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RATE OF CONVERGENCE IN THE INVARIANCE PRINCIPLE FOR RANDOM SUMS

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SUMMARY. A rate of convergence in the invariance principle for random sums is obtained using the results on the rate of convergence for the invariance principle due to Heyde (1969) and Borovkov (1973).

1. Introduction

Let $\{X_n\}$ be a sequence of i.i.d. randam variables (r.v.) with $EX_1 = 0$ and $\text{var}(X_1) < \infty$. Let $S_k = \sum_{j=1}^{k} X_j$. Erdos and Kac (1946) considered the problem of finding the limit distributions of the three functions of $S_1, \ldots, S_n$ viz. $\max_{1 \leq k \leq n} S_k$, $\max_{1 \leq k \leq n} |S_k|$ and the number of positive sums among $(S_1, \ldots, S_n)$. These results were later generalized by Donsker (1951), Prohorov (1956) and to some dependent cases by Billingsley (1956). Their technique is now known as "The invariance principle".

One of the important problems in probability theory is to obtain the rate of convergence in limit theorems. Rosenkrantz (1967), Heyde (1969) and Borovkov (1973) investigated the problem of rate of convergence in the invariance principle proved by Donsker (1951).

Another problem of interest is the limit distributions of functions of sums of random number of random variables. Let $\{N_n\}$ be a sequence of positive integer valued r.v.s. such that $\frac{N_n}{n}$ converges in probability to a positive r.v. N. The problem of obtaining the limit distributions of functions of randomly selected partial sums such as S_{N_n}, $\max_{1 \leq k \leq N_n} S_k$ has been considered by Renyi (1960), Blum *et al.* (1963), Mogyorodi (1962), Billingsley (1962, 1968), Prakasa Rao (1969), Sreehari (1968a) and others. Billingsley (1968) and Sreehari (1968b) independently extended the invariance principle of Billingsley

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(1956) to random sum case. Further, the rate of convergence in the limit distribution of S_{N_n} has been studied for independent r.v.s. by Sreehari (1975) and for dependent r.v.s. by Prakasa Rao (1974, 1975). Landers and Rogge (1976) have recently obtained an exact rate of convergence in the random central limit theorem and it has been extended by Ahmad and Basu (1979) and Ahmad (1979). The purpose of this paper is to extend the results of Rosenkrantz, Heyde and Borovkov to the random sum case in the light of the work of Ahmad (1979).

2. Assumptions and known results

Let $\{X_n\}$ be a sequence of i.i.d. r.v.s. with $EX_n = 0$ and $\text{var}(X_n) = 1$. Let $S_n = \sum_{j=1}^{n} X_j$. Let $\{N_n\}$ be a sequence of non-negative integer valued r.v.s. such that

$$\frac{N_n}{n} \to N \text{ in probability as } n \to \infty$$

where N is a positive r.v. We assume N to be independent of the r.v.s. X_n.

Consider $C[0, 1]$ the space of continuous functions on $[0, 1]$ with uniform topology. For $v \in [0, \infty)$, let $p(v) = S_{[v]}+(v-[v])X_{[v]+1}$, where $[v]$ denotes the greatest integer less than or equal to v. Define on $C[0, 1]$ the random processes

$$p_n(t) = p\,(nt)n^{-1/2}, \qquad 0 \leqslant t \leqslant 1,$$

$$q_n(t) = p\,(tN_n)\,(nN)^{-1/2}, \qquad 0 \leqslant t \leqslant 1,$$

and

$$h_n(t) = p\,(t[nN])\,(nN)^{-1/2}, \quad 0 \leqslant t \leqslant 1.$$

Let $W(t)$ be the standard Wiener process on $[0, 1]$. Let $\mathcal{F}$ be a real valued functional defined on $C[0, 1]$ and let $\psi(x) = P[\mathcal{F}(W(\cdot)) \leqslant x]$ and $\psi_n(x) = P[\mathcal{F}(p_n(\cdot)) \leqslant x]$. Suppose $\mathcal{F}$ is uniformly continuous and there exists a constant $L > 0$ such that

$$|\psi(x+h)-\psi(x)| \leqslant L|h|.$$

Let $0 < E|X_1|^{2+\alpha} < \infty$ for some α, $0 \leqslant \alpha \leqslant 2$. Then Rosenkrantz (1967) proved that there exists a constant $A_1 > 0$ such that

$$\Delta_n = \sup_x |\psi_n(x)-\psi(x)| \leqslant A_1 n^{-\alpha/8}(\log n)^{1/2} \qquad \ldots \quad (1)$$

where $\mu = \alpha/2(\alpha+3)$. Heyde (1969) extended the result of Rosenkrantz (1967) under the assumption $0 < E|X_1|^{2+\alpha} < \infty$ for some $\alpha > 0$. He proved that there exists a constant $A_2 > 0$ such that

$$\Delta_n = \sup_x |\psi_n(x) - \psi(x)| \leqslant A_2 n^{-\mu} (\log n)^{\lambda} \qquad \ldots \quad (2)$$

where $\mu = \min(2\alpha, \alpha+2)/4(\alpha+3)$ and $\lambda = (\alpha+2)/2(\alpha+3)$. Borovkov (1973) proved that, if $0 < E|X_1|^{2+\alpha} < \infty$ for some $0 < \alpha \leqslant 1$ then there exists a constant $A_3 > 0$ such that

$$\Delta_n = \sup_x \; |\psi_n(x) - \psi(x)| \leqslant A_3 \, n^{-\mu} \qquad \ldots \quad (3)$$

where $\mu = \alpha/2(\alpha+3)$.

We need the following two results in the sequel.

Theorem 1 : (Gikhman and Shorokhod, 1974, p.70) : *Let* $\{X_n\}$ *be i.i.d r.v.s. and* $S_n = \sum_1^n X_j$, $S_0 = 0$. *If*

$$P(|S_n - S_k| \leqslant t) \geqslant 1-a > 0 \quad \textit{for } k = 0, 1, \ldots, n,$$

then

$$P\left(\max_{1 \leqslant k \leqslant n} |S_k| > 2t \right) \leqslant \frac{P(|S_n| > t)}{1-a} \leqslant \frac{a}{1-a}.$$

Theorem 2 : (Petrov, 1975, p. 251) : *Let* $\{X_n\}$ *be i.i.d. r.v.s. with* $EX_1 = 0$, *var* $X_1 = 1$ *and* $S_n = \sum_{j=1}^n X_j$. *If*

$$E[\exp\{|X_1|^{4\gamma/(2\gamma+1)}\}] < \infty \textit{ for some } 0 < \gamma \leqslant 1/6$$

then

$$\frac{P(S_n \geqslant xn^{1/2})}{1-\Phi(x)} \to 1 \;\; \textit{as } n \to \infty,$$

uniformly in x *in the interval* $0 < x \leqslant n^{\gamma}/\rho(n)$ *where* $\rho(n) \to \infty$ *as* $n \to \infty$.

Here Φ is the standard normal distribution function.

We make the following assumptions in the sequel on the sequence $\{N_n\}$ and N and the functional $\mathscr{F}$.

(A1) $P\left(N \leqslant \frac{1}{n\theta_n}\right) = O(\varepsilon_n)$ where $\theta_n = n^{-1}(\log\log n)^d$, $d > 0$ and

$\varepsilon_n = n^{-\mu}$.

(A2) $P\left(\left|\frac{N_n}{[nN]} - 1\right| \geqslant \delta_n\right) = O(\varepsilon_n)$ for $\delta_n = \varepsilon_n^{2/4}$.

(A3) $\mathscr{F}$ is a uniformly continuous functional on $C[0, 1]$.

(A4) Let $\psi(x) = P(\mathscr{F}(W(\cdot)) \leqslant x)$. Then there exists $L > 0$ such that

$|\psi(x+h) - \psi(x)| \leqslant L|h|$ for all x and h.

(A5) $E\{\exp(|X_1|^{2\mu/(4-8\mu)})\} < \infty$ for some $0 < \mu < 1/4$.

In view of (A5), all the moments of X_1 exist. Hence

$$\Delta_n = \begin{cases} O(n^{-\mu}) & \text{for } 0 < \mu \leqslant 1/8 \\ O(n^{-\mu}(\log n)^{2\mu}) & \text{for } 1/8 < \mu < 1/4 \end{cases} \quad \ldots \quad (4)$$

from the results of Heyde (1969) and Borovkov (1973). In the following $c, c_1, c_2, \ldots$ are all nonnegative constants.

3. Main result

The aim of this paper is to prove the following theorem.

Theorem 3 : *Under the assumptions* (A1) *to* (A5),

$$\sup_x |P(\mathscr{F}(q_n(\cdot)) \leqslant x) - P(\mathscr{F}(W(\cdot)) \leqslant x)| = \begin{cases} O[n^{-\mu}(\log\log n)^{d\mu}] \\ \qquad \text{for } 0 < \mu \leqslant 1/8 \\ O[n^{-\mu}(\log n)^{2\mu}(\log\log n)^{d\mu}] \\ \qquad \text{for } 1/8 < \mu < 1/4. \end{cases}$$

At first, we prove a couple of lemmas. Let

$$G_n(x) = P(\mathscr{F}(q_n(\cdot)) \leqslant x)$$

and

$$H_n(x) = P(\mathscr{F}(h_n(\cdot)) \leqslant x).$$

Lemma 1: *Under assumptions* (A1), (A2), (A4) *and* (A5),

$$\sup_x |G_n(x) - H_n(x \pm \epsilon_n)| = O(\epsilon_n).$$

Proof: Let $B_n = [|\mathcal{F}(q_n(\cdot)) - \mathcal{F}(h_n(\cdot))| > \epsilon_n]$ and B_n^c denote the complement of B_n.

Then

$$G_n(x) \leq P[\mathcal{F}(q_n(\cdot)) \leq x, B_n^c] + P(B_n)$$

$$\leq P[\mathcal{F}(h_n(\cdot)) \leq x + \epsilon_n] + P(B_n).$$

Similarly

$$G_n(x) \geq P[\mathcal{F}(h_n(\cdot)) \leq x - \epsilon_n] - P(B_n).$$

The lemma follows if we show that $P(B_n) = O(\epsilon_n)$. There exists $c > 0$ such that

$$P(B_n) \leq P[\|q_n(\cdot) - h_n(\cdot)\| > c\epsilon_n]$$

$$= P\left[\sup_{0 \leq t \leq 1} |q_n(t) - h_n(t)| > c\epsilon_n\right]$$

$$\leq P\left[\sup_{0 \leq t \leq 1} |q_n(t) - h_n(t)| > c\epsilon_n, N > \frac{1}{n\theta_n}, \left|\frac{N_n}{[nN]} - 1\right| < \delta_n\right] + O(\epsilon_n)$$

$$= O(\epsilon_n) + \sum_{k > 1/\theta_n} P\left[\sup_{0 \leq t \leq 1} |q_n(t) - h_n(t)| > c\epsilon_n, [nN] = k, |N_n - k| < k\delta_n\right]$$

$$= O(\epsilon_n) + \sum_{k > 1/\theta_n} P\left[\sup_{\substack{0 \leq t \leq 1 \\ |N_n - k| < k\delta_n}} |p(tN_n) - p(t[nN])| > c\epsilon_n k^{1/2}, [nN] = k\right]$$

$$= O(\epsilon_n) + \sum_{k > 1/\theta_n} P\left[\sup_{\substack{0 \leq t \leq 1 \\ |j - k| < k\delta_n}} |p(tj) - p(tk)| < c\,\epsilon_n k^{1/2}, ([nN] = k)\right]$$

$$= O(\epsilon_n) + \sum_{k > 1/\theta_n} P\left[\sup_{\substack{0 \leq t \leq 1 \\ |j - k| < k\delta_n}} |p(tj) - p(tk)| < c\epsilon_n k^{1/2}\right] P([nN] = k). \quad \ldots (5)$$

Consider

$$\sup_{\substack{0\le t\le 1\\ |j-k|<k\delta_n}} |p(tj)-p(tk)| \le 2 \sup_{\substack{0\le t\le 1\\ 1<j/k<1+\delta_n}} |p(tj)-p(tk)|$$

$$= 2 \max_{1\le m\le 1/\delta_n} \sup_{\substack{(m-1)\delta_n\le t<m\delta_n\\ 1<j/k<1+\delta_n}} |p(tj)-p(tk)|$$

$$\le 2 \max_{1\le m<1/\delta_n} \sup_{\substack{(m-1)k\delta_n<tj<mk\delta_n(1+\delta_n)\\ k<j<k(1+\delta_n)\\ (m-1)\delta_n<t<m\delta_n}} |p(tj)-p(tk)|$$

so that

$$P\Big[\sup_{\substack{|j-k|<k\delta_n\\ 0\le t\le 1}} |p(tj)-p(tk)| > c\epsilon_n k^{1/2}\Big]$$

$$\le \sum_{m=1}^{[1/\delta_n]} P\left\{\sup_{\substack{(m-1)k\delta_n<tj<mk\delta_n(1+\delta_n)\\ (m-1)\delta_n<t<m\delta_n\\ k<j<k(1+\delta_n)}} |p(tj)-p(tk)|>c_1\epsilon_n k^{1/2}\right\}$$

$$\le \sum_{m=1}^{[1/\delta_n]} P\Big[\sup_{(m-1)k\delta_n<tj<mk\delta_n(1+\delta_n)} |p(tj)-p(((m-1)k\delta_n)(m-1)k\delta_n)| > c_2\epsilon_n k^{1/2}\Big]$$

$$\le \sum_{m=1}^{[1/\delta_n]} P\Big[\sup_{(m-1)k\delta_n<r<mk\delta_n(1+\delta_n)} |S_r-S_{[(m-1)k\delta_n]}| > c_2\epsilon_n k^{1/2}\Big].$$

Note that central limit theorem holds for $\{X_n\}$. Using Theorem 1 we get that the last term is less than or equal to

$$\sum_{m=1}^{[1/\delta_n]} \frac{1}{1-\left(\dfrac{m\delta_n^2+\delta_n}{c_2^2\,\epsilon_n^2}\right)} P(|S_{[mk\delta_n^2+k\delta_n]}| > c_3\epsilon_n k^{1/2})$$

$$\le \frac{c_2^2\,\epsilon_n^2}{c_2^2\,\epsilon_n^2-2\delta_n} \sum_{m=1}^{[1/\delta_n]} P(|S_{[mk\delta_n^2+k\delta_n]}| > c_3\epsilon_n k^{1/2}). \qquad \dots \quad (6)$$

Write $u_n=[mk\delta_n^2+k\delta_n]$ and $x_n=c_3\,\epsilon_n k^{1/2}u_n^{-1/2}$. Note that $u_n\to\infty$. Choosing $\gamma=\dfrac{\mu}{\mu(2-5\mu)}$, we obtain that $\gamma<1/6$ since $0<\mu<1/4$ and $\dfrac{x_n}{u_n^{\gamma}}=O(n^{\rho}(\log n)^{\tau})$ for some $\tau>0$ where $\rho=\mu\Big[\dfrac{9}{4}\gamma+\dfrac{1}{8}\Big]-\gamma<0$. Hence $\dfrac{x_n}{u_n^{\gamma}}\to 0$ as $n\to\infty$. Furthermore $E[\exp\{|X_1|^{4\gamma/(2\gamma+1)}\}]<\infty$ by (A5). Therefore, applying Theorem 2, we get that

$$P[|S_{u_n}|>x_n u_n^{1/2}] \le 2\{1-\Phi(x_n)\}$$

for n large, $m \leqslant 1/\delta_n$ and $k \geqslant 1/\theta_n$. Thus, from (6), we have

$$P\left[\sup_{\substack{0 \leqslant t \leqslant 1 \\ |j-k| \leqslant k\delta_n}} |p(tj)-p(tk)| \leqslant c\epsilon_n k^{1/2}\right] \leqslant \left(\frac{c_4}{c_5-2\epsilon_n^{1/4}}\right)\frac{1}{\delta_n}\{1-\Phi(x_n)\}$$

$$= \frac{c_4\epsilon_n^{-9/4}}{c_5-2\epsilon_n^{1/4}}\{1-\Phi(x_n)\}.$$

Since $x_n \sim \epsilon_n^{1/8}$ and $1-\Phi(x_n) \sim x_n^{-1}e^{-\frac{1}{2}x_n^2}$ as $n \to \infty$, it follows that the last term is $O(\epsilon_n)$. Thus, from (5), we have

$$P(B_n) \leqslant O(\epsilon_n)+\sum_{k>1/\theta_n} O(\epsilon_n)P([nN]=k) = O(\epsilon_n)$$

completing the proof of Lemma 1.

Lemma 2: *Under the assumptions* (A1) *to* (A5),

$$\sup_x |P(\mathcal{F}(h_n(\cdot)) \leqslant x)-P(\mathcal{F}(W(\cdot)) \leqslant x)| = \begin{cases} O(\theta_n^\mu) & \text{if } 0<\mu\leqslant 1/8 \\ O(\theta_n^\mu(-\log\theta_n)^{2\mu}) & \text{if } \frac{1}{8}<\mu<\frac{1}{4} \end{cases}$$

Proof: Note that

$$H_n(x)-\psi(x)$$

$$= \sum_{k>1/\theta_n} P(\mathcal{F}(h_n(\cdot)) \leqslant x, [nN]=k)-P([nN]=k, \mathcal{F}(W(\cdot)) \leqslant x)$$

$$+\sum_{k<1/\theta_n} P(\mathcal{F}(h_n(\cdot)) \leqslant x, [nN]=k)-P([nN]=k, \mathcal{F}(W(\cdot)) \leqslant x)$$

so that

$$|H_n(x)-\psi(x)| \leqslant \left|\sum_{k>1/\theta_n} P([nN]=k)\{P(\mathcal{F}(p_k(\cdot)) \leqslant x)-\psi(x)\}\right|$$

$$+2\sum_{k<1/\theta_n} P([nN]=k)$$

$$\leqslant \sum_{k>1/\theta_n} P([nN]=k)\,|P(\mathcal{F}(p_k(\cdot)) \leqslant x)-\psi(x)|$$

$$+2P\left(N \leqslant \frac{1}{n\theta_n}\right).$$

Assumption (A5) implies that $E|X_1|^{2+\alpha} < \infty$ for $\alpha > 0$. Hence assumptions (A3) and (A4) prove (4) which in turn gives the relation

$$\sup_x |H_n(x)-\psi(x)| \leqslant \begin{cases} \sum_{k > 1/\theta_n} P([nN]=k)Ak^{-\mu}+O(\varepsilon_n) \text{ for } 0<\mu\leqslant 1/8 \\ \sum_{k > 1/\theta_n} P([nN]=k)Ak^{-\mu}(\log k)^{2\mu}+O(\varepsilon_n) \text{ for } \tfrac{1}{8}<\mu<\tfrac{1}{4} \end{cases}$$

for some $A > 0$. Therefore,

$$\sup_x |H_n(x)-\psi(x)| \leqslant \begin{cases} c_1\theta_n^{\mu} \text{ for } 0<\mu\leqslant 1/8 \\ c_2\theta_n^{\mu}(-\log \theta_n)^{2\mu} \text{ for } \tfrac{1}{8}<\mu<\tfrac{1}{4}. \end{cases}$$

This completes the proof of Lemma 2.

Combining Lemmas 1 and 2 and using (A4), we complete the proof of Theorem 3.

Remarks: It may be noted that since the estimate (3) is valid if one replaces Δ_n by the Levy-Prohorov distance between the measures associated with p_n and W (vide Borovkov (1973)), a similar extension should be possible in Theorem 3. After the original submission of our paper we have come to know of better estimates than (3) for Δ_n due to Sahanenko. This, too, is likely to extend to the present case. We shall return to these questions later. An excellent review of the literature up to 1978 is given in Borovkov (1978).

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