A novel "magnetic" field and its dual non-commutative phase space

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Abstract

In this Letter we have studied a new form of non-commutative (NC) phase space with an operatorial form of noncommutativity. A point particle in this space feels the effect of an interaction with an *"internal"* magnetic field, that is singular at a specific position θ^{-1}. By "internal" we mean that the effective magnetic fields depends essentially on the particle properties and modifies the symplectic structure. Here θ is the NC parameter and induces the coupling between the particle and the "internal" magnetic field. The magnetic moment of the particle is computed. Interaction with an *external* physical magnetic field reveals interesting features induced by the inherent fuzziness of the NC phase space: introduction of non-trivial structures into the charge and mass of the particle and possibility of the particle dynamics collapsing to a Hall type of motion. The dynamics is studied both from Lagrangian and symplectic (Hamiltonian) points of view. The canonical (Darboux) variables are also identified. We briefly comment, that the model presented here, can play interesting role in the context of (recently observed) *real* space Berry curvature in material systems.

Particle motion in an electromagnetic field and its *dual* description in terms of an effective non-commutative (NC) (phase)space is very well known. However, with the advent of various forms of *intrinsically* NC phase spaces [1–3] in recent times, the above scenario has turned in to an actively discussed topic. To give the simplest of examples, the Lorentz force law,

$$m\ddot{x}_i = e\epsilon_{ijk}\dot{x}_j B_k,$$
(1)

is derivable from either of the two Hamiltonian systems given below:

$$\{x_i, p_j\} = \delta_{ij}; \qquad \{x_i, x_j\} = \{p_i, p_j\} = 0; \qquad H = \frac{1}{2m}(p_i - eA_i)^2,$$
(2)

and

$$\{x_i, P_j\} = \delta_{ij}; \qquad \{P_i, P_j\} = e\epsilon_{ijk}B_k; \qquad \{x_i, x_j\} = 0; \qquad H = \frac{1}{2m}(P_i)^2,$$
(3)

where time evolution is given by $\dot{O} = \{O, H\}$. The equivalence is complete if one invokes validity of the Jacobi identity[1] in the latter algebra,

$$\epsilon_{ijk}\{p_i, \{p_j, p_k\}\} \sim \partial_i B_i = 0,$$

which demands source free magnetic fields with $B_i = \epsilon_{ijk}\partial_j A_k$. The two systems are related by $P_i \equiv p_i - eA_i$. However, we emphasize that this identification is not necessary and both symplectic structures (2) and (3) are each of them can be utilized independently. On the other hand, in quantum mechanics, associativity—the key property—can be maintained even with $\partial_i B_i \neq 0$ but the source has to be monopole like [4].

More close to the framework of NC geometry is the Landau problem that deals with a point charge e in a plane with a constant magnetic field B normal to the plane. In the limit of large B, the Lagrangian and the corresponding symplectic structure reduces to,

$$L = -\frac{B}{2}\epsilon_{ij}\dot{x}_i x_j \quad \Rightarrow \quad \{x_i, x_j\} = \frac{\epsilon_{ij}}{B}, \tag{4}$$

which is the NC plane. Although the above are not examples of an intrinsically NC phase space, the same mechanism will work in an intrinsically NC phase space, that can be simulated by an "internal" magnetic field. This is the subject matter of the present Letter.

It is quite obvious, at least mathematically, that an NC phase space, complimentary to (3), is possible where the coordinates are non-commutative (instead of the momenta), which will yield a monopole in momentum space. An explicit construction of such a spacetime first appeared in $(2 + 1)$-dimensional spinning particle models [5] of anyons [6]—excitations with arbitrary spin and statistics. Very interestingly, singular structures of precisely the above monopole form have been experimentally observed [7] in momentum space. The theoretical description relies on the momentum space Berry curvature and associated monopole singularity in anomalous Hall effect [8]. This is closely connected to the exciting areas of spin Hall effect and spintronics [9]. The condensed matter phenomena and NC spacetime were first connected in a coherent framework by Berard and Mohrbach [10] who have considered an NC coordinate space with a singularity in momentum variable. This is in fact complimentary to the NC momentum space of [4] mentioned above, that describes a charged particle in the presence of a magnetic monopole in coordinate space. The idea of [10] has been further elucidated in [11] who demonstrate that Bloch electron dynamics, developed in [8] is indeed Hamiltonian in nature (contrary to recent claims in [12]), provided the proper symplectic phase space volume is taken into account.

This has naturally given rise to the question of existence of singular structures in coordinate space and a recent paper [13] has suggested observation of the same. The theoretical framework [14] is based on the non-vanishing local spin chirality.

In this perspective we put forward the motivations of our work. Our primary aim is to construct an NC phase space that induces an effective singular magnetic field in *coordinate* space, that is structurally different from a monopole. We claim that this result can be relevant in the observation of coordinate space singularity in [13]. The NC phase space we have presented here is new and has not appeared before. As we will elaborate, it is complimentary to the well-known NC spacetime model proposed by Snyder [2]. For this reason we refer to our construction as the "Snyder" space and since we develop a point particle picture, we term the latter as a "Snyder" particle. The other aim is a generalization of the work of [10], which is restrictive in the sense that the noncommutativity depends *only* on momentum variables and this is due to the fact that in [10] the mixed coordinate–momentum bracket is taken as canonical: $\{x_i, p_j\} = \delta_{ij}$.[2] In our work, Berry phase in coordinate space appear. We emphasize that our generalizations (here and in [15]) *maintains rotational invariance* in the sense that the angular momentum operator and algebra as well as rotational transformation laws of position and momentum are unchanged. This is explained later in terms of Darboux variables. However, the singular nature of the intrinsic magnetic field is *not* of the form of a monopole. We deal with a non-relativistic classical model but the major part of our work goes through in the quantized version.

After studying dynamics of the non-interacting "Snyder" particle we couple it with an *external* physical electromagnetic field. Now the inherent fuzziness of the NC phase space is revealed whereby the localized properties of the particle, its charge and mass, acquire non-trivial structures. In fact, we find a quadrupole moment like behavior in the particle charge distribution which is interesting if we recall [16] that fundamental particles behave like dipoles in spaces with constant noncommutativity.

Let us start by considering a (first order) Lagrangian,

$$L = \dot{X}_i P_i + \theta \dot{X}_i \mathcal{A}_i - H, \tag{5}$$

where θ denotes the particle charge, H is some unspecified Hamiltonian and the vector potential $\mathcal{A}_i$ is given by,

$$\mathcal{A}_i = \frac{(X \cdot P)}{1 - \theta X^2} X_i. \tag{6}$$

We have used the notation $(X \cdot P) = X_i P_i$, $X^2 = X_i X_i$. This leads to a magnetic field of the form,

$$\mathcal{B}_i = \epsilon_{ijk}\partial_j^{(X)}\mathcal{A}_k = \epsilon_{ijk}\frac{P_j X_k}{1 - \theta X^2} \equiv -\frac{L_i}{1 - \theta X^2}, \tag{7}$$

where $L_i = \epsilon_{ijk}X_j P_k$ is the particle angular momentum.[3] It might be possible to express $\vec{\mathcal{B}}$ as a combination of multipole fields. Notice that $\partial_i \mathcal{B}_i = 0$ but for the singular surface $X^2 = \theta^{-1}$.

Let us consider a (magneto)static property of the particle, i.e. its magnetic moment. Mimicking the classical electrodynamics laws for magnetostatics, for a steady current $\mathcal{J}_i$ one has $\mathcal{J}_i = \epsilon_{ijk}\partial_j \mathcal{B}_k$, which leads to an effective magnetic moment,

$$\mathcal{M}_i = \frac{1}{2}\epsilon_{ijk}X_j \mathcal{J}_k = \frac{1}{(1 - \theta X^2)^2}L_i. \tag{8}$$

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[2] We have shown in [15] that more general forms of NC phase space yields structurally different forms of singular magnetic fields, in momentum space.

[3] We will establish later that L_i is truly the conserved angular momentum.

Quite interestingly, even for this "Snyder" particle residing in an NC space (to be elaborated later), the above agrees with the conventional result for a moving particle [17].

Before studying the dynamics, let us see the effect of introducing the particular form of "internal" magnetic field. It is termed internal because it actually reflects the property of the particle in question, rather than a physical external magnetic field. In fact it modifies the symplectic structure, as we now demonstrate.

For a generic first order Lagrangian, as in (5),

$$L = a_\alpha(\eta)\dot\eta^\alpha - H(\eta),\tag{9}$$

with η being phase space variables, one has the Euler–Lagrange equations of motion,

$$\omega_{\alpha\beta}\dot\eta^\beta = \partial_\alpha H \quad \Rightarrow \quad \dot\eta^\alpha = \omega^{\alpha\beta}\partial^\beta H.\tag{10}$$

In the above, $\omega_{\alpha\beta}$ denotes the symplectic two form [18]. In the present model (5) we thus compute,

$$\omega_{\alpha\beta} = \begin{pmatrix} -\frac{\theta}{1-\theta X^2}(X_i P_j - X_j P_i) & -\left(\delta_{ij} + \frac{\theta}{1-\theta X^2}X_i X_j\right) \\ \left(\delta_{ij} + \frac{\theta}{1-\theta X^2}X_i X_j\right) & 0 \end{pmatrix}.\tag{11}$$

We adhere to our earlier notation (9) [11,18], that defines the symplectic structure for a first order Lagrangian model in the following way:

$$\{f, g\} = \omega^{\alpha\beta}\partial_\alpha f \partial_\beta g.\tag{12}$$

In the above, f and g constitute two generic operators and $\omega^{\alpha\beta}$ is the inverse of the symplectic matrix,

$$\omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^\alpha_\gamma, \qquad \omega_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha.\tag{13}$$

For the model (5) we compute,

$$\omega^{\alpha\beta} = \begin{pmatrix} 0 & (\delta_{ij} - \theta X_i X_j) \\ -(\delta_{ij} - \theta X_i X_j) & -\theta(X_i P_j - X_j P_i) \end{pmatrix}.\tag{14}$$

It induces the non-canonical Poisson brackets in a straightforward way:

$$\{X_i, X_j\} = 0, \qquad \{X_i, P_j\} = \delta_{ij} - \theta X_i X_j, \qquad \{P_i, P_j\} = -\theta(X_i P_j - X_j P_i).\tag{15}$$

This is the new form of NC space, or "Snyder" space, that we have advertised and this constitutes the other major result in our Letter. For $\theta = 0$ we recover the canonical phase space. This can be compared with the spatial part of Snyder algebra [2] which reads,

$$\{X_i, X_j\} = -\theta(X_i P_j - X_j P_i), \qquad \{X_i, P_j\} = \delta_{ij} - \theta P_i P_j, \qquad \{P_i, P_j\} = 0,\tag{16}$$

and the effects of which have been studied in various contexts [20]. The duality between (15) and (16) is due to the fact we had in mind the construction of an NC phase space that would be rotationally invariant and have a singular structure in the effective magnetic field in *coordinate* space. From our previous experience with Snyder algebra in [15], we knew that Snyder phase space does have this property but with the singularity in *momentum* space. Thus, we guessed (which turns out to be correct) that a form of NC spacetime will be needed that is dual to the Snyder form.

Notice that the structures (15) (and (16) as well) survive upon quantization since there are no operator ordering ambiguities.[4] More important for us is the fact that these NC algebras satisfy the Jacobi identity,

$$J(A, B, C) \equiv \{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0,\tag{17}$$

with A, B and C being X_i or P_j.

Indeed, from the viewpoint of symplectic analysis, it is not surprising that the induced phase space (15) is associative (in the sense of classical brackets). However, at the same time we should remember that in the Lagrangian framework (5) of the same system, we had a particle interacting with a singular magnetic field and singular potentials tend to violate Jacobi identities [4,10]. But there is no controversy involved[5] since, although the final result (15) has no ambiguity regarding associativity, the intermediate steps ($\omega_{\alpha\beta}$ or its inverse) are defined only for $X^2 \neq \theta^{-1}$.

We now consider the dynamics in the Faddeev–Jackiw symplectic formalism [18] (see also [11]). We consider a simple form of Hamiltonian

$$H = \frac{P^2}{2m},\tag{18}$$

in analogy with the normal non-interacting particle Hamiltonian and the particle dynamics follows:

$$\dot{X}_i = \frac{i}{m}(\delta_{ij} - X_i X_j)P_j \quad \Rightarrow \quad P_i = m\left(\delta_{ij} + \theta \frac{X_i X_j}{1 - \theta X^2}\right)\dot{X}_j, \tag{19}$$

$$\dot{P}_i = \frac{\theta}{m}((X \cdot P)P_i - P^2 X_i) \quad \Rightarrow \quad m\ddot{X}_i = -m\theta\left[(2 - \theta X^2)\dot{X}^2 + \frac{\theta}{1 - \theta X^2}(X \cdot \dot{X})^2\right]X_i. \tag{20}$$

It is worthwhile to obtain the rotation generator. We posit it to be of the form

$$L_i = \epsilon_{ijk} X_j P_k. \tag{21}$$

We find that it transforms the position X_i and momentum P_i properly:

$$\{L_i, X_j\} = \epsilon_{ijk} X_k, \qquad \{L_i, P_j\} = \epsilon_{ijk} P_k. \tag{22}$$

It satisfies the correct angular momentum algebra and is conserved,

$$\{L_i, L_j\} = \epsilon_{ijk} L_k; \qquad \{L_i, H\} = 0. \tag{23}$$

This demonstrates the consistency of our derivation of the magnetic moment of the particle in (8).

Since, as shown in (8), there is an effective magnetic moment involved even with the non-interacting particle, it is more interesting to study the behavior of the "Snyder" particle in the presence of external electromagnetic field. This is done by introducing the external $U(1)$ gauge field C_i in (5),

$$a_i = \theta \mathcal{A}_i + eC_i. \tag{24}$$

We thus obtain,

$$\omega_{\alpha\beta}^{(e)} = \begin{pmatrix} -\frac{\theta}{1-\theta X^2}(X_i P_j - X_j P_i) + eF_{ij} & -\left(\delta_{ij} + \frac{\theta}{1-\theta X^2} X_i X_j\right) \\ \left(\delta_{ij} + \frac{\theta}{1-\theta X^2} X_i X_j\right) & 0 \end{pmatrix}, \tag{25}$$

and its inverse,

$$\omega^{(e)\alpha\beta} = \begin{pmatrix} 0 & (\delta_{ij} - \theta X_i X_j) \\ -(\delta_{ij} - \theta X_i X_j) & -\theta(X_i P_j - X_j P_i) + e[F_{ij} + \theta(F_{ki}X_j - F_{kj}X_i)]X_k \end{pmatrix}, \tag{26}$$

where $F_{ij} = \partial_i C_j - \partial_j C_i$ and $\omega^{(e)}$ signifies the presence of external interaction. This modifies the original symplectic structure to:

$$\{X_i, X_j\} = 0; \qquad \{X_i, P_j\} = \delta_{ij} - \theta X_i X_j, \qquad \{P_i, P_j\} = -\theta(X_i P_j - X_j P_i) + e[F_{ij} + \theta(X_i F_{jk} - X_j F_{ik})X_k]. \tag{27}$$

The new equations of motion are,

$$\dot{X}_i = (\delta_{ij} - \theta X_i X_j)\frac{P_j}{m}, \tag{28}$$

$$\dot{P}_i = e\left(E_i + \frac{1}{m}F_{ij}P_j\right) + \frac{\theta}{m}[(X \cdot P)P_i - P^2 X_i] + \theta e\left[\frac{1}{m}(X \cdot P)X_k F_{ki} - \frac{1}{m}X_k P_j F_{kj}X_i - (X \cdot E)X_i\right], \tag{29}$$

where the Hamiltonian is taken as,

$$H = \frac{P^2}{2m} + e\phi(X). \tag{30}$$

The equation of motion for X_i throws up some interesting features which are due to the NC phase space with its inherent fuzziness

$$m\ddot{X}_i = e[\delta_{ij} - \theta(2 - \theta X^2)X_i X_j](E_j + F_{jk}\dot{X}_k) - m\theta\left[(2 - \theta X^2)\dot{X}^2 + \frac{\theta}{1 - \theta X^2}(X \cdot \dot{X})^2\right]X_i. \tag{31}$$

The Lorentz force term in the R.H.S. shows that we are no longer dealing with a simple point charge but an effective charge tensor that reminds us of a quadrupole moment structure, which, in the normal spacetime is expressed as,

$$Q_{ij} = \int (3x_i' x_j' - r'^2 \delta_{ij})\rho(\vec{x}')d^3x' = 3X_i X_j - X^2 \delta_{ij}, \tag{32}$$

where the last relation is for a localized distribution. One should recall [16] that an NC spacetime with a constant form of noncommutativity also induces a dipole like behavior in the basic excitations.

In an earlier interesting work, Duval and Horvathy [19] had considered an "exotic" particle in the two-dimensional plane, that is characterized by *two* parameters, the mass and another scalar parameter, the latter being related to the anyon spin in the non-relativistic limit of [5]. The work [19] showed that, when placed in a constant magnetic field normal to the plane (the classical

Landau problem), the particle acquires a dressed mass and the condition of vanishing of this effective mass forces the particle to follow the Hall law. It might be interesting to see whether a similar situation prevails here. With this motivation we recast (31) in the form,

$$m\left(\delta_{ij} + \theta\frac{2 - \theta X^2}{(1 - \theta X^2)^2}X_iX_j\right)\ddot{X}_j = e(E_i + F_{ij}\dot{X}_j) - \frac{m\theta}{(1 - \theta X^2)^2}\left[(2 - \theta X^2)\dot{X}^2 + \theta\frac{(X.\dot{X})^2}{1 - \theta X^2}\right]X_i. \tag{33}$$

Once again we note that there is an effective tensorial structure instead of a scalar point mass. Let us simplify (33) by considering $O(\theta)$ effects only and in the mass-tensor we make a spatial averaging $\frac{X_iX_j}{X^2} = 3\delta_{ij}$ and obtain,

$$m^*\ddot{X}_i = e(E_i + F_{ij}\dot{X}_j) - 2\theta m^*X_i + O(\theta^2). \tag{34}$$

In the above, the effective mass m^* is given by,

$$m^* = m(1 + 6\theta X^2). \tag{35}$$

Taking cue from [19], we conclude that for a negative value of θ, m^* can vanish on the surface $x^2 \sim (6\theta)^{-1}$ in which case we find the Hall law of motion,

$$E_i + F_{ij}\dot{X}_j \Rightarrow \dot{X}_i = \frac{1}{B}\epsilon_{ij}E_j, \tag{36}$$

where for the planar case $F_{ij} = B\epsilon_{ij}$.

Rotational transformation of the phase space variables reveals that, with $L^{ij} = X^iP^j - X^jP^i$, the coordinates behave as before,

$$\{L_{ij}, X_k\} = \delta_{ik}X_j - \delta_{jk}X_i. \tag{37}$$

As expected, the momentum transformation is modified by the gauge field term but there is an additional $e\theta$-contribution which appears due to the "Snyder" NC phase space but vanishes for $e = 0$. This is qualitatively different from the previously studied result (22) with no $U(1)$ interaction

$$\{L_{ij}, P_k\} = \delta_{ik}P_j - \delta_{jk}P_i + e(X_iF_{jk} - X_jF_{ik}) - e\theta(X_iF_{jl} - X_jF_{il})X_lX_k. \tag{38}$$

We will end our discussion on the particle dynamics scenario with a few words on canonical (Darboux) coordinates corresponding to the phase space (15) of non-interacting "Snyder" particle. It is easy to check that the algebra (15) can be reproduced by expressing the NC variables (X_i, P_j) in terms of a canonical set (x_i, p_j) obeying

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{x_i, x_j\} = \{p_i, p_j\} = 0,$$

via the transformations:

$$X_i \equiv x_i, \qquad P_i \equiv p_i - \theta(x.p)x_i. \tag{39}$$

Quite obviously, these transformations will require proper operator ordering upon quantization. Exploiting (39) we find,

$$L_{ij} = X_iP_j - X_jP_i = x_ip_j - x_jp_i, \tag{40}$$

which shows that the angular momentum operator remains structurally unchanged. In terms of the (x, p) set, the Lagrangian (5) will reduce to,

$$L = \dot{x}_ip_i - H(x, p). \tag{41}$$

The symplectic matrix has turned in to a canonical one as expected and the interactions will be manifested through the Hamiltonian. From the structure of the Hamiltonian in (x_i, p_j) canonical space, it might seem that the singular behavior of the "Snyder" particle, discussed throughout this Letter, has disappeared. However, we reemphasize that the inverse of the above Darboux transformation (39) exists provided $X^2 \ne \theta^{-1}$.

Finally, we come to the works [11] (related to [10]), that have discussed the condensed matter problem of Berry phase effect in the context of anomalous Hall effect [7,8] from a fluid dynamical framework in phase space. It has been shown in [11] that the modified equations of motion [8] for a Bloch electron, in the presence of a Berry curvature, are indeed Hamiltonian in nature provided one exploits an NC phase space and uses the appropriate (phase space) volume-form. One can attempt similar analysis in our "Snyder" NC phase space (15), which is quite distinct from the ones considered in [10,11]. In the present case, the correct volume form [11] is,

$$\sqrt{\det \omega_{\alpha\beta}} = \frac{1}{1 - \theta X^2}. \tag{42}$$

Notice that the same singularity has reappeared in the volume element.

Let us comment on the possible physical realization of our model. A very recent paper [13] has suggested the possible observation of real space Berry phase structure in the anomalous Hall effect in re-entrant AuFe alloys. As has been pointed out in [13,14], this real space Berry phase contribution depends on topologically nontrivial contributions of spin and involves the magnetization explicitly. Hence clearly the structure of the singularity will be more complicated than a monopole type and will have a chiral nature. It should also be remembered that [21] in principle, complicated structures of the vortex are indeed possible depending on the particular nature of a sample, although so far the only numerical work concerns the simple monopole form, as observed in [7]. In the present Letter, the "Snyder" space has yielded an internal magnetic field that is singular in *coordinate space* and not in the form of a monopole. In fact it depends on the particle magnetic moment. Also it should be noted that the angular momentum occurs explicitly in the "Snyder" phase space algebra (15). This indicates the presence of an inherent chiral nature in the phase space. Hence, we believe that it will be very interesting to study the response of this particle to an external magnetic field, which we intend to study in near future. These observations tend to suggest that the novel NC model studied here, can serve as an effective theory for the physical phenomena analyzed in [13,14].

Acknowledgements

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