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Sequential Estimation for Fractional Ornstein–Uhlenbeck Type Process

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ABSTRACT

We investigate the asymptotic properties of the sequential maximum likelihood estimator of the drift parameter for fractional Ornstein–Uhlenbeck type process satisfying a linear stochastic differential equation driven by a fractional Brownian motion.

Key Words: Fractional Ornstein–Uhlenbeck type process; Fractional Brownian motion; Sequential maximum likelihood estimation.

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1. INTRODUCTION

Long range dependence phenomenon is said to occur in a stationary time series $\{X_n, n \geq 0\}$ if the $\mathrm{Cov}(X_0, X_n)$ of the time series tend to zero as $n \to \infty$ and yet it satisfies the condition

$$\sum_{n=0}^{\infty} |\mathrm{Cov}(X_0, X_n)| = \infty.$$

In other words $\mathrm{Cov}(X_0, X_n)$ tends to zero but so slowly that their sums diverge. This phenomenon was first observed by the hydrologist Hurst[3] on projects involving the design of reservoirs along the Nile river (cf. Montanari[9]) and by others in hydrological time series. It was recently observed that a similar phenomenon occurs in problems concerned with traffic patterns of packet flows in high-speed data net works such as the Internet (cf. Willinger et al.,[20] Norros[11]). The long range dependence pattern is also observed in macroeconomics and finance (cf. Henry and Zafforoni[2]). Long range dependence is also related to the concept of self-similarity for a stochastic process. A stochastic process $\{X(t), t \in R\}$ is said to be H-self-similar with index $H > 0$ if for every $a > 0$, the processes $\{X(at), t \in R\}$ and the process $\{a^H X(t), t \in R\}$ have the same finite dimensional distributions. Suppose a self-similar process has stationary increments. Then the increments form a stationary time series which exhibits long range dependence. A Gaussian H-self-similar process with stationary increments with $0 < H < 1$ is called a *fractional Brownian motion* (fBm). A recent monograph by Doukhan et al.[1] discusses the theory and the applications of long range dependence and the properties of fractional brownian motion (see Taqqu[19]). If $H = 1/2$, then the fractional Brownian motion reduces to the standard Brownian motion also called the Wiener process.

Diffusion processes and diffusion type processes satisfying stochastic differential equations driven by Wiener processes are used for stochastic modeling in wide variety of sciences such as population genetics, economic processes, signal processing as well as for modeling sunspot activity and more recently in mathematical finance. Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes has been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao.[14] There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion to model processes involving long range dependence. Le Breton[8] studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. In a recent paper, Kleptsyna and Le Breton[6] studied parameter estimation problems for fractional Ornstein–Uhlenbeck type process. This is a fractional analogue of the Ornstein–Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in (1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = \theta \int_0^t X_s\, ds + \sigma W_t^H, \quad t \geq 0. \tag{1.1}$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

Parametric estimation for more general classes of stochastic processes, satisfying linear stochastic differential equations driven by a fractional Brownian motion and observed over a fixed period of time T, is studied in Prakasa Rao.[16,17] It is well known that the sequential estimation methods might lead to efficient estimators from a process observed possibly over a shorter expected period of observation time as compared to estimators based on predetermined fixed observation time. We now investigate the conditions for such a phenomenon. Novikov[12] investigated the asymptotic properties of a sequential maximum likelihood estimator for the drift parameter in the Ornstein–Uhlenbeck process. We now discuss analogous results for the fractional Ornstein–Uhlenbeck type process.

2. PRELIMINARIES

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions. The natural filtration of a process is understood as the P-completion of the filtration generated by the process.

Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion (fBm) defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with the Hurst parameter $H \in [1/2, 1)$ that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}\left[s^{2H} + t^{2H} - |s - t|^{2H}\right], \quad t \geq 0, \ s \geq 0. \tag{2.1}$$

Let us consider a stochastic process $\{X_t, t \geq 0\}$ defined by the stochastic integral equation

$$X_t = \theta \int_0^t X(s)ds + \sigma W_t^H, \quad t \geq 0 \tag{2.2}$$

where θ and σ^2 are the unknown constant drift and the diffusion parameters respectively. For convenience, we write the above integral equation in the form of a stochastic differential equation

$$dX_t = \theta X(t)dt + \sigma\, dW_t^H, \quad X_0 = 0, \quad t \geq 0 \tag{2.3}$$

driven by the fractional Brownian motion W^H. Even though the process X is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration $(\mathcal{Z}_t)$ of the process Z coincides with the natural filtration $(\mathcal{X}_t)$ of the process X (see Kleptsyna et al.[7]). Define, for $0 < s < t$,

$$k_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(H + \frac{1}{2}\right), \tag{2.4}$$

$$k_H(t, s) = k_H^{-1} s^{\frac{1}{2}-H}(t - s)^{\frac{1}{2}-H}, \tag{2.5}$$

$$\lambda_H = \frac{2H\,\Gamma(3 - 2H)\Gamma(H + \frac{1}{2})}{\Gamma(\frac{3}{2} - H)}, \tag{2.6}$$

$$w_t^H = \lambda_H^{-1} t^{2-2H}, \tag{2.7}$$

and

$$M_t^H = \int_0^t k_H(t, s)dW_s^H, \quad t \geq 0. \tag{2.8}$$

The process M^H is a gaussian martingale, called the *fundamental martingale* (cf. Norros et al.[10]), and its quadratic variation $\langle M_t^H \rangle = w_t^H$. Further more the natural filtration of the martingale M^H coincides with the natural filtration of the fBm W^H. Let

$$K_H(t, s) = H(2H - 1)\frac{d}{ds}\int_s^t r^{H-\frac{1}{2}}(r - s)^{H-\frac{3}{2}}dr, \quad 0 \leq s \leq t. \tag{2.9}$$

The sample paths of the process $\{X_t, t \geq 0\}$ are smooth enough so that the process Q defined by

$$Q(t) = \frac{d}{dw_t^H}\int_0^t k_H(t, s)X_s\,ds, \quad t \in [0, T] \tag{2.10}$$

is well-defined where w^H and k_H are as defined in (2.7) and (2.5) respectively and the derivative is understood in the sense of absolute continuity with respect to the measure generated by w^H. More over the sample paths of the process Q belong to $L^2([0,T], dw^H)$ a.s. $[P]$. The following theorem due to Kleptsyna et al.[7] associates a *fundamental semimartingale* Z associated with the process X such that the natural filtration $(\mathcal{Z}_t)$ of Z coincides with the natural filtration $(\mathcal{X}_t)$ of X.

Theorem 2.1. *Let the process* $Z = (Z_t, t \in [0,T])$ *be defined by*

$$Z_t = \int_0^t k_H(t,s)dX_s \tag{2.11}$$

where the function $k_H(t,s)$ *is as defined by* (2.5). *Then the following results hold*:

(i) *The process Z is an* $(\mathcal{F}_t)$*-semimartingale with the decomposition*

$$Z_t = \theta \int_0^t Q(s)dw_s^H + \sigma M_t^H \tag{2.12}$$

where M^H *is the Gaussian martingale defined by* (2.8),

(ii) *The process X admits the representation*

$$X_t = \int_0^t K_H(t,s)dZ_s \tag{2.13}$$

where the function K_H *is as defined in* (2.9), *and*

(iii) *The natural filtrations of* $(\mathcal{Z}_t)$ *and* $(\mathcal{X}_t)$ *coincide.*

Kleptsyna et al.[7] derived a Girsanov type formula for the fractional Brownian motion. As an application, it follows that the Radon–Nikodym derivative of the measure P_θ^T, generated by the stochastic process X when θ is the true parameter, with respect to the measure generated by the process X when $\theta = 0$, is given by

$$\frac{dP_\theta^T}{dP_0^T} = \exp\left[\theta \int_0^T Q(s)dZ_s - \frac{1}{2}\theta^2 \int_0^T Q^2(s)dw_s^H\right]. \tag{2.14}$$

From the representation (2.12), it follows that the quadratic variation $\langle Z \rangle_T$ of the process Z on $[0,T]$ is equal to $\sigma^2 w_T^H$ a.s. and hence the parameter σ^2 can be estimated by the relation

$$\lim_n \sum \left[Z_{t_{i+1}^{(n)}} - Z_{t_i^{(n)}}\right]^2 = \sigma^2 w_T^H \quad \text{a.s.} \tag{2.15}$$

where $(t_i^{(n)})$ is an appropriate partition of $[0, T]$ such that

$$\sup |t_{i+1}^{(n)} - t_i^{(n)}| \to 0$$

as $n \to \infty$. Hence we can estimate σ^2 almost surely from any small interval as long as we have a continuous observation of the process. For further discussion, we assume that $\sigma^2 = 1$.

3. MAXIMUM LIKELIHOOD ESTIMATION

We consider the problem of estimation of the parameter θ based on the observation of the process $X = \{X_t, 0 \le t \le T\}$ for a fixed time T and study its asymptotic properties as $T \to \infty$. These results are due to Kleptsyna and Le Breton[6] and Prakasa Rao.[13,16,17]

Theorem 3.1. *The maximum likelihood estimator θ based on the observation $X = \{X_t, 0 \le t \le T\}$ is given by*

$$\hat{\theta}_T = \left\{ \int_0^T Q^2(s) dw_s^H \right\}^{-1} \int_0^T Q(s) dZ_s \tag{3.1}$$

where the processes Q and Z are as defined by (2.10) *and* (2.11) *respectively. Furthermore the estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$, that is,*

$$\lim_{T \to \infty} \hat{\theta}_T = \theta \quad a.s. \ [P_\theta] \tag{3.2}$$

for every $\theta \in R$.

We now discuss the limiting distribution of the MLE $\hat{\theta}_T$ as $T \to \infty$.

Theorem 3.2. *Let*

$$R_T = \int_0^T Q(s) dZ_s. \tag{3.3}$$

Assume that there exists a norming function I_t, $t \ge 0$ such that

$$I_T^2 \int_0^T Q(t)^2 \, dw_t^H \xrightarrow{p} \eta^2 \quad as \ T \to \infty \tag{3.4}$$

where $I_T \to 0$ *as* $T \to \infty$ *and* η *is a random variable such that* $P(\eta > 0) = 1$. *Then*

$$(I_T R_T, I_T^2 \langle R_T \rangle) \xrightarrow{\mathcal{D}} (\eta Z, \eta^2) \quad \text{as } T \to \infty \tag{3.5}$$

where the random variable Z has the standard normal distribution and the random variables Z and η *are independent.*

Proof. This theorem follows as a consequence of the central limit theorem for martingales (cf. Theorem 1.49, Remark 1.47 in p. 65 of Ref.[15]).

Observe that

$$I_T^{-1}(\hat{\theta}_T - \theta_0) = \frac{I_T R_T}{I_T^2 \langle R_T \rangle} \tag{3.6}$$

Applying the Theorem 3.2, we obtain the following result.

Theorem 3.3. *Suppose the conditions stated in the Theorem 3.2 hold. Then*

$$I_T^{-1}(\hat{\theta}_T - \theta_0) \to \frac{Z}{\eta} \quad \text{in law as } t \to \infty \tag{3.7}$$

where the random variable Z has the standard normal distribution and the random variables Z and η *are independent.*

Remarks. If the random variable η is a constant with probability one, then the limiting distribution of the maximum likelihood estimator is normal with mean 0 and variance η^{-2}. Otherwise it is a mixture of the normal distributions with mean zero and variance η^{-2} with the mixing distribution as that of η.

4. SEQUENTIAL MAXIMUM LIKELIHOOD ESTIMATION

We now consider the problem of sequential maximum likelihood estimation of the parameter θ. Let h be a nonnegative number. Define the stopping rule $\tau(h)$ by the rule

$$\tau(h) = \inf\left\{t : \int_0^t Q^2(s) dw_s^H \geq h\right\}. \tag{4.1}$$

Kletptsyna and Le Breton[6] have shown that

$$\lim_{t\to\infty}\int_0^t Q^2(s)dw_s^H = +\infty \quad \text{a.s. } [P_\theta] \tag{4.2}$$

for every $\theta \in R$. Then it can be shown that $P_\theta(\tau(h) < \infty) = 1$. If the process is observed up to a previously determined time T, we have observed that the maximum likelihood estimator is given by

$$\hat{\theta}_T = \left\{\int_0^T Q^2(s)dw_s^H\right]^{-1}\int_0^T Q(s)dZ_s. \tag{4.3}$$

The estimator

$$\begin{aligned}\hat{\theta}(h) &\equiv \hat{\theta}_{\tau(h)} \\ &= \left\{\int_0^{\tau(h)} Q^2(s)dw_s^H\right]^{-1}\int_0^{\tau(h)} Q(s)dZ_s \\ &= h^{-1}\int_0^{\tau(h)} Q(s)dZ_s\end{aligned} \tag{4.4}$$

is called the *sequential maximum likelihood estimator* of θ. We now study the asymptotic properties of the estimator $\hat{\theta}(h)$.

We shall first prove a lemma which is an analogue of the Cramer–Rao inequality for sequential plans $(\tau(X), \hat{\theta}_\tau(X))$ for estimating the parameter θ satisfying the property

$$E_\theta\{\hat{\theta}_\tau(X)\} = \theta \tag{4.5}$$

for all θ.

Lemma 4.1. *Suppose that differentiation under the integral sign with respect to θ on the left side of Eq.* (4.5) *is permissible. Further suppose that*

$$E_\theta\left\{\int_0^{\tau(X)} Q^2(s)dw_s^H\right\} < \infty \tag{4.6}$$

for all θ. Then

$$\mathrm{Var}_\theta\{\hat{\theta}_\tau(X)\} \geq \left(E_\theta\left\{\int_0^{\tau(X)} Q^2(s)dw_s^H\right\}\right)^{-1} \tag{4.7}$$

for all θ.

Proof. Let P_θ be the measure generated by the process $X(t), 0 \leq t \leq \tau(X)$ for given θ. It follows from the results discussed above that

$$\frac{dP_\theta}{dP_{\theta_0}} = \exp\left\{(\theta - \theta_0)\int_0^{\tau(X)} Q(s)dZ_s - \frac{1}{2}(\theta^2 - \theta_0^2)\int_0^{\tau(X)} Q^2(s)dw_s^H\right\} \quad \text{a.s. } [P_{\theta_0}]. \tag{4.8}$$

Differentiating (4.5) with respect to θ under the integral sign, we get that

$$E_\theta\left[\hat{\theta}_\tau(X)\left\{\int_0^{\tau(X)} Q(s)dZ_s - \theta\int_0^{\tau(X)} Q^2(s)dw_s^H\right\}\right] = 1. \tag{4.9}$$

Theorem 2.1, implies that

$$dZ_s = \theta Q_s\, dw_s^H + dM_s^H \tag{4.10}$$

and hence

$$\int_0^T Q(s)dZ_s = \theta\int_0^T Q^2(s)dw_s^H + \int_0^T Q(s)dM_s^H. \tag{4.11}$$

The above relation in turn implies that

$$E_\theta\left\{\int_0^{\tau(X)} Q(s)dZ_s - \theta\int_0^{\tau(X)} Q^2(s)dw_s^H\right\} = 0 \tag{4.12}$$

and

$$E_\theta\left\{\int_0^{\tau(X)} Q(s)dZ_s - \theta\int_0^{\tau(X)} Q^2(s)dw_s^H\right\}^2 = E_\theta\left\{\int_0^{\tau(X)} Q^2(s)dw_s^H\right\} \tag{4.13}$$

from the properties of the fundamental martingale M^H and the fact that the quadratic variation $\langle M^H\rangle_t$ of the process M_t^H is w_t^H. Applying the Cauchy–Schwartz inequality to the left side of Eq. (4.9), we obtain that

$$\mathrm{Var}_\theta\{\hat{\theta}_\tau(X)\} \geq \left(E_\theta\left\{\int_0^{\tau(X)} Q^2(s)dw_s^H\right\}\right)^{-1} \tag{4.14}$$

for all θ.

Definition. A sequential plan $(\tau(X), \hat{\theta}_\tau(X))$ is said to be *efficient* if there is equality in (4.7) for all θ.

We now prove the main result.

Theorem 4.2. *Consider the fractional Ornstein–Uhlenbeck type process governed by the stochastic differential Eq.* (2.3) *with* $\sigma = 1$ *driven by the fractional Brownian motion* W^H *with* $H \in [1/2, 1)$. *Then the sequential plan* $(\tau(h), \hat{\theta}(h))$ *defined by Eqs.* (4.1) *and* (4.4) *has the following properties for all* θ:

(i) $\hat{\theta}(h) \equiv \hat{\theta}_{\tau(h)}$ *is normally distributed with* $E_\theta(\hat{\theta}(h)) = \theta$ *and* $\mathrm{Var}_\theta(\hat{\theta}(h)) = h^{-1}$;
(ii) *the plan is efficient; and*
(iii) *the plan is closed, that is,* $P_\theta(\tau(h) < \infty) = 1$.

Proof. Let

$$J_T = \int_0^T Q(s)dM_s^H. \tag{4.15}$$

From the results in Karatazas and Shreve,[5] Revuz and Yor[18] and Ikeda and Watanabe,[4] it follows that there exists a standard Wiener process W such that

$$J_T = W(\langle J\rangle_T) \quad \text{a.s.} \tag{4.16}$$

with respect to the filtration $\{\mathcal{F}_{\tau_t}, t \geq 0\}$ under P where $\tau_t = \inf\{s : \langle J\rangle_s > t\}$. Hence the process

$$\int_0^{\tau(h)} Q(s)dM_s^H \tag{4.17}$$

is a standard Wiener process. Observe that

$$\begin{aligned}\hat{\theta}(h) &= h^{-1}\int_0^{\tau(h)} Q(s)dZ_s \\ &= h^{-1}\left\{\theta\int_0^{\tau(h)} Q^2(s)dw_s^H + \int_0^{\tau(h)} Q(s)dM_s^H\right\} \\ &= \theta + h^{-1}\int_0^{\tau(h)} Q(s)dM_s^H \\ &= \theta + h^{-1}J_{\tau(h)} \\ &= \theta + h^{-1}W(\langle J\rangle_{\tau(h)})\end{aligned} \tag{4.18}$$

which proves that the estimator $\hat{\theta}(h)$ is normally distributed with mean θ and variance h^{-1}. Since

$$E_\theta\left\{\int_0^{\tau(h)} Q^2(s)dw_s^H\right\} = h, \tag{4.19}$$

it follows that the plan is efficient by the Lemma 4.1. Since

$$P_\theta(\tau(h) \geq T) = P_\theta\left\{ \int_0^T Q^2(s)\,dw_s^H < h \right\} \tag{4.20}$$

for every $T \geq 0$, it follows that $P_\theta(\tau(h) < \infty) = 1$ from the observation

$$P_\theta\left(\int_0^\infty Q^2(s)\,dw_s^H = \infty \right) = 1. \tag{4.21}$$

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