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ORDER STATISTICS FOR NONIDENTICALLY DISTRIBUTED VARIABLES AND PERMANENTS

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SUMMARY. Theory of permanents provides an effective tool in dealing with order statistics corresponding to random variables which are independent but possibly nonidentically distributed. This is illustrated by giving a characterization of symmetric random variables in terms of order statistics and by generalizing some known recurrence relations. It is shown that the distribution function of one or more order statistics can be represented in terms of permanents and this fact combined with the Alexandroff inequality is used to demonstrate the log-concavity of certain sequences. The case of order statistics corresponding to independent exponential random variables is considered and the m.g.f. and moments of an order statistic and those of the range are derived explicitly.

1. Introduction

Let $X_1, \ldots, X_n$ be independent random variables with distribution functions $F_1, \ldots, F_n$ respectively and let $Y_1 \leqslant Y_2 \leqslant \ldots \leqslant Y_n$ denote the corresponding order statistics. In the theory of order statistics it is usually assumed that $X_1, \ldots, X_n$ are identically distributed. However in many practical situations it is necessary to allow for nonidentical $F_1, \ldots, F_n$. This is the case, for example, if there is a possibility of one or more outliers being present. Also in some instances $F_1, \ldots, F_n$ may be believed to be of the same functional form but with different values of the parameters involved.

In this paper we consider the case where $F_1, \ldots, F_n$ are not necessarily identical. The paper is organized as follows. In Section 2, we give some elementary facts concerning permanents. It is pointed out that the theory of permanents provides an effective technique to handle the case of order statistics from nonidentical parents. This is illustrated by the results in the next three sections.

In Section 3, a characterization of symmetric distributions is obtained. In Section 4 we show that the distribution function of a subset of $Y_1, \ldots, Y_n$ can be expressed in terms of permanents. This is used to show that the sequences $P(Y_r \leqslant y)$ and $P(Y_r > y)$, $r = 1, 2, \ldots, n$ are log-concave for any

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real y. In Section 5 we generalize certain recurrence relations to the case of nonidentically distributed variables. In Section 6 we consider the case of independent nonidentical exponential random variables. The m.g.f. of the order statistic Y_r and that of the range $Y_n - Y_1$ are derived explicitly.

2. Permanents

Let S_n denote the set of permutations of $1, 2, \ldots, n$. If A is an $n\times n$ matrix, the permanent of A, denoted by per A, is defined as :

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

The permanent clearly remains unchanged if the rows or columns of the matrix are permuted. Furthermore the permanent admits a Laplace expansion along any row or column of the matrix. Thus if we denote by $A(i,j)$ the matrix obtained by deleting row i and column j of the $n\times n$ matrix A, then

$$\text{per } A = \sum_{j=1}^{n} a_{ij} \text{ per } A(i,j), \quad i = 1, 2, \ldots, n$$

and

$$\text{per } A = \sum_{i=1}^{n} a_{ij} \text{ per } A(i,j), \quad j = 1, 2, \ldots, n.$$

For a detailed account of permanents see Minc (1979) and the survey papers of Minc (1983, 1987).

If $a_1, a_2, \ldots$ are column vectors, then

$$[\underbrace{a_1,}_{i_1} \ \underbrace{a_2,}_{i_2} \ \ldots]$$

will denote the matrix obtained by taking i_1 copies of a_1, i_2 copies of a_2 and so on.

Let $X_1, \ldots, X_n$ be independent random variables with absolutely continuous distribution functions $F_1, \ldots, F_n$ and densities $f_1, \ldots, f_n$ respectively. Then Vaughan and Venables (1972) have shown that the density of the order statistic Y_r or the joint density of a subset of $Y_1, \ldots, Y_n$ is conveniently expressed in terms of a permanent. For example, the density of Y_r is given by

$$g_r(y) = \frac{1}{(r-1)!\,(n-r)!} \text{ per} \begin{bmatrix} f_1(y) & F_1(y) & 1-F_1(y) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \underbrace{f_n(y)}_{1} & \underbrace{F_n(y)}_{r-1} & \underbrace{1-F_n(y)}_{n-r} \end{bmatrix}, \quad -\infty < y < \infty \ \ldots \quad (1)$$

It will be seen in Section 4 that the distribution function of one or more order statistics can also be given in terms of permanents.

It must be remarked that the representation (1) and the analogous formula for joint densities have not been sufficiently exploited in the literature. One reason for this is perhaps the fact that the permanent does not lend itself to manipulation as easily as the determinant. However, there have been some significant advances in the theory of permanents in the last few years and we believe that it must be thought of as an essential tool in the theory of order statistics. In the next three sections we illustrate by means of examples that the theory of permanents facilitates generalizations of results known for the case of identically distributed variables as well as produces some new inequalities.

3. Symmetric random variables

A random variable X with distribution function F is said to be symmetric about a real number μ if

$$F(\mu+x)+F(\mu-x) = 1, \text{ for all } x.$$

In this section we consider the case $\mu = 0$ for convenience. It is well known (David, 1981, p. 24) that if $X_1, \ldots, X_n$ are i.i.d. continuous random variables which are symmetric about zero, then for any r, $1 \leqslant r \leqslant n$, $-Y_r$ and Y_{n-r+1} are identically distributed. In the next result this fact is generalized and a partial converse is given.

Theorem 3.1 : *Let $X_1, \ldots, X_n$ be independent random variables with absolutely continuous distribution functions $F_1, \ldots, F_n$ and densities $f_1, \ldots, f_n$ respectively. Suppose X_i is symmetric about zero, $i = 2, \ldots, n$. Let r be fixed, $1 \leqslant r \leqslant n$. Then $-Y_r$ and Y_{n-r+1} are identically distributed if X_1 is also symmetric about zero. Conversely, if $0 < F_i(x) < 1$ for all x, $i = 1, \ldots, n$ and if $-Y_r$ and Y_{n-r+1} are identically distributed, then X_1 is symmetric about zero.*

Proof : By (1), the density of $-Y_r$ is given by

$$g(y) = \frac{1}{(r-1)!\,(n-r)!} \operatorname{per} \begin{bmatrix} f_1(-y) & F_1(-y) & 1-F_1(-y) \\ \vdots & \vdots & \vdots \\ f_n(-y) & F_n(-y) & 1-F_n(-y) \end{bmatrix}, \quad -\infty < y < \infty \quad \ldots (2)$$

$$\underbrace{}_{1} \quad \underbrace{}_{r-1} \quad \underbrace{}_{n-r}$$

real y. In Section 5 we generalize certain recurrence relations to the case of nonidentically distributed variables. In Section 6 we consider the case of independent nonidentical exponential random variables. The m.g.f. of the order statistic Y_r and that of the range $Y_n - Y_1$ are derived explicitly.

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It will be seen in Section 4 that the distribution function of one or more order statistics can also be given in terms of permanents.

It must be remarked that the representation (1) and the analogous formula for joint densities have not been sufficiently exploited in the literature. One reason for this is perhaps the fact that the permanent does not lend itself to manipulation as easily as the determinant. However, there have been some significant advances in the theory of permanents in the last few years and we believe that it must be thought of as an essential tool in the theory of order statistics. In the next three sections we illustrate by means of examples that the theory of permanents facilitates generalizations of results known for the case of identically distributed variables as well as produces some new inequalities.

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Similarly the density of Y_{n-r+1} is

$$h(y) = \frac{1}{(n-r)!\,(r-1)!} \operatorname{per} \begin{bmatrix} f_1(y) & F_1(y) & 1-F_1(y) \\ \vdots & \vdots & \vdots \\ \underbrace{f_n(y)}_{1} & \underbrace{F_n(y)}_{n-r} & \underbrace{1-F_n(y)}_{r-1} \end{bmatrix}, -\infty < y < \infty \quad \ldots \quad (3)$$

First suppose that X_1 is also symmetric about zero. Then $F_i(x) = 1-F_i(-x)$ for all x, $i = 1, \ldots, n$. Hence $f_i(x) = f_i(-x)$ for almost all x, $i = 1, \ldots, n$. It follows from (2), (3) that $g(y) = h(y)$ for almost all y, since the permanent remains unchanged if the columns of the matrix are permuted.

To prove the converse, suppose that $0 < F_i(x) < 1$ for all x, $i = 1, \ldots, n$ and that $-Y_r$ and Y_{n-r+1} are indentically distributed. Then $g(y) = h(y)$ for all y and from (2), (3), for all y,

$$\operatorname{per} \begin{bmatrix} f_1(-y) & F_1(-y) & 1-F_1(-y) \\ \vdots & & \\ \underbrace{f_n(-y)}_{1} & \underbrace{F_n(-y)}_{r-1} & \underbrace{1-F_n(-y)}_{n-r} \end{bmatrix} = \operatorname{per} \begin{bmatrix} f_1(y) & F_1(y) & 1-F_1(y) \\ \vdots & & \\ \underbrace{f_n(y)}_{1} & \underbrace{F_n(y)}_{n-r} & \underbrace{1-F_n(y)}_{r-1} \end{bmatrix} \quad \ldots \quad (4)$$

Expand the permanents on both sides of (4) in terms of the first row and use the fact that $f_i(-x) = f_i(x)$ for all x, $i = 2, \ldots, n$ and that $F_i(-x) = 1-F_i(x)$ for all x, $i = 2, \ldots, n$. Then we get

$$\alpha\{f_1(-y)-f_1(y)\}+\beta(r-1)\{F_1(-y)-1+F_1(y)\}$$
$$+\gamma(n-r)\{1-F_1(-y)-F_1(y)\} = 0, \quad \ldots \quad (5)$$

where α, β, γ, are respectively, the permanents of the matrices obtained by deleting row 1 and column 1 ; row 1 and column 2 ; and row 1 and column $r+1$ of the matrix appearing on the left hand side of (5).

Fix a real number y and suppose it satisfies $\beta(r-1) = \gamma(n-r)$. Then from (5),

$$\alpha\{f_1(-y)-f_1(y)\} = 0$$

Since $0 < F_i(x) < 1$ for all $x, i = 1, \ldots, n$, α is nonzero and hence $f_1(-y) = f_1(y)$.

If $\beta(r-1) \neq \gamma(n-r)$, then replace y by $-y$ in (5) and add the resulting equation to (5) to get

$$F_1(-y)-1+F_1(y) = 0.$$

We have therefore shown that for each real y, either $f_1(-y) = f_1(y)$ or $F_1(y)+F_1(-y) = 1$. A simple calculus argument shows that then for all y, $F_1(y)+F_1(-y) = 1$ and hence X_1 is symmetric about zero.

An argument similar to that used in Theorem 3.1 and the permanent representation of joint densities of order statistics (Vaughan and Venables, 1972) may be used to show that if $X_1, \ldots, X_n$ are independent continuous random variables symmetric about zero, then for any $1 \leqslant j_1 < \ldots < j_k \leqslant n$, the joint density of $-Y_{j_1}, \ldots, -Y_{j_k}$ is the same as that of $Y_{n-j_k+1}, \ldots, Y_{n-j_1+1}$. Observations of this type have been used in the identical variables case to reduce the computational effort in calculating covariances. For example, we have

$$E(Y_r) = -E(Y_{n-r+1}) \quad \ldots \quad (6)$$

$$\operatorname{cov}(Y_r, Y_s) = \operatorname{cov}(Y_{n-s+1}, Y_{n-r+1}) \quad \ldots \quad (7)$$

The relations (6), (7) have been used for tabulation in the case of identically distributed $X_1, \ldots, X_n$. (See David, 1981, p. 36).

4. Log-concavity

It is possible to represent the distribution function of Y_r or the joint distribution function of a subset of $Y_1, \ldots, Y_n$ in terms of permanents. Although this fact has not been explicitly stated in the literature, the basic idea is the same as the one used by Vaughan and Venables (1972). We state the representation in the next two results.

Theorem 4.1 : *Let $X_1, \ldots, X_n$ be independent random variables with distribution fuhctions $F_1, \ldots, F_n$ respectively. Then the distribution function of Y_r, $1 \leqslant r \leqslant n$, is given by*

$$P(Y_r \leqslant y) = \sum_{i=r}^{n} \frac{1}{i\,!(n-i)\,!} \operatorname{per} \begin{bmatrix} F_1(y) & 1-F_1(y) \\ \vdots & \vdots \\ \underbrace{F_n(y)}_{i} & \underbrace{1-F_n(y)}_{n-i} \end{bmatrix}, -\infty < y < \infty \quad \ldots \quad (8)$$

Proof : A simple argument shows (David, 1981, p. 22) that

$$P(Y_r \leqslant y) = \sum_{i=r}^{n} P(\text{exactly } i \text{ variables from } X_1, \ldots, X_n \text{ are} \leqslant y)$$

$$= \sum_{i=r}^{n} \sum_{T_i} \prod_{l=1}^{i} F_{j_l}(y) \prod_{l=i+1}^{n} [1-F_{j_l}(y)],$$

where the summation T_i extends over all permutations $j_1, \ldots, j_n$ of $1, \ldots, n$ for which $j_1 < \ldots < j_i$ and $j_{i+1} < \ldots < j_n$. The result now follows by the definition of the permanent.

The next result can be proved by a similar argument.

Theorem 4.2 : *Let $X_1, \ldots, X_n$ be independent random variables with distribution function $F_1, \ldots, F_n$ respectively and let $1 \leqslant i_1 < \ldots < i_k \leqslant n$. Then for $y_1 \leqslant \ldots \leqslant y_k$,*

$$P(Y_{i_1} \leqslant y_1, \ldots, Y_{i_k} \leqslant y_k)$$

$$= \Sigma \frac{1}{j_1! \ldots j_{k+1}!} \operatorname{per} \begin{bmatrix} F_1(y_1) & F_1(y_2)-F_1(y_1) & \ldots & 1-F_1(y_k) \\ \vdots & & & \\ \underbrace{F_n(y_1)}_{j_1} & \underbrace{F_n(y_2)-F_n(y_1)}_{j_2} & \ldots & \underbrace{1-F_n(y_k)}_{j_{k+1}} \end{bmatrix}$$

where the summation is over $j_1, \ldots, j_{k+1}$ satisfying $j_1 \geqslant i_1$, $j_1+j_2 \geqslant i_2, \ldots,$ $j_1+ \ldots + j_k \geqslant i_k$, $j_1+ \ldots +j_{k+1} = n$.

It may be remarked that if the condition $y_1 \leqslant \ldots \leqslant y_k$ is not imposed in Theorem 4.2 then some of the inequalities in $Y_{i_1} \leqslant y_1 \ldots, Y_{i_k} \leqslant y_k$ may be redundant and the probability can be evaluated after making the necessary reduction.

Definition : A sequence of nonnegative numbers $\alpha_1, \alpha_2, \ldots$ is said to be log-concave if $\alpha_i^2 \geqslant \alpha_{i-1}\, \alpha_{i+1}$, $i = 2, 3, \ldots$. A finite sequence $\alpha_1, \ldots, \alpha_n$ will be said to be log-concave if $\alpha_1, \ldots, \alpha_n, 0, \ldots$ is log-concave.

Log-concave sequences arise frequently in statistics and in combinatorics. It is easy to see that a log-concave sequence must be unimodal (see, for example, Comtet, 1974, p. 270).

In the next result we state some properties of log-concave sequences that will be used. The first property is well-known but a proof is included for completeness.

Lemma 4.3 : *Let $\alpha_1, \alpha_2, \ldots$ be log-concave. Then the following statements are true* :

(a) *For $1 < j \leqslant k$, $\alpha_k\, \alpha_j \geqslant \alpha_{k+1}\, \alpha_{j-1}$;*

(b) *Let k be a positive integer and let*

$$\beta_j = \sum_{i=j}^{j+k-1} \alpha_i, \; j = 1, 2, \ldots .$$

Then $\beta_1, \beta_2, \ldots$ is log-concave.

(c) *For any positive integer n, the following sequences are log-concave.*

$$\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\ldots+\alpha_n$$

$$\alpha_n, \alpha_n+\alpha_{n-1}, \ldots, \alpha_n+\ldots+\alpha_1$$

Proof : (a) First suppose that $\alpha_i > 0$ for all i. Since $\alpha_1, \alpha_2, \ldots$ is logconcave,

$$\frac{\alpha_i}{\alpha_{i+1}} \geqslant \frac{\alpha_{i-1}}{\alpha_i}, \quad i = 2, 3, \ldots$$

Therefore
$$\frac{\alpha_k}{\alpha_{k+1}} \geqslant \frac{\alpha_{k-1}}{\alpha_k} \geqslant \ldots \geqslant \frac{\alpha_{j-1}}{\alpha_j}$$

and the result follows. The case of nonnegative $\alpha_1, \alpha_2, \ldots$ is then settled by a continuity argument.

(b) We will show that $\beta_2^2 \geqslant \beta_1 \beta_3$ and since $\alpha_{i-1}, \alpha_i, \ldots$ is also log-concave it will follow that $\beta_i^2 \geqslant \beta_{i-1} \beta_{i+1}$, $i = 2, 3, \ldots$ We have

$$\beta_2^2 - \beta_1 \beta_3 = \left(\sum_{i=2}^{k+1} \alpha_i\right)^2 - \left(\sum_{i=1}^{k} \alpha_i\right)\left(\sum_{i=3}^{k+2} \alpha_i\right)$$

$$= \alpha_2(\alpha_2+\ldots+\alpha_{k+1}) + \alpha_{k+1}(\alpha_3+\ldots+\alpha_{k+1})$$

$$- \alpha_1(\alpha_3+\ldots+\alpha_{k+2}) - \alpha_{k+2}(\alpha_2+\ldots+\alpha_k) \qquad \ldots \quad (9)$$

By (a), $\alpha_2 \alpha_j \geqslant \alpha_1 \alpha_{j+1}$, $j = 2, \ldots, k+1$ and $\alpha_{k+1}\alpha_j \geqslant \alpha_{k+2}\alpha_{j-1}$, $j = 3, \ldots, k+1$. Using this in (9) we get $\beta_2^2 \geqslant \beta_1 \beta_3$.

(c) For this part we may assume $\alpha_i = 0$, $i > n$. Then the sequence

$$\underbrace{0, \ldots, 0}_{(n-1) \text{ times}}, \alpha_1, \ldots, \alpha_n$$

is clearly log-concave and by applying (b) to this sequence (with $k = n$) we see that

$$\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\ldots+\alpha_n$$

is log-concave. The log-concavity of the second sequence now follows since $\alpha_n, \ldots, \alpha_1$ is log-concave.

We now state an important inequality for permanents due to A. D. Alexandroff who proved it in 1938 in a more general setting. The inequality was used around 1980 by Egorychev to prove the famous van der Waerden conjecture for permanents (see Minc, 1983).

Theorem 4.4 : *Let A be a nonnegative $n \times n$ matrix and let a_j denote the j-th column of A, $j = 1, 2, \ldots, n$. Then*

$$(per\ A)^2 \geqslant per(a_1, a_1, a_3, \ldots, a_n)\ per\ (a_2, a_2, a_3, \ldots, a_n)$$

Theorem 4.4 was used by Bapat (1988) to prove the log-concavity of certain sequences associated with order statistics. Here we obtain yet another result along similar lines using permanent representation of the distribution function.

Theorem 4.5 : *Let $X_1, \ldots, X_n$ be independent random variables with distribution functions $F_1, \ldots, F_n$ respectively. Then for any real y, the sequences $P(Y_r \leqslant y)$, $r = 1, \ldots, n$ and $P(Y_r > y)$, $r = 1, \ldots, n$ are log-concave.*

Proof : Define

$$\alpha_i = \frac{1}{i!\,(n-i)!} \text{per} \begin{bmatrix} F_1(y) & 1-F_1(y) \\ \vdots & \\ F_n(y) & 1-F_n(y) \end{bmatrix}_{\substack{\underbrace{\ }_{i}\ \underbrace{\ }_{n-i}}}, \quad i = 1, \ldots, n$$

By Theorem 4.4,

$$\left\{ \text{per} \begin{bmatrix} F_1(y) & 1-F_1(y) \\ \vdots & \\ F_n(y) & 1-F_n(y) \end{bmatrix}_{\substack{\underbrace{\ }_{i}\ \underbrace{\ }_{n-i}}} \right\}^2 \geqslant \text{per} \begin{bmatrix} F_1(y) & 1-F_1(y) \\ \vdots & \\ Fn(y) & 1-F_n(y) \end{bmatrix}_{\substack{\underbrace{\ }_{i-1}\ \underbrace{\ }_{n-i+1}}}$$

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Also, $\{i\,!\,(n-i)\,!\}^{-1}$, $i = 1, \ldots, n$ is easily verified to be log-concave and hence it follows that $\alpha_1, \ldots, \alpha_n$ is log-concave. Now the result is obtained by (c) of Theorem 4.3 and the representation (8).

5. Recurrence relations

Several recurrence relations involving order statistics are in the literature. A good number of these have been documented as exercises in David (1981). In this section we illustrate how permanents can be used to generalize some of the known recurrence relations.

The following notation will be used throughout this section. Let $X_1, \ldots, X_n$ be independent random variables with distribution functions $F_1, \ldots, F_n$ respectively. Let Y_r denote the r-th order statistic corresponding to

$X_1, \ldots, X_n$. For $1 \leqslant j \leqslant n$, let Y_r^j denote the r-th order statistic corresponding to $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n$. Let y be a real number. For convenience we will write F_j for $F_j(y)$ and F will denote the column vector $(F_1, \ldots, F_n)'$. Also, $\mathbf{1}$ will denote the column vector of all ones. As in Section 2, $A(i, j)$ will denote the matrix obtained by deleting row i and column j of A. We will denote by $A(j,.)$, the matrix obtained by deleting row j of A.

The next result generalizes a recurrence relation obtained by David and Shu (1978) who consider the set up in which the variables $X_1, \ldots, X_{n-1}$ are identically distributed and only X_n is supposed to have a possibly different distribution.

Theorem 5.1 : (a) $$P(Y_r \leqslant y) = \frac{1}{r!\,(n-r)!}\, per\Big[\underbrace{F,}_{r}\ \underbrace{\mathbf{1}-F}_{n-r}\Big] + P(Y_{r+1} \leqslant y)$$

(b) $$P(Y_r \leqslant y) = \frac{1}{n!}\binom{n-1}{r-1} per\Big[\underbrace{F,}_{r}\ \underbrace{\mathbf{1}-F}_{n-r}\Big] + \frac{1}{n}\sum_{j=1}^{n} P(Y_r^j \leqslant y)$$

Proof : (a) By Theorem 4.1,

$$P(Y_r \leqslant y) = \sum_{i=r}^{n} \frac{1}{i!(n-i)!}\ \text{per}\Big[\underbrace{F,}_{i}\ \underbrace{\mathbf{1}-F}_{n-i}\Big]$$

$$= \frac{1}{r!\,(n-r)!}\ \text{per}\Big[\underbrace{F,}_{r}\ \underbrace{\mathbf{1}-F}_{n-r}\Big] + \sum_{i=r+1}^{n} \frac{1}{i!(n-i)!}\ \text{per}\Big[\underbrace{F,}_{i}\ \underbrace{\mathbf{1}-F}_{n-i}\Big]$$

$$= \frac{1}{r!\,(n-r)!}\ \text{per}\Big[\underbrace{F,}_{r}\ \underbrace{\mathbf{1}-F}_{n-r}\Big] + P(Y_{r+1} \leqslant y)$$

(b) Using the relation

$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$$

and the first step in (a) we get

$$n\,!\,P(Y_r \leqslant y) = \sum_{i=r}^{n}\left\{\binom{n-1}{i-1} + \binom{n-1}{i}\right\} \text{per}\Big[\underbrace{F,}_{i}\ \underbrace{\mathbf{1}-F}_{n-i}\Big]$$

$$= \binom{n-1}{r-1}\ \text{per}\Big[\underbrace{F,}_{r}\ \underbrace{\mathbf{1}-F}_{n-r}\Big] + \sum_{i=r+1}^{n}\binom{n-1}{i-1}\ \text{per}\Big[\underbrace{F,}_{i}\ \underbrace{\mathbf{1}-F}_{n-i}\Big]$$

$$+ \sum_{i=r}^{n-1}\binom{n-1}{i}\ \text{per}\Big[\underbrace{F,}_{i}\ \underbrace{\mathbf{1}-F}_{n-i}\Big]$$

Using the Laplace expansion for permaments the sum of the last two terms in the above expression is seen to be

$$\sum_{i=r+1}^{n}\binom{n-1}{i-1}\sum_{j=1}^{n} F_j \operatorname{per}\Big[\underbrace{F,}\ \underbrace{1-F}_{n-i}\Big](j,1)$$

$$+\sum_{i=r}^{n-1}\binom{n-1}{i}\sum_{j=1}^{n}(1-F_j)\operatorname{per}\Big[\underbrace{F,}_{i}\ \underbrace{1-F}_{n-i}\Big](j,n)$$

$$=\sum_{i=r}^{n}\binom{n-1}{i}\sum_{j=1}^{n}\{F_j+1-F_j\}\operatorname{per}\Big[\underbrace{F,}_{i}\ \underbrace{1-F}_{n-i-1}\Big](j,.)$$

$$=(n-1)!\sum_{j=1}^{n}\sum_{i=r}^{n-1}\frac{1}{i!(n-i-1)!}\operatorname{per}\Big[\underbrace{F,}_{i}\ \underbrace{1-F}_{n-i-1}\Big](j,.)$$

$$=\frac{n!}{n}\sum_{j=1}^{n}P(Y_r^j\leqslant y)$$

and the proof is complete.

We now obtain a recurrence relation for densities of order statistics. With the same notation as before, suppose X_i is a continuous random variable with density f_i, $i=1,2,\ldots,n$. Let $g_r(g_r^j)$ denote the density of $Y_r(Y_r^j)$. Then we have the following.

Theorem 5.2 : $(n-r)\,g_r(y)+rg_{r+1}(y)=\sum_{j=1}^{n} g_r^j(y)$, $-\infty<y<\infty$.

Proof : Fix a real number y and for convenience, let f denote the column vector $(f_1(y),\ldots,f_n(y))'$.

By (1), we have

$$g_r(y)=\frac{1}{(r-1)!(n-r)!}\operatorname{per}\,[\underbrace{f,}_{1}\ \underbrace{F,}_{r-1}\ \underbrace{1-F}_{n-r}]$$

$$=\frac{1}{(r-1)!(n-r)!}\Big\{\operatorname{per}\,[\underbrace{f,}_{1}\ \underbrace{F,}_{r-1}\ \underbrace{1-F,}_{n-r-1}\ \underbrace{1}_{1}]-\operatorname{per}\,[\underbrace{f,}_{1}\ \underbrace{F}_{r}\ \underbrace{1-F}_{n-r-1}]\Big\}$$

$$=\frac{1}{(r-1)!(n-r)!}\Big\{\sum_{j=1}^{n}\operatorname{per}\Big[\underbrace{f,}_{1}\ \underbrace{F,}_{r-1}\ \underbrace{1-F}_{n-r-1}\Big](j,.)-\operatorname{per}\,[\underbrace{f,}_{1}\ \underbrace{F,}_{r}\ \underbrace{1-F}_{n-r-1}]\Big\} \quad \ldots (10)$$

Since $\quad g_{r+1}(y)=\dfrac{1}{r!(n-r-1)!}\operatorname{per}\Big[\underbrace{f,}_{1}\ \underbrace{F,}_{r}\ \underbrace{1-F}_{n-r-1}\Big]$

and $$g_r^j(y) = \frac{1}{(r-1)!\,(n-r-1)!} \text{ per}\left[\underbrace{f,}_{1}\ \underbrace{F,}_{r-1}\ \underbrace{1-F}_{n-r-1}\right](j,.),$$

the result follows by multiplying equation (10) by $n-r$ and by making a simple rearrangement of the terms.

In the statement of Theorem 5.2 multiply both sides by y^k, $k = 1, 2, \ldots$ and integrate with respect to y. Then we obtain a recurrence relation for moments which generalizes a known result for identically distributed $X_1, \ldots, X_n$ (David, 1981, p. 46).

6. Nonidentical exponential variables

In this section we consider the situation where $X_1, \ldots, X_n$ are independent random variables and X_i has the exponential distribution with parameter $\lambda_i > 0$ i.e., X_i has the density

$$f_i(x) = \lambda_i e^{-\lambda_i x},\ x > 0, \quad i = 1, \ldots, n$$

and the distribution function

$$F_i(x) = 1 - e^{-\lambda_i x},\ x > 0, \quad i = 1, \ldots, n.$$

As before, let $Y_1 \leqslant \ldots \leqslant Y_n$ denote the corresponding order statistics. We first derive the joint m.g.f. of $Y_1, \ldots, Y_n$. Then we obtain a formula for the m.g.f. of Y_r, $1 \leqslant r \leqslant n$, which is best suited to derive the moments of Y_r. we also obtain the m.g.f. of the range $Y_n - Y_1$. Results obtained in this section are a significant improvement over earlier work by Gross, Hunt and Odeh (1986) where the case of only one λ_i being different from the remaining is mainly considered.

In the remainder of this section the range of a summation and a product is from $1, \ldots, n$ unless specified otherwise.

Lemma 6.1 : *Let $X_i \sim$ exponential (λ_i), $i = 1, \ldots, n$ be independent. Then the m.g.f. of $Y_1, \ldots, Y_n$ exists in a sufficiently small neighbourhood of the origin and is given by*

$$\phi(t_1, \ldots, t_n) = \left(\prod_i \lambda_i\right) \sum_{\sigma \in S_n} \frac{1}{(\lambda_{\sigma(n)} - t_n)\,(\lambda_{\sigma(n)} + \lambda_{\sigma(n-1)} - t_n - t_{n-1}) \ldots (\Sigma\lambda_i - \Sigma t_i)} \quad \ldots (11)$$

Proof : The joint density of $Y_1, \ldots, Y_n$ is given by

$$f(y_1, \ldots, y_n) = \sum_{\sigma \in S_n} \prod_i \{\lambda_{\sigma(i)} e^{-\lambda_{\sigma(i)} y_i}\},\ 0 < y_1 < \ldots < y_n$$

Hence

$$\phi(t_1, \ldots, t_n) = E(e^{\Sigma t_i Y_i}) = (\pi \lambda_i) \int_{y_1 < \cdots < y_n} \int e^{\Sigma t_i y_i} \sum_{\sigma \in S_n} e^{-\Sigma \lambda_{\sigma(i)} y_i} \pi dy_i$$

$$= (\pi\lambda_i) \sum_{\sigma \in S_n} \int_0^\infty \ldots \int_{y_{n-2}}^\infty \int_{y_{n-1}}^\infty e^{-\Sigma(\lambda_{\sigma(i)}-t_i)y_i} dy_n, \ldots, dy_1.$$

The result follows after a routine integration.

It is possible to obtain the m.g.f. of Y_r by setting $t_i = 0$, $i \neq r$ in (11). However, we now obtain another formula for the m.g.f. of Y_r which can readily be used to calculate moments. The following notation will be used. Let $N = \{1, \ldots, n\}$. If $S \subset N$ then S' will denote the complement of S in N while $|S|$ will denote the cardinality of S. If $S \subset N$, define

$$\lambda(S) = \sum_{i \in S} \lambda_i$$

Theorem 6.2 : *Let $X_i \sim$ exponential (λ_i), $i = 1, \ldots, n$ be independent and let r be fixed, $1 \leqslant r \leqslant n$. Then the m.g.f. of Y_r is given, for sufficiently small t, by*

$$\psi(t) = \sum_{k=n-r+1}^{n} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{|S|=k} \frac{\lambda(S)}{\lambda(S)-t} \qquad \ldots \quad (12)$$

Proof : The result will be proved by induction on n. The result is trivial if $n = 1$. Suppose the result is true for $n-1$. If $r = 1$, then since Y_1 is Exponential $(\Sigma\lambda_i)$, (12) clearly holds. So suppose $r > 1$. Let S_n^j denote the set of permutations of the elements of $N^j = \{1, \ldots, j-1, j+1, \ldots, n\}$

The m.g.f. of Y_r is obtained, by setting $t_i = 0$ for all $i \neq r$ in (11), as

$$\psi(t) = (\pi\lambda_i) \sum_{\sigma \in S_n} \frac{1}{\lambda_{\sigma(n)}(\lambda_{\sigma(n)}+\lambda_{\sigma(n-1)}) \ldots (\lambda_{\sigma(n)}+ \ldots +\lambda_{\sigma(r)}-t) \ldots (\Sigma\lambda_i-t)}$$

By induction hypothesis, we can write

$$\psi(t) = \frac{(\pi\lambda_i)}{\Sigma\lambda_i - t} \sum_{j=1}^{n} \frac{1}{\underset{i \neq j}{(\pi \ \lambda_i)}} \sum_{k=n-r+1}^{n-1} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{\substack{S \subset N^j \\ |S|=k}} \frac{\lambda(S)}{\lambda(S)-t}$$

$$= \sum_{j=1}^{n} \lambda_j \sum_{k=n-r+1}^{n-1} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{\substack{N \subset S^j \\ |S|=k}} \frac{\lambda(S)}{\lambda(S')} \left\{ \frac{1}{\lambda(S)-t} - \frac{1}{\Sigma\lambda_i - t} \right\} \qquad \ldots \quad (13)$$

Consider $S \subset N$. If $|S| = k < n$, the coefficient of $\frac{\lambda(S)}{\lambda(S)-t}$ in (13) as well as in (12) is seen to be

$$(-1)^{k-n+r-1} \binom{k-1}{n-r}$$

Now we show that the coefficient of $(\Sigma\lambda_i - t)^{-1}$ is also identical in (13) and (12).

The coefficient of $(\Sigma\lambda_i - t)^{-1}$ in (13) is

$$\sum_{k=n-r+1}^{n-1} (-1)^{k-n+r} \binom{k-1}{n-r} \sum_{|S|=k} \lambda(S)$$

$$= (\Sigma\lambda_i) \sum_{k=n-r+1}^{n-1} (-1)^{k-n+r} \binom{k-1}{n-r} \binom{n-1}{k-1}$$

$$= \frac{(\Sigma\lambda_i)}{(n-r)\,!} \sum_{z=0}^{r-z} (-1)^{z+1} \frac{(n-1)\,!}{z!(r-z-1)!}$$

$$= (\Sigma\lambda_i)\ (-1)^{r-1} \binom{n-1}{n-r} \quad \ldots \quad (14)$$

where the last step follows by an application of the Binomial theorem. The coefficient of $(\Sigma\lambda_i - t)^{-1}$ in (12) is also given by (14) and the proof is complete.

From (12), we obtain by differentiation,

$$E(Y_r) = \sum_{k=n-r+1}^{n} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{|S|=k} \frac{1}{\lambda(S)} \quad \ldots \quad (15)$$

$$E(Y_r^2) = \sum_{k=n-r+1}^{n} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{|S|=k} \frac{2}{\lambda(S)^2} \quad \ldots \quad (16)$$

From (15), (16) we can get an expression for the variance of Y_r. In the special case when $\lambda_1, \ldots, \lambda_{n-1}$ are all equal, a different formula for the variance of Y_n has been obtained by Gross, Hunt and Odeh (1986).

If $X_i \sim$ exponential (1), $i = 1, \ldots, n$ are independent then it is well known (David 1981, p. 49) that

$$E(Y_r) = \sum_{k=n-r+1}^{n} \frac{1}{k} \quad \ldots \quad (17)$$

Note that if $\lambda_i = 1$, $i = 1, \ldots, n$, then $\lambda(S) = |S|$ for any $S \subset N$ and since there are $\binom{n}{k}$ subsets of N of cardinality k,

$$\sum_{|S|=k} \frac{1}{\lambda(S)} = \binom{n}{k} \frac{1}{k}$$

This can be substituted in (15) to get another expression for $E(Y_r)$. Equating the expression obtained to (17) we get the following binomial identity·

$$\sum_{k=n-r+1}^{n} (-1)^{k-n+r-1} \binom{k-1}{n-r} \binom{n}{k} \frac{1}{k} = \sum_{k=n-r+1}^{n} \frac{1}{k} \qquad \ldots \quad (18)$$

The case $r = n$ of the identity (18) has been mentioned by Feller (1968, p. 65), but we have not been able to locate the general case in the literature.

Theorem 6.3 : *Let $X_i \sim$ exponential (λ_i), $i = 1, \ldots, n$ be independent and let r be fixed, $1 < r \leqslant n$. Then the m.g.f. of $Y_r - Y_1$ is given by*

$$\psi(t) = \frac{1}{\Sigma\lambda_i} \sum_{k=n-r+1}^{n-1} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{|S|=k} \frac{\lambda(S)\lambda(S')}{\lambda(S)-t} \qquad \ldots \quad (19)$$

Proof : The m.g.f. of $Y_r - Y_1$ is obtained by setting $t_r = 1$, $t_1 = -1$, $t_i = 0$, $i \neq 1$, r in (11) as

$$\psi(t) = (\pi\lambda_i) \sum_{\sigma \in S_n} \frac{1}{\lambda_{\sigma(n)}(\lambda_{\sigma(n)}+\lambda_{\sigma(n-1)}) \ldots (\lambda_{\sigma(n)} + \ldots + \lambda_{\sigma(r)} - t) \ldots (\Sigma\lambda_i)}$$

$$= \frac{\pi\lambda_i}{\Sigma\lambda_i} \sum_{j=1}^{n} \frac{1}{\pi_{i \neq j} \lambda_i} h_j(t),$$

where, by (11), $h_j(t)$ is the m.g.f. of the $(r-1)$-th order statistic for the random variables $X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n$.

By Theorem 6.2,

$$\psi(t) = \frac{\pi\lambda_i}{\Sigma\lambda_i} \sum_{j=1}^{n} \frac{1}{\pi_{i \neq j} \lambda_i} \sum_{k=n-r+1}^{n-1} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{\substack{S \subset N^j \\ |S|=k}} \frac{\lambda(S)}{\lambda(S)-t}$$

$$= \frac{1}{\Sigma\lambda_i} \sum_{k=n-r+1}^{n-1} (-1)^{k-n+r-1} \binom{k-1}{n-r} \sum_{|S|=k} \frac{\lambda(S)\lambda(S')}{\lambda(S)-t}.$$

That completes the proof of the theorem.

Setting $r = n$ in Theorem 6.3 we obtain the m.g.f. of the range $Y_n - Y_1$ from (19) as

$$\psi(t) = \frac{1}{\Sigma\lambda_i} \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{|S|=k} \frac{\lambda(S)\lambda(S')}{\lambda(S)-t} \qquad \ldots \quad (20)$$

The raw moments of the range can easily be obtained from (20) by differentiation. Thus we have

$$E(Y_n - Y_1) = \frac{1}{\Sigma\lambda_i} \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{|S|=k} \frac{\lambda(S')}{\lambda(S)}$$

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