot)$ is a prior density for the parameter θ. It is easy to see that $\boldsymbol{Z} = (\zeta, \delta)$ has the joint density given by

$$h(\boldsymbol{z}; \theta) = h(x, y; \theta) = [\bar{G}(x, \theta) f(x, \theta)]^y [\bar{F}(x, \theta) g(x, \theta)]^{1-y}$$

with respect to the product measure $\mu \times \nu$ where μ is the Lebesgue measure on the real line and ν is the measure with mass one at the points $\{0\}$ and $\{1\}$. Here $\bar{G} = 1 - G$ and $\bar{F} = 1 - F$.

Assume that $h(\cdot)$ is differentiable with respect to θ and the set $\{x : \bar{G}f = 0\} \cup \{x : \bar{F}g = 0\}$ does not depend on the parameter θ. It is easy to see that the score function is given by

$$\frac{\partial \log h}{\partial \theta} = \frac{y}{\bar{G}f} \frac{\partial}{\partial \theta}(\bar{G}f) + \frac{(1-y)}{\bar{F}g} \frac{\partial}{\partial \theta}(\bar{F}g) \tag{2.1}$$

and the Fisher information is given by

$$\begin{aligned}
I(\theta) &\equiv E_{Z|\theta}\left[\frac{\partial \log h(Z,\theta)}{\partial\theta}\right]^2 \\
&= E_{Z|\theta}\left[\left(\frac{\partial}{\bar{G}f}\right)\frac{\partial(\bar{G}f)}{\partial\theta}\right]^2 + E_{Z|\theta}\left[\frac{(1-\delta)}{\bar{F}g}\frac{\partial(\bar{F}g)}{\partial\theta}\right]^2 \\
&\quad +2\, E_{Z|\theta}\left[\frac{\delta(1-\delta)}{\bar{G}f\,\bar{F}g}\frac{\partial(\bar{G}f)}{\partial\theta}\frac{\partial(\bar{F}g)}{\partial\theta}\right] \\
&= E_{Z|\theta}\left[\frac{\delta^2}{\bar{G}^2 f^2}\left(\frac{\partial(\bar{G}f)}{\partial\theta}\right)^2\right] + E_{Z|\theta}\left[\frac{(1-\delta)^2}{\bar{F}^2 g^2}\left(\frac{\partial(\bar{F}f)}{\partial\theta}\right)^2\right] \qquad (2.2)
\end{aligned}$$

since $\partial(1-\delta)=0$. It is easy to check that

$$E_{Z|\theta}\left[\frac{\delta^2}{\bar{G}^2 f^2}\left(\frac{\partial(\bar{G}f)}{\partial\theta}\right)^2\right] = \int_R \left[\frac{\partial \log(\bar{G}f)}{\partial\theta}\right]^2 \bar{G}f\,dx$$

and

$$E_{Z|\theta}\left[\frac{(1-\delta)^2}{\bar{F}^2 g^2}\left(\frac{\partial(\bar{F}g)}{\partial\theta}\right)^2\right] = \int_R \left[\frac{\partial \log(\bar{F}g)}{\partial\theta}\right]^2 \bar{F}g\,dx.$$

Hence

$$I(\theta) = \int_R \left[\frac{\partial \log(\bar{G}f)}{\partial\theta}\right]^2 \bar{G}f\,dx + \int_R \left[\frac{\partial \log(\bar{F}g)}{\partial\theta}\right]^2 \bar{F}g\,dx. \qquad (2.3)$$

Two cases are of special interest in survival analysis and reliabilkity studies: Case (A) The censoring distribution G does not depend on the parameter θ; Case (B) The pair (F,G) follows a *proportional hazards model* (PHM) i.e., there exists a constant $\beta > 0$ such that

$$\bar{G}(x,\theta) = [\bar{F}(x,\theta)]^\beta, \quad -\infty < x < \infty. \qquad (2.4)$$

In the following discussion, we assume that β is known and is *independent* of $\theta \in \Theta$.

Case (A) The Fisher information is given by

$$I(\theta) = \int_R \left[\frac{\partial \log f}{\partial\theta}\right]^2 \bar{G}f\,dx + \int_R \left[\frac{\partial \log \bar{F}}{\partial\theta}\right]^2 \bar{F}g\,dx. \qquad (2.5)$$

Case (B) (PHM) In this case, it is easy to see that

$$\log \bar{G} = \beta \log \bar{F}, \qquad (2.6)$$

and

$$g\,\bar{F} = \beta\, f\bar{G}. \tag{2.7}$$

Differentiating (2.6) and (2.7) with respect to θ, it follows that

$$\frac{\partial \log \bar{G}}{\partial \theta} = \beta \frac{\partial \log \bar{F}}{\partial \theta} \tag{2.8}$$

and

$$\frac{\partial}{\partial \theta}(\log(g\bar{F})) = \frac{\partial}{\partial \theta}(\log(\bar{G}f)). \tag{2.9}$$

Hence, the relations (2.7) and (2.9) imply that

$$\begin{aligned} I(\theta) &= \int_R \left[\frac{\partial \log (\bar{G}f)}{\partial \theta}\right]^2 \bar{G}f\, dx + \int_R \left[\frac{\partial \log (\bar{F}g)}{\partial \theta}\right]^2 \bar{F}g\, dx \\ &= (1+\beta)\int_R \left[\frac{\partial \log (\bar{G}f)}{\partial \theta}\right]^2 \bar{G}f\, dx. \end{aligned} \tag{2.10}$$

3 Lower bound for the risk for the class of unbiased estimators

Suppose the following regularity conditions hold :

(C1) $f(x,\theta)$ and $g(x,\theta)$ are differentiable with respect θ and

$$\int_R \left|\frac{\partial f}{\partial \theta}\right| dx < \infty, \quad \int_R \left|\frac{\partial g}{\partial \theta}\right| dx < \infty;$$

(C2) $E_{\xi|\theta}\left|\frac{\partial \log f(\xi;\theta)}{\partial \theta}\right|^2 < \infty,\ E_{\eta|\theta}\left|\frac{\partial \log g(\eta;\theta)}{\partial \theta}\right|^2 < \infty;$

(C3) $0 < I(\theta) < \infty$;

(C4) the set $\{x : \bar{G}f = 0\} \cup \{x : \bar{F}g = 0\}$ does not depend on θ; and

(C5) suppose that $\hat{\phi}(\zeta,\delta)$ is an unbiased estimator of $\phi(\theta)$ and differentiation with respect to θ under integral sign is permissible in the relation

$$\int_{RX}\int_{\{0,1\}} \phi(x,y)h(x,y;\theta)\ dx\, d\nu(y) = \phi(\theta),\ \ \theta \in \Theta. \tag{3.1}$$

Let $\phi'(\theta)$ denote the derivative of $\phi(\theta)$ with respect to θ.

Theorem 3.1 (Abdushukurov and Kim (1987)): Suppose the conditions (C1) to (C5) hold. Then

$$\text{var}\ \hat{\phi}(\zeta,\delta) \geq \frac{[\phi'(\theta)]^2}{I(\theta)}. \tag{3.2}$$

In particular, for $\phi(\theta) \equiv \theta$,

$$E_{\boldsymbol{Z}|\theta}(\hat{\theta}-\theta)^2 \geq \frac{1}{I(\theta)} \tag{3.3}$$

for any unbiased estimator of θ based on $\boldsymbol{Z}$. It is again easy to check that if $\boldsymbol{Z}_i$, $1 \leq i \leq n$ are n i.i.d. random vectors each distributed as $\boldsymbol{Z} = (\zeta, \delta)$, then

$$E_{\boldsymbol{Z}|\theta}(\hat{\theta}-\theta)^2 \geq \frac{1}{nI(\theta)} \tag{3.4}$$

for any unbiased estimator $\hat{\theta}$ of θ based on $\boldsymbol{Z}_i$, $1 \leq i \leq n$. Taking expectations over both sides with respect to θ, it follows that the risk

$$\begin{aligned} R(\hat{\theta},\theta) &= E_\theta(E_{\boldsymbol{Z}|\theta}(\hat{\theta}-\theta)^2) \\ &\geq E_\theta\left(\frac{1}{nI(\theta)}\right) \\ &\geq \frac{1}{nE_\theta(I(\theta))} \end{aligned} \tag{3.5}$$

by the elementary inequality

$$E\left(\frac{1}{Y}\right) \geq \frac{1}{E(Y)} \tag{3.6}$$

for any positive random variable Y. Here $E_\theta(\cdot)$ denotes the expectation with respect to the prior density $\lambda(\cdot)$ such that $E_\theta(I(\theta)) < \infty$.

Hence we have the following result.

Theorem 3.2 : Suppose the conditions (C1) to (C5) hold. Then

$$\inf_{\hat{\theta}\in\mathcal{D}_0} E(\hat{\theta}-\theta)^2 \geq \frac{1}{nE_\theta(I(\theta))} \tag{3.7}$$

where $I(\theta)$ is given by (2.3) and $\mathcal{D}_0$ denotes the class of unbiased estimators for θ.

4 Lower bound for the risk for the Bayes estimator

Suppose $\underset{\sim}{\theta}$ is the Bayes estimator of θ given $\boldsymbol{Z}_i = (\zeta_i, \delta_i)$, $1 \leq i \leq n$ with respect to the squared error loss function corresponding to the prior density $\lambda(\cdot)$ for θ. Then

$$\underset{\sim}{\theta} = \int_\Theta \theta h_{\theta|\boldsymbol{Z}}(\theta|\boldsymbol{z})d\theta \tag{4.1}$$

where

$$h_{\theta|Z}(\theta|z) = \frac{\prod_{i=1}^{n} h(z_i;\theta)\lambda(\theta)}{\int_\Theta \prod_{i=1}^{n} h(z_i;\theta)\lambda(\theta)d\theta}. \tag{4.2}$$

Furthermore

$$E[(\underset{\sim}{\theta} - \theta)^2|Z = z) = \int_\Theta (\underset{\sim}{\theta} - \theta)^2 h_{\theta|Z}(\theta|z)d\theta \equiv \nu_z^2 \text{ (say)}. \tag{4.3}$$

Theorem 4.1 : Assume that the classical regularity conditions hold for the validity of Cramer-Rao inequality for the family of conditional densities $h_{\theta|Z}(\theta|z)$, $\theta \in \Theta$ for every $z \in R \times \{0,1\}$. Let $Z_i = (\zeta_i, \delta_i)$, $1 \leq i \leq n$ be i.i.d. as defined earlier. Then

$$E[\underset{\sim}{\theta} - \theta]^2 \geq \frac{1}{F + nE(I(\theta))} \tag{4.4}$$

where

$$I(\theta) = E_{Z|\theta}\left[\frac{\partial \log h(Z_i,\theta)}{\partial\theta}\right]^2$$

and

$$F = E\left[\frac{d \log \lambda(\theta)}{\partial\theta}\right]^2 \tag{4.5}$$

with the prior density $\lambda(\cdot)$ restricted to the class of priors for which $F < \infty$.

Proof : Note that $E(\underset{\sim}{\theta} - \theta]^2 = E_z[\nu_Z^2]$. The theorem follows now as a consequence of the application of Cramer-Rao inequality for conditional set up given z, taking expectations on both sides of the Cramer-Rao inequality and using the inequality (3.6). For details, see Schutzenberger (1959) or Prakasa Rao (1991).

5 Lower bounds for the posterior risk for the class of all estimators

Method 1 (Weinstein and Weiss (1985))

Let

$$g(z;\theta) = h(z;\theta)\ \lambda(\theta). \tag{5.1}$$

Note that $g(\cdot)$ is the joint density of $(Z;\theta) = (\zeta,\delta;\theta)$. Define

$$L(z;\theta + d,\ \theta) \equiv \frac{g(z;\theta + d)}{g(z;\theta)} \tag{5.2}$$

for any d and $0 < s < 1$. Let

$$\mu(s,d) = \log\ E[L^s(\boldsymbol{Z};\theta + d,\ \theta)]. \tag{5.3}$$

Then, Weinstein and Weiss (1985) proved that, for any estimator $\hat{\theta}$ of θ based on $\boldsymbol{Z}_i$, $1 \leq i \leq n$,

$$E(\hat{\theta} - \theta)^2 \geq \frac{d^2 e^{2\mu(s,d)}}{e^{\mu(2s,d)} + e^{\mu(2s-1,d)} - 2e^{\mu(s,2d)}} \tag{5.4}$$

for all d and $0 < s < 1$. One can get a better bound by taking the supremum of the right side of the inequality (5.4) over all d and $0 < s < 1$. One obvious choice for s is $s = \frac{1}{2}$.

We now compute $\mu(s,d)$ for the problem under consideration.

Note that

$$\begin{aligned} \mu(s,d) &= \log\ E[L^s(\boldsymbol{Z};\theta + d,\theta)] \\ &= \log\ E_\theta\left\{\left(\frac{\lambda(\theta+d)}{\lambda(\theta)}\right)^s \left(E_{\boldsymbol{Z}|\theta}\left[\frac{h(\boldsymbol{Z}_1;\theta+d)}{h(\boldsymbol{Z}_1;\theta)}\right]^s\right)^n\right\} \\ &= \log\ E_\theta\left\{\left(\frac{\lambda(\theta+d)}{\lambda(\theta)}\right)^s \tau^n(\theta;d,s)\right\} \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} \tau(\theta;d,s) &\equiv \int\limits_{RX}\int\limits_{\{0,1\}} \left\{\frac{h(x,y;\theta+d)}{h(x,y;\theta)}\right\}^s h(x,y;\theta)\ dx\ d\nu(y) \\ &= \int\limits_{RX}\int\limits_{\{0,1\}} h^s(x,y;\theta+d)h^{1-s}(x,y;\theta)\ dx\ d\nu(y) \\ &= \int\limits_R h^s(x,0;\theta+d)h^{1-s}(x,0;\theta)\ dx \\ &\quad + \int_R h^s(x,1;\theta+d)h^{1-s}(x,1;\theta)\ dx \\ &= \int_R (\bar{F}g)^s_{\theta+d}(\bar{F}g)^{1-s}_\theta dx + \int_R (\bar{G}f)^s_{\theta+d}(\bar{G}f)^{1-s}_\theta dx. \end{aligned} \tag{5.6}$$

Case (A) (G does not depend on θ) : Observe that

$$\tau(\theta;d,s) = \int_R (\bar{F})^s_{\theta+d}(\bar{F})^{1-s}_\theta g\ dx + \int_R (f)^s_{\theta+d}(f)^{1-s}_\theta \bar{G}\ dx \tag{5.7}$$

where $(R)_\theta$ denotes $R(x,\theta)$.

Case (B) (PHM) : Here

$$\tau(\theta; d, s) = (1+\beta)\int_R (\bar{G}f)^s_{\theta+d}(\bar{G}f)^{1-s}_\theta dx. \tag{5.8}$$

Method 2 (Babrovsky and Zakai (1976))

Following the approach used in Hammersley - Chapman - Robbins inequality (cf. Lehmann (1983)), when the differentiability assumption does not hold, Babrovsky and Zakai (1976) obtained the inequality

$$E[\hat{\theta}-\theta]^2 \geq \frac{1}{E\left[\frac{1}{d^2}\left(1-\frac{K(\boldsymbol{Z};\theta+d)}{K(\boldsymbol{Z};\theta)}\right)^2\right]} \tag{5.9}$$

for any $d \neq 0$ where $K(\boldsymbol{Z};\theta)$ is the joint density of $\boldsymbol{Z}_i = (\zeta_i, \delta_i)$, $1 \leq i \leq n$ and θ and $\hat{\theta}$ is any estimator of θ. Note that

$$\frac{K(\boldsymbol{Z};\theta+d)}{K(\boldsymbol{Z};\theta)} = \frac{\prod_{i=1}^{n} h(\boldsymbol{Z}_i;\theta+d)\ \lambda(\theta+d)}{\prod_{i=1}^{n} h(\boldsymbol{Z}_i;\theta)\ \lambda(\theta)} \tag{5.10}$$

Case (B) (PHM) : In this case

$$\begin{aligned} h(\boldsymbol{Z}_i;\theta) &= (\bar{G}f)^{\delta_i}(\bar{F}g)^{1-\delta_i} \\ &= (\bar{G}f)^{\delta_i}(\beta\bar{G}f)^{1-\delta_i} \\ &= \beta^{1-\delta_i}(\bar{G}f) \end{aligned}$$

and hence

$$\frac{K(\boldsymbol{Z};\theta+d)}{K(\boldsymbol{Z};\theta)} = \frac{\prod_i (\bar{G}f)_{\theta+d}}{\prod_i (\bar{G}f)_\theta}\ \frac{\lambda(\theta+d)}{\lambda(\theta)} \tag{5.11}$$

where $(q)_\theta$ denotes $q(\boldsymbol{Z}_i;\theta)$.

Method 3 (Borovkov and Sakhanenko (1980))

Following Borovkov and Sakhanenko (1980) (cf. Theorem 4.1, Prakasa Rao (1991)), it follows that

$$\inf_{\theta^*\in\mathcal{D}} E(\theta^*-\theta)^2 \;\geq\; \frac{\left(E\left(\frac{1}{I(\theta)}\right)\right)^2}{nE\left(\frac{1}{I(\theta)}\right)+E\left(\frac{d}{d\theta}(\lambda(\theta)/I(\theta))/\lambda(\theta)\right)^2}$$

$$= \frac{J^2}{nJ+H} \geq \frac{J}{n} - \frac{H}{n^2} \tag{5.12}$$

where

$$J = E(\frac{1}{I(\theta)}) \text{ and } H = E\left(\frac{d}{d\theta}\left(\frac{\lambda(\theta)}{I(\theta)}\right)/\lambda(\theta)\right)^2 \tag{5.13}$$

and $\mathcal{D}$ is the class of *all* estimators θ^* of θ. Detailed conditions for (5.12) to hold are given in Prakasa Rao (1991). the inequality (5.12) can be weakened to obtain a computable possibly weaker lower bound. In fact

$$\inf_{\theta^* \in \mathcal{D}} E(\theta^* - \theta)^2 \geq \frac{1}{nE(I(\theta))} - \frac{H}{n^2} \tag{5.14}$$

by the inequality (3.6). This inequality can be compared with (3.7) for the class $\mathcal{D}_0$ of unbiased estimators θ^* of θ.

6 Locally asymptotic minimax bounds

Extending the Borovkov-Sakhanenko bound, Prakasa Rao (1991) proved that, for any estimator $q(\theta^*)$ of $q(\theta)$,

$$E_\theta[q(\theta^*) - q(\theta)]^2 \geq \frac{(E_\theta[q'(\theta)])^2}{nE_\theta(I(\theta)) + E_\theta\left[\frac{\partial \log \lambda(\theta)}{\partial\theta}\right]^2} \tag{6.1}$$

under some regularity conditions on $q(\cdot)$ and the family of distributions, where

$$I(\theta) = E_{Z|\theta}\left[\frac{\partial \log h(Z;\theta)}{\partial\theta}\right]^2. \tag{6.2}$$

As a consequence, it follows that, for any $\theta_0 \in \Theta^0$

$$\underline{\lim}_{n\to\infty} \sup_{\theta\in D_n} E_{Z|\theta}[\sqrt{n}(q(\theta^*) - q(\theta))]^2 \geq \frac{[q'(\theta_0)]^2}{I(\theta_0)} \tag{6.3}$$

where $D_n = [|\theta - \theta_0| \leq \varepsilon_n]$ and $0 < \varepsilon_n$ with $\varepsilon_n^{-2}\, n^{-1} \to 0$ as $n \to \infty$. Here $q'(\theta)$ denotes the derivative of $q(\theta)$ with respect to θ and Θ^0 denotes the interior of Θ.

Case (B) (PHM) : If $q(\theta) \equiv \theta$, then the lower bound is $\frac{1}{I(\theta_0)}$ and hence

$$\underline{\lim}_{n\to\infty} \sup_{\theta\in D_n} E_{Z|\theta}[\sqrt{n}(\theta^* - \theta)]^2$$
$$\geq \frac{1}{(1+\beta)\int_R \left[\frac{\partial \log \bar{G}f}{\partial\theta}\right]^2_{\theta=\theta_0} (\bar{G}f)_{\theta_0} dx}. \tag{6.4}$$

7 Example

Example 7.1 Suppose $F \simeq \exp(\theta)$ and $G \simeq \exp(\beta\theta)$ where β is a known constant. Then

$$
\begin{aligned}
I(\theta) &= (1+\beta)\int_{R_+}\left[\frac{\partial \log\,(e^{-\beta\theta x}\theta e^{-\theta x})}{\partial\theta}\right]^2 \theta\, e^{-(1+\beta)x\theta}dx \\
&= \int_0^{\infty}(\frac{1}{2}-y)^2\theta e^{-y\theta}dy = \frac{1}{\theta^2}
\end{aligned}
$$

and hence

$$\inf_{\hat\theta\in\mathcal{D}_0} E(\hat\theta-\theta)^2 \geq \frac{1}{nE_\theta(\frac{1}{\theta^2})}$$

where $\mathcal{D}_0$ is as defined earlier in Section 3. In this case, it can be checked that

$$\tau(\theta;d,s) = \frac{(\theta+d)^s\,\theta^{1-s}}{\theta+ds}$$

and

$$\mu(s,d) = \log\,E_\theta\left[\left(\frac{\lambda(\theta+d)}{\lambda(\theta)}\right)^s\left\{\frac{(\theta+d)^s\theta^{1-s}}{\theta+ds}\right\}^n\right]$$

to compute the Weinstein-Weiss lower bound given in Section 5. Furthermore

$$\frac{K(Z;\theta+d)}{K(Z;\theta)} = e^{-(1+\beta)d\sum_{i=1}^{n}\zeta_i}\left(\frac{\theta+d}{\theta}\right)^n\frac{\lambda(\theta+d)}{\lambda(\theta)}$$

and hence

$$
\begin{aligned}
&E\left[\frac{1}{d^2}\left(1-\frac{K(Z;\theta+d)}{K(Z;\theta)}\right)^2\right] \\
&= E_\theta\left[E_{Z|\theta}\left\{\frac{1}{d^2}\left(1-e^{-(1+\beta)d\Sigma\zeta_i}(\frac{\theta+d}{\theta})^n\,\frac{\lambda(\theta+d)}{\lambda(\theta}\right)^2\right\}\right] \\
&= \frac{1}{d^2}E_\theta\left[1-2\left(\frac{\theta+d}{\theta}\right)^n\frac{\lambda(\theta+d)}{\lambda(\theta)}\,E_{Z|\theta}(e^{-(1+\beta)d\Sigma\zeta_i})\right. \\
&\quad\left.+\left(\frac{\theta+d}{d}\right)^{2n}\left(\frac{\lambda(\theta+d)}{\lambda(\theta)}\right)^2 E_{Z|\theta}\left(e^{-2(1+\beta)d\Sigma\zeta_i}\right)\right].
\end{aligned}
$$

Note that

$$E_{Z|\theta}\left[e^{t\zeta_1}\right] = \frac{\theta(1+\beta)}{(1+\beta)\theta-t}.$$

Therefore

$$E_{Z|\theta}\left[e^{-(1+\beta)d\Sigma\zeta_i}\right] = \left[\frac{\theta(1+\beta)}{(1+\beta)\theta+(1+\beta)d}\right]^n = \left(\frac{\theta}{\theta+d}\right)^n$$

and

$$E_{Z|\theta}\left[e^{-2(1+\beta)d\Sigma\zeta_i}\right] = \left[\frac{\theta(1+\beta)}{(1+\beta)\theta+2(1+\beta)d}\right]^n = \left(\frac{\theta}{\theta+2d}\right)^n.$$

Hence the Babrovsky-Zakai lower bound defined in Section 5 is

$$\frac{1}{d^2}E_\theta\left\{1-2\frac{\lambda(\theta+d)}{\lambda(\theta)}+\left(\frac{\theta+d}{\theta}\right)^{2n}\left(\frac{\theta}{\theta+2d}\right)^n\frac{\lambda^2(\theta+d)}{\lambda^2(\theta)}\right\}$$

for any $d \neq 0$.

The Borovkov-Sakhanenko lower bound as defined in Section 5 is given by

$$\begin{aligned}\frac{J}{n}-\frac{H}{n^2} &= \frac{1}{n}E_\theta\left(\frac{1}{I(\theta)}\right)-\frac{1}{n^2}E_\theta\left(\frac{d}{d\theta}\left(\frac{\lambda(\theta)}{I(\theta)}\right)/\lambda(\theta)\right)^2\\ &= \frac{1}{n}E_\theta(\theta^2)-\frac{1}{n^2}E_\theta\left(\frac{d}{d\theta}(\lambda(\theta)\theta^2)/\lambda(\theta)\right)^2.\end{aligned}$$

In this example, $I(\theta)=\theta^{-2}$ and hence

$$\underline{\lim}_{n\to\infty}\sup_{\theta\in D_n} E_{Z|\theta}[\sqrt{n}(\theta^*-\theta)]^2 \geq \theta_0^2$$

where D_n is as defined in Section 6.

It is easy to check that

$$\hat{\theta}=\frac{n}{(1+\beta)\sum_{i=1}^{n}\zeta_i}$$

is the MLE of θ in this example where $\zeta_i \simeq \exp((1+\beta)\theta)$ and

$$E_{Z|\theta}(\hat{\theta})=\theta,\quad \mathrm{var}_{Z|\theta}(\hat{\theta})=\frac{\theta^2}{n-1}.$$

Here minimax lower bound is attained and the estimator $\hat{\theta}$ is a *locally asymptotic minimax* estimator of θ in the model.

Remark : The results in Example 7.1 can be easily extended to the case when F is the Weibull distribution with density function

$$\begin{aligned} f(x,\theta) &= \theta\gamma x^{\gamma-1}e^{-\theta x^\gamma} &, \quad x>0\\ &= 0 &, \quad x\leq 0\end{aligned}$$

and the censoring distribution G is the Weibull distribution with density function

$$\begin{aligned} g(x,\theta) &= \beta\,\theta\gamma x^{\gamma-1}\,e^{-\beta\theta x^\gamma} &, \quad x>0\\ &= 0 &, \quad x\leq 0\end{aligned}$$

where γ and β are known. This can be seen by applying the transformation $Y=X^\gamma$ and the problem reduces to Example 7.1.

8 Remarks

At the time of presentation of this paper, Prof. Jon Wellner has brought the work of Gill and Levit (1992) to the author's attention where they use Van Trees inequality (Van Trees (1968)) to obtain minimax convergence rates in various non – and semiparametric problems. As was pointed out in the introduction, inequality derived by Van Trees (1968) has been obtained earlier by Schutzenberger (1959). Applications to obtain locally asymptotic minimax estimators in several examples through a Bayesian version of Cramer-Rao bound, extending the work of Borovkov and Sakhanenko (1980), were discussed in Prakasa Rao (1992) and Bhattacharya and Prakasa Rao (1995).

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