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THIRD-ORDER COMPARISON OF UNBIASED TESTS : A SIMPLE FORMULA FOR THE POWER DIFFERENCE IN THE ONE-PARAMETER CASE

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SUMMARY. This paper develops a simple and readily applicable formula for third order comparison of tests in the one-parameter case and extends some earlier results in this area. This leads to a simple proof of the intuitively plausible result on the optimality of the locally most powerful unbiased test in a very large class of tests. As a consequence, the optimality of Rao's test is proved in a class much larger than that in Chandra and Mukerjee (1984, *Commun. Statist. —Theor. Meth.*, **13**, 1507-1515). A new identity for the multivariate normal distribution turns out to be very helpful.

1. Introduction

For a sequence $\{X_n\}$, $n \geqslant 1$, of i.i.d. random variables with a common density $f(x, \theta)$, $\theta \in \Theta$, an open subset of $\mathcal{R}^1$, consider the problem of testing $H_0 : \theta = \theta_0$ ($\in \Theta$) against the alternative $\theta \neq \theta_0$. In such a setting, Chandra and Joshi (1983) made a third-order comparison of the likelihood ratio, Rao's and Wald's tests (see Rao, 1973, 417–418) ; for the last two tests Chandra and Joshi considered modified versions which are locally unbiased up to $o(n^{-1})$ under contiguous alternatives $\theta_0+\delta n^{-\frac{1}{2}}$, and showed that for sufficiently small but reasonable common size of the tests, the power of Rao's test is higher than those of the other two tests provided δ is also small. Subsequently, Chandra and Mukerjee (1984, 1985) established the optimality of Rao's test, in the same sense, within a larger class of tests. Furthermore, it was demonstrated by Mukerjee and Chandra (1987) that, under a third-order comparison, the locally most powerful unbiased (LMPU) test is only marginally superior to Rao's test.

The objective in this paper is to develop a simple formula using which the third-order powers of any two tests in a large family can be easily compared. As a by-product, in section 5, the optimality of Rao's test is established in a very large class of tests (much wider than that in Chandra and Mukerjee, 1984 ; 1985). Also, a simple proof of the intuitively expected

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optimality of the LMPU test within a large class of tests is available and this has been presented in section 5. In passing, it is observed that identity of power up to the first order implies that up to the second order—a phenomenon analogous to that of first-order efficiency implying second-order efficiency for one-sided tests.

Recently, Kumon and Amari (1983, 1985) and Amari (1985) developed an elegant differential geometric approach for the third-order comparison of tests in multiparameter curved exponential families. Although the present work deals with the one-parameter situation, the derivations require no assumption regarding curved exponentiality. Also, compared to the differential geometric approach, the final results here can be presented in a rather elementary form which possibly makes applications to particular situations simpler. There is some similarity between our main results and Theorem 2 in Pfanzagl and Wefelmeyer (1978) for one-sided tests. However, our results do not seem to follow from Pfanzagl and Wefelmeyer (1978) without considerable algebra (vide remark following Theorem 1). For further literature references in this area, see Chandra and Joshi (1983) and Kumon and Amari (1985).

Earlier, Peers (1971) compared the likelihood ratio, Rao's and Walds' tests without correcting the last two for bias up to $o(n^{-1})$ and, consequently, the tests differed in terms of their second-order power. Our results differ from those of Peers because we are comparing modified versions of the tests (in particular, of Rao's and Wald's tests), the objective being to make them unbiased up to $o(n^{-1})$. Chandra and Joshi (1983), who initiated such studies following a suggestion of J. K. Ghosh, discuss why this may be a more satisfactory way of comparing tests for two-sided alternatives.

2 Notation, preliminaries and main results

Let $X_1, X_2, \ldots,$ be i.i.d. random variables each with p.d.f. $f(x, \theta)$, where $\theta \in \mathcal{R}^1$, or an open subset thereof. For the problem of testing $H_0 : \theta = \theta_0$ against $\theta \neq \theta_0$, let

$$H_1 = n^{-1/2} \sum_{j=1}^{n} \frac{d}{d\theta} \log f(X_j, \theta_0), \qquad \ldots \quad (2.1)$$

and consider a family $\mathcal{F}$ of test procedures as described below. For contiguous alternatives $\theta_n = \theta_0 + \delta n^{-1/2}$ and for every test procedure in $\mathcal{F}$, a set A_n with $P_{\theta_n}(A_n) = 1 + o(n^{-1})$, uniformly over compact subsets of δ, can be obtained (cf. Chandra and Joshi, 1983) such that over A_n the test procedure is given by a critical region of the form :

$$\left.\begin{array}{ll} & W_n > z+n^{-1/2}b_1+n^{-1}c_1+o(n^{-1}) \\ \text{or} & W_n^* < -z+n^{-1/2}b_2+n^{-1}c_2+o(n^{-1}) \end{array}\right\} \quad \ldots \quad (2.2)$$

where z is the upper $(\alpha/2)$-point of a normal deviate and

$$\left.\begin{array}{l} W_n = H_1I^{-1/2}+n^{-1/2}Q_1+n^{-1}Q_2,\; W_n^* = H_1I^{-1/2}+n^{-1/2}Q_3+n^{-1}Q_4, \\ I = E_{\theta_0}\left[\left(\frac{d}{d\theta}\log f(X,\theta_0)\right)^2\right], \end{array}\right\} \quad \ldots \quad (2.3a)$$

$$\left.\begin{array}{l} Q_v = g_v(Q_{v1}, Q_{v2}, \ldots, Q_{vr_v}), \\ Q_{vs} = n^{-1/2}\sum\limits_{j=1}^{n}(q_{vs}(X_j)-\beta_{vs}(\theta_0))\;(1 \leqslant s \leqslant r_v,\; 1 \leqslant v \leqslant 4). \end{array}\right\} \quad \ldots \quad (2.3b)$$

Here $g_v(.)$ are polynomials and $q_{vs}(.)$ are such that

$$E_\theta(q_{vs}(X_j)) = \beta_{vs}(\theta),\; \forall\; \theta\; (1 \leqslant s \leqslant r_v,\; 1 \leqslant v \leqslant 4), \quad \ldots \quad (2.4)$$

which are assumed to exist.

In the above, b_1, b_2, c_1, c_2 are constants, free from n, to be so chosen that the test procedure has size $\alpha+o(n^{-1})$ and is locally unbiased up to $o(n^{-1})$.

It may be made explicit that Q_1, Q_2, Q_3, Q_4, and also the q_{vs}'s depend on the particular test procedure in $\mathcal{F}$ under consideration. The family $\mathcal{F}$ is very rich and includes the likelihood ratio, LMPU and unbiased (up to $o(n^{-1})$) versions of Rao's and Wald's test procedures. Let

$$\left.\begin{array}{l} Q_{vs}^* = n^{-1/2}\sum\limits_{j=1}^{n}(q_{vs}(X_j)-\beta_{vs}(\theta_n))\;(1 \leqslant s \leqslant r_v,\; 1 \leqslant v \leqslant 4), \\ H_2 = n^{-1/2}\left(\sum\limits_{j=1}^{n}\frac{d^2}{d\theta^2}\log f(X_j,\theta_0)+nI\right), \\ H_2^* = n^{-1/2}\left(\sum\limits_{j=1}^{n}\frac{d^2}{d\theta^2}\log f(X_j,\theta_0)-nE_{\theta_n}\left(\frac{d^2}{d\theta^2}\log f(X_j,\theta_0)\right)\right) \\ l_u = E_{\theta_n}\left(\frac{d^u}{d\theta^u}\log f(X,\theta_n)\right), \\ \Delta_u = n^{-1/2}\left(\sum\limits_{j=1}^{n}\frac{d^u}{d\theta^u}\log f(X_j,\theta_n)-nl_u\right),\quad (u = 1, 2, \ldots) \end{array}\right\} \quad \ldots \quad (2.5a)$$

$$\left.\begin{aligned} h^{(u)} &= \frac{d^u}{d\theta^u} \log f(X, \theta_0), \\ L_{ujk} &= E_{\theta_0} [(h^{(1)})^u (h^{(2)})^j (h^{(3)})^k], \\ L_{uj} &= L_{uj0},\ L_u = L_{u0}\ (u, j, k = 0, 1, 2, \ldots). \end{aligned}\right\} \quad \ldots \quad (2.5b)$$

It may be remarked that among the $\{l_u\}$ and the $\{L_{ujk}\}$ only those which have been used later are assumed to exist.

The following assumptions are made to avoid justification of various formal manipulations. Since we are working with polynomials in sample means, they should not be hard to verify.

Assumption 1 : (i) Under θ_n, for each v, adequately many moments exist for $(\Delta_1, Q^*_{v1}, \ldots, Q^*_{vr_v})'$ and its characteristic function admits an expansion up to $o(n^{-\frac{1}{2}})$.

(ii) Under θ_n, for each v, adequately many moments exist for $(\Delta_1, H^*_2, Q^*_{v1}, \ldots, Q^*_{vr_v})'$ and its limiting distribution is multivariate normal.

Assumption 2 : For each v, $E_{\theta_n}(Q_v)$, $E_{\theta_n}(H_2 Q_v)$ and $E_{\theta_n}(Q_v^2)$ exist and the following can be calculated up to stated orders of approximation, using the expansion for characteristic function considered in Assumption 1 above :

$$E_{\theta_n}(Q_v) = C_v(\delta)+n^{-1/2}M_v(\delta)+o(n^{-1/2}) \ (1 \leqslant v \leqslant 4),$$

$$E_{\theta_n}(Q_{v-4}H_2) = C_v(\delta)+O(n^{-1/2}) \ (5 \leqslant v \leqslant 8),$$

$$E_{\theta_n}(Q^2_{v-8}) = C_v(\delta)+O(n^{-1/2}) \ (9 \leqslant v \leqslant 12),$$

where the $C_v(\delta)$ and $M_v(\delta)$ are free from n.

Since each $g_v(.)$ is a polynomial, it follows from (2.3b) that for each v, $C_v(.)$, $M_v(.)$ are polynomials (see also the proof of Lemma 1 below). For each v, let $C_v^{(r)}(\delta)$ be the rth derivative of $C_v(\delta)$ with respect to δ ($r = 0, 1, 2, \ldots$). Also, for $r = 0, 1, 2, \ldots$, let $J_r(x)$ be the Hermite polynomial of degree r in x. Note that for each v and for $u = z$ or $-z$, the expression $\{\Sigma\, C_v^{(r)}(\delta)\, I^{-r/2} J_r(u-\delta I^{1/2})/r\,!\}$ is essentially a finite sum as $C_v(.)$ is a polynomial. Hence noting that $dJ_r(x)/dx = rJ_{r-1}(x)$ ($r = 1, 2, \ldots$), it follows readily through term-by-term differentiation that

$$\frac{d}{d\delta}\{\sum_r C_v^{(r)}(\delta)\, I^{-r/2} J_r(u-\delta I^{1/2})/r\,!\} = 0, \quad \ldots \quad (2.6)$$

so that one may write

$$\sum_r C_v^{(r)}(\delta)\, I^{-r/2} J_r(u-\delta I^{1/2})/r\,! = C_v^{(00)}(u) \;(1 \leqslant v \leqslant 12,\; u = z \text{ or } -z), \;\ldots \quad (2.7)$$

where $\{C_v^{(00)}(u)\}$ are free from δ (and also from n). We are now in a position to state one of the main theorems of the paper which is as follows :

Theorem 1 : *If b_1, b_2, c_1, c_2 are chosen subject to the conditions of size and local unbiasedness up to $o(n^{-1})$, then the (local) power function of a test procedure of the form (2.2) is given by*

$$P = P_0 + n^{-1/2} P_1 + n^{-1} P_2 + o(n^{-1}),$$

where P_0, P_1, P_2 are free from n and

(i) $$P_0 = \int_{z-\delta I^{1/2}}^{\infty} \phi(y)dy + \int_{-\infty}^{-z-\delta I^{1/2}} \phi(y)dy,$$

$\phi(y)$ being the standard normal density,

(ii) *P_1 is free from Q_1, Q_2, Q_3, Q_4 and hence the same for all test procedures in the family $\mathcal{F}$,*

(iii) $$P_2 = V(\delta) + \phi(z-\delta I^{1/2})\{-(2zI^{1/2})^{-1}(\gamma_{11}+\gamma_{21}) + \gamma_{11}\delta + \gamma_{12}\delta^2\}$$
$$+\, \phi(z+\delta I^{1/2})\{(2zI^{1/2})^{-1}(\gamma_{11}+\gamma_{21}) + \gamma_{21}\delta + \gamma_{22}\delta^2\},$$

with $V(\delta)$ free from Q_1, Q_2, Q_3, Q_4 and

$$\gamma_{11} = \frac{1}{2} I^{1/2}\{-C_9^{(00)}(z) + (C_1^{(00)}(z))^2\}.$$

$$\gamma_{21} = \frac{1}{2} I^{1/2}\{C_{11}^{(00)}(-z) - (C_3^{(00)}(-z))^2\}$$

$$\gamma_{12} = \frac{1}{2} - L_{11} I^{-1/2} z C_1^{(00)}(z) + \frac{1}{2} C_5^{(00)}(z),$$

$$\gamma_{22} = \frac{1}{2} - L_{11} I^{-1/2} z C_3^{(00)}(-z) - \frac{1}{2} C_7^{(00)}(-z)$$

Remark : From the expression for P_2, it follows that the right- and left-tailed "sizes" up to $o(n^{-1})$ may vary over the family of tests of the form (2.2) since $\gamma_{11}+\gamma_{21}$ may vary from one test to another. Moreover, examples show that this holds even if one restricts to a subfamily of tests based only on the first two derivatives of the log-likelihood. Hence a possible application of Theorem 2 in Pfanzagl and Wefelmeyer (1978) to the present context will also require explicit evaluation of $\gamma_{11}+\gamma_{21}$ and for that almost the same computations as in the present paper are required. Thus it appears that an application of the findings in Pfanzagl and Wefelmeyer (1978) cannot possibly lead to a considerable reduction in the algebra.

In passing we note that (ii) in Theorem 1 gives a formal demonstration of the phenomenon that identity of power up to first order implies that up to second order (cf. Pfanzagl, 1979; Bickel, Chibishov and van Zwet, 1981) for two-sided tests in the one-parameter case. Also, (iii) in Theorem 1 shows that P_2 does not depend on Q_2 and Q_4.

A further reduction is obtained as follows. With reference to (2.3b), let for $v = 1, 3, \tilde{\Sigma}_v$ denote the dispersion matrix of $\left(\frac{d^2}{d\theta^2} \log f(X, \theta_0),\right.$ $\left.q_{v1}(X), \ldots, q_{vr_v}(X)\right)'$ under θ_0 and $\boldsymbol{\beta}^{(v)} = I^{-1/2} \; (\beta'_{v1}(\theta_0), \ldots, \beta'_{vr_v}(\theta_0))'$, where $\beta'_{vs}(\theta_0)$ is the first derivative of $\beta_{vs}(\theta)$ at θ_0 $(1 \leqslant s \leqslant r_v)$. Also, let $\beta_0 = I^{-1/2}L_{11}$ and $\tilde{\boldsymbol{\beta}}^{(v)} = (\beta_0, \boldsymbol{\beta}^{(v)\prime})'$. Then the following holds :

Theorem 2 : *P_2, and in Theorem 1, can be expressed as*

$$P_2 = V(\delta)+\delta^2 \phi(z)R+O(\delta^3),$$

where $V(\delta)$ *is free from* Q_1, Q_2, Q_3, Q_4,

$$R = \tfrac{1}{2}[cov(Z_0, g_1(\mathbf{Z}^{(1)}+z\boldsymbol{\beta}^{(1)}))-Iz\, var(g_1(\mathbf{Z}^{(1)}+z\boldsymbol{\beta}^{(1)}))$$
$$-cov(Z_0, g_3(\mathbf{Z}^{(3)}-z\boldsymbol{\beta}^{(3)}))-Iz\, var(g_3(\mathbf{Z}^{(3)}-z\boldsymbol{\beta}^{(3)}))],$$

$g_1(.)$, $g_3(.)$ *are as in* (2.3b), *and for* $v = 1, 3$, *the* (r_v+1)*-component random vector* $(Z_0, \mathbf{Z}^{(v)\prime})'$ *is multivariate normal with a zero mean vector and a dispersion matrix* $\tilde{\Sigma}_v-\tilde{\boldsymbol{\beta}}^{(v)}\tilde{\boldsymbol{\beta}}^{(v)\prime}$.

It may be remarked that the explicit evaluation of R, for a given test procedure, is quite straightforward. Hence Theorem 2 serves as a simple but powerful tool for third-order comparison of tests in the family $\mathcal{F}$. An illustrative example in this connexion will be presented later in Section 5 where applications of Theorem 2 in studying the performance of the LMPU and Rao's tests have been considered. The proofs of Theorems 1, 2 have been presented in Sections 3 and 4. The following lemmas, which have been proved in the Appendix, will be helpful in the sequel.

Lemma 1 : (*a*) $E_{\theta_n}(Q_v e^{\xi\Delta_1}) = e^{I\xi^2/2}\Big[C_v(\delta+\xi)+n^{-1/2}\Big\{M_v(\delta+\xi)$

$$-\frac{\xi^3}{2}C_{v+4}(\delta+\xi)-\Big(\frac{\delta\xi^2}{2}+\frac{\xi^3}{6}\Big)L_{001}C_v(\delta+\xi)\Big\}\Big]+o(n^{-1/2}).$$

(*b*) $E_{\theta_n}(Q_v^2 e^{\xi\Delta_1}) = e^{I\xi^2/2}C_{v+8}(\delta+\xi)+O(n^{-1/2}),$

(*c*) $E_{\theta_n}(Q_v H_2 e^{\xi\Delta_1}) = e^{I\xi^2/2}C_{v+4}(\delta+\xi)+O(n^{-1/2}),$

$(1 \leqslant v \leqslant 4)$ *where* $\xi = it$, $i^2 = -1$.

Lemma 2. *Let* $\mathbf{Z}^* = (Z_1^*, ..., Z_r^*)'$ *be r-variate normal with a null mean vector and a dispersion matrix* Σ, $g(.)$ *be a polynomial in r variates and* $\boldsymbol{\mu} = (\mu_1, ..., \mu_r)'$, $\boldsymbol{\beta} = (\beta_1, ..., \beta_r)'$ *be fixed vectors such that* $\Sigma-\boldsymbol{\beta\beta}'$ *is non-negative definite. Then*

$$E\Big[\exp\Big\{-\frac{1}{2}\Big(\sum_{s=1}^{r}\beta_s D_s\Big)^2\Big\}g(\mathbf{Z}^*+\boldsymbol{\mu})\Big] = E(g(\mathbf{Z}+\boldsymbol{\mu})),$$

where D_s *is the operator of partial diffreentiation with respect to* $Z_s^*(1 \leqslant s \leqslant r)$ *and* $\mathbf{Z} =(Z_1, ..., Z_r)'$ *is r-variate normal with a null mean vector and a dispersion matrix* $\Sigma-\boldsymbol{\beta\beta}'$.

3. Derivation of the power function

Let

$$\begin{aligned} T_n &= W_n-n^{-1/2}b_1-n^{-1}c_1-\delta I^{1/2} \\ &= H_1 I^{-1/2}+n^{-1/2}(Q_1-b_1)+n^{-1}(Q_2-c_1)-\delta I^{1/2}, \qquad \text{... (3.1a)} \end{aligned}$$

$$\begin{aligned} T_n^* &= W_n^*-n^{-1/2}b_2-n^{-1}c_2-\delta I^{1/2} \\ &= H_1 I^{-1/2}+n^{-1/2}(Q_3-b_2)+n^{-1}(Q_4-c_2)-\delta I^{1/2}. \qquad \text{... (3.1b)} \end{aligned}$$

In this section, Theorem 1 will be proved considering formal Edgeworth expansions for the distributions of T_n and T_n^* under θ_n.

Note that by a formal expansion (cf. Chandra and Joshi, 1983),

$$\left.\begin{aligned} H_1 I^{-1/2} &= \Delta I^{-1/2}+\delta I^{1/2}+n^{-1/2}m_1+n^{-1}m_2+o(n^{-1}), \\ \text{where} \quad \Delta &= \Delta_1-n^{-1/2}\delta\Delta_2+n^{-1}\frac{\delta^2}{2}\Delta_3, \end{aligned}\right\} \qquad \text{... (3.2)}$$

and m_1, m_2 are free from n but may involve δ.

Assumption 3 : The characteristic function of Δ, under θ_n, admits an expansion up to $o(n^{-1})$ and for that adequately many moments exist.

By Assumption 3, let

$$E_{\theta_n}(e^{\xi\Delta}) = [1+n^{-1/2}F_1(\xi, \delta)+n^{-1}F_2(\xi, \delta)]e^{I\xi^2/2}+o(n^{-1}), \quad \ldots \quad (3.3)$$

where $\xi = it$ and F_1 (.), F_2(.) are polynoimials in ξ and δ which are free from n(cf. Chandra and Joshi, 1983). Then the following lemma, which has been proved in the Appendix, holds :

Lemma 3 : $E_{\theta_n}(e^{\xi T_n}) = X(\xi, \delta)\, e^{\xi^2/2}+o(n^{-1})$,

where $X(\xi, \delta) = 1+n^{-1/2}[(m_1-b_1)\xi+\xi C_1(\delta+I^{-1/2}\xi)+F_1(I^{-1/2}\xi, \delta)]$

$$+n^{-1}[(m_2-c_1)\xi+(m_1-b_1)\xi F_1(I^{-1/2}\xi, \delta)+\frac{1}{2}(m_1-b_1)^2\xi^2$$

$$+\xi M_1(\delta+I^{-1/2}\xi)+\xi C_2(\delta+I^{-1/2}\xi)+\frac{1}{2}\xi^2C_3(\delta+I^{-1/2}\xi)$$

$$-\xi^2\left(\frac{1}{2}I^{-1}\xi+I^{-1/2}\delta\right) C_5(\delta+I^{-1/2}\xi)$$

$$-L_{001}\xi^3\left(\frac{1}{2}\delta I^{-1}+\frac{1}{6}I^{-3/2}\xi\right) C_1(\delta+I^{-1/2}\xi)+F_2(I^{-1/2}\xi, \delta)$$

$$+\delta^2 I^{-1/2}\xi^2 L_{11}C_1(\delta+I^{-1/2}\xi)+(m_1-b_1)\xi^2C_1(\delta+I^{-1/2}\xi)].$$

Hence through a formal Edgeworth expansion,

$$P_{\theta_n}(T_n \leqslant x) = \int_{-\infty}^{x} X(-D, \delta)\, \phi(y)\, dy+o(n^{-1}), \quad \ldots \quad (3.4)$$

where D is the operator of differentiation with respect to y. A similar expression may be obtained considering T_n^*, instead of T_n, proceeding along the same line. It is assumed that the formal Edgeworth expansions considered here are valid. Now by Lemma 3, (3.1a, b,), (3.4) and the analogous expression for T_n^*, the (local) power function of the test procedure (2.2) is given by

$$P = P_{\theta_n}(T_n > z-\delta I^{1/2})+P_{\theta_n}(T_n^* < -z-\delta I^{1/2})$$

$$= \int_{e}^{\infty}\phi(y)\,dy+\int_{-\infty}^{e^*}\phi(y)\,dy+n^{-1/2}[(m_1-b_1)\phi(e)$$

$$+\int_{e}^{\infty}F_1(-I^{-1/2}D, \delta)\phi(y)dy-(m_1-b_2)\phi(e^*)+\int_{-\infty}^{e^*}F_1(-I^{-1/2}D, \delta)\phi(y)dy$$

$$+\{C_1(\delta-I^{-1/2}D)\phi(y)\}_{y=e}-\{C_3(\delta-I^{-1/2}D)\phi(y)\}_{y=e^*}]$$

$$+n^{-1}[(m_2-c_1)\phi(e)-(m_1-b_1)\int_e^\infty DF_1(-I^{-1/2}D,\delta)\phi(y)dy$$

$$+\int_e^\infty F_2(-I^{-1/2}D,\delta)\phi(y)dy+\Big\{-\frac{1}{2}(m_1-b_1)^2D+M_1(\delta-I^{-1/2}D)$$

$$+C_2(\delta-I^{-1/2}D)-\frac{1}{2}DC_2(\delta-I^{-1/2}D)+D\left(-\frac{1}{2}I^{-1}D+I^{-1/2}\delta\right)C_6(\delta-I^{-1/2}D)$$

$$-L_{001}D^2\left(\frac{1}{2}\delta I^{-1}-\frac{1}{6}I^{-3/2}D\right)C_1(\delta-I^{-1/2}D)$$

$$-\delta^2I^{-1/2}L_{11}DC_1(\delta-I^{-1/2}D)-(m_1-b_1)DC_1(\delta-I^{-1/2}D)\Big\}\phi(y)|_{y=e}$$

$$-(m_2-c_2)\phi(e^*)-(m_1-b_2)\int_{-\infty}^{e^*}DF_1(-I^{-1/2}D,\delta)\phi(y)dy$$

$$+\int_{-\infty}^{e^*}F_2(-I^{-1/2}D,\delta)\phi(y)dy+\Big\{\frac{1}{2}(m_1-b_2)^2D-M_2(\delta-I^{-1/2}D)$$

$$-C_4(\delta-I^{-1/2}D)+\frac{1}{2}DC_{11}(\delta-I^{-1/2}D)-D\Big(-\frac{1}{2}I^{-1}D+I^{-1/2}\delta\Big)C_7(\delta-I^{-1/2}D\Big)$$

$$+L_{001}D^2\left(\frac{1}{2}\delta I^{-1}-\frac{1}{6}I^{-3/2}D\right)C_3(\delta-I^{-1/2}D)$$

$$+\delta^2I^{-1/2}L_{11}DC_3(\delta-I^{-1/2}D)+(m_1-b_2)DC_3(\delta-I^{-1/2}D)\Big\}\phi(y)|_{y=e^*}]$$

$$+o(n^{-1}), \quad \ldots \quad (3.5)$$

where $e=z-\delta I^{1/2}$, $e^*=-z-\delta I^{1/2}$. In deriving (3.5), it has been noted that the functions $C_v(.)$, $M_v(.)$ are polynomials and hence for $s=0,1,2,3$, $D^sC_v(\delta-I^{-1/2}D)\phi(y)$, $D^sM_v(\delta-I^{-1/2}D)\phi(y)$ tend to zero as $y\to\pm\infty$.

In order to simplify (3.5) further, note that for $s=0,1,2,3$; $1\leqslant v\leqslant 12$, $u=z$ or $-z$,

$$D^sC_v(\delta-I^{-1/2}D)\phi(y)|_{y=u-\delta I^{1/2}}=(-1)^s\sum_r C_v^{(r)}(\delta)\,I^{-r/2}\,J_{r+s}(u-\delta I^{1/2})\phi(u-\delta I^{1/2})/r! \quad \ldots \quad (3.6)$$

where $C_v^{(r)}(.)$ and $J_{r+s}(.)$ are as defined in the preceding section. The fact that $C_v(.)$ is a polynomial implies that the right-hand member of (3.6) is a finite sum and hence, analogously to (2.6), term-by-term differentiation yields

$$\frac{d^{s+1}}{d\delta^{s+1}}\left\{\sum_r C_v^{(r)}(\delta)\,I^{-r/2}J_{r+s}(u-\delta I^{1/2})/r!\right\}=0,$$

so that (analogously to (2.7))

$$\sum_r C_g^{(r)}(\delta)\, I^{-r/2}\, J_{r+s}(u-\delta I^{1/2})/r\,! = \sum_{j=0}^{s} C_g^{(sj)}(u)\,\delta^j, \quad \ldots \quad (3.7)$$

where $C_g^{(sj)}(u)$ are free from δ (and also from n). Similarly,

$$\left.\begin{aligned} M_1(\delta-I^{-1/2}D)\phi(y)\,|_{y=e} &= M_1^{(00)}(z)\phi(e), \\ M_3(\delta-I^{-1/2}D)\phi(y)\,|_{y=e^*} &= M_3^{(00)}(-z)\,\phi(e^*), \end{aligned}\right\} \quad \ldots \quad (3.8)$$

where $M_1^{(00)}(z)$, $M_3^{(00)}(-z)$ are free from δ and n. By (3.5)—(3.8),

$$P = P_0+n^{-1/2}\,P_1+n^{-1}P_2+o(n^{-1}), \quad \ldots \quad (3.9)$$

where P_0, P_1, P_2 are free from n (but involve δ), P_0 is as in Theorem 1, and P_1, P_2 are given by

$$P_1 = \int_{z-\delta I^{1/2}}^{\infty} F_1(-I^{-1/2}D,\ \delta)\,\phi(y)dy+\int_{-\infty}^{-z-\delta I^{1/2}} F_1(-I^{-1/2}D,\delta)\,\phi(y)dy$$

$$+(m_1-b_1+C_1^{(00)}(z))\,\phi(z-\delta I^{1/2})-(m_1-b_2+C_2^{(00)}(-z))\,\phi(z+\delta I^{1/2}), \quad \ldots \quad (3.10)$$

$$P_2 = \int_{z-\delta I^{1/2}}^{\infty} F_2(-I^{-1/2}D,\delta)\,\phi(y)dy+\int_{-\infty}^{-z-\delta I^{1/2}} F_2(-I^{-1/2}D,\delta)\,\phi(y)dy$$

$$-(m_1-b_1)\int_{-z-\delta I^{1/2}} DF_1(-I^{-1/2}D,\delta)\,\phi(y)dy$$

$$-(m_1-b_2)\int_{-\infty}^{-z-\delta I^{1/2}} DF_1(-I^{-1/2}D,\delta)\,\phi(y)dy$$

$$+\phi(z-\delta I^{1/2})\Big[(m_2-c_1)+\frac{1}{2}(m_1-b_1)^2(z-\delta I^{1/2})+M_1^{(00)}(z)$$

$$+C_2^{(00)}(z)+\frac{1}{2}\{C_2^{(10)}(z)+C_2^{(11)}(z)\delta\}$$

$$-\frac{1}{2}I^{-1}\{C_5^{(20)}(z)+C_5^{(21)}(z)\delta+C_5^{(22)}(z)\delta^2\}$$

$$-I^{-1/2}\delta\{C_6^{(10)}(z)+C_6^{(11)}(z)\delta\}$$

$$-\tfrac{1}{2}L_{001}\delta I^{-1}\{C_1^{(20)}(z)+C_1^{(21)}(z)\delta+C_1^{(22)}(z)\delta^2\}$$

$$-\frac{1}{6}\,L_{001}I^{-3/2}\{C_1^{(30)}(z)+C_1^{(31)}(z)\delta+C_1^{(32)}(z)\delta^2+C_1^{(33)}(z)\delta^3\}$$

$$+\delta^2 I^{-1/2} L_{11}\{C_1^{(10)}(z)+C_1^{(11)}(z)\delta\}$$

$$+(m_1-b_1)\{C_1^{(10)}(z)+C_1^{(11)}(z)\delta\}\Big]$$

$$+\phi(z+\delta I^{1/2})\Big[-(m_2-c_2)+\frac{1}{2}(m_1-b_2)^2(z+\delta I^{1/2})-M_3^{(00)}(-z)$$

$$-C_4^{(00)}(-z)-\frac{1}{2}\{C_{11}^{(10)}(-z)+C_{11}^{(11)}(-z)\delta\}$$

$$+\frac{1}{2}I^{-1}\{C_7^{(20)}(-z)+C_7^{(21)}(-z)\delta+C_7^{(22)}(-z)\delta^2\}$$

$$+I^{-1/2}\delta\{C_7^{(10)}(-z)+C_7^{(11)}(-z)\delta\}$$

$$+\frac{1}{2}L_{001}\delta I^{-1}\{C_3^{(20)}(-z)+C_3^{(21)}(-z)\delta+C_3^{(22)}(-z)\delta^2\}$$

$$+\frac{1}{6}L_{001}I^{-3/2}\{C_5^{(30)}(-z)+C_5^{(31)}(-z)\delta+C_5^{(32)}(-z)\delta^2+C_5^{(33)}(-z)\delta^3\}$$

$$-\delta^2 I^{-1/2}L_{11}\{C_3^{(10)}(-z)+C_3^{(11)}(-z)\delta\}$$

$$-(m_1-b_2)\{C_3^{(10)}(-z)+C_3^{(11)}(-z)\delta\}\Big] \quad \ldots \quad (3.11)$$

Writing $P_1 = P_1(\delta)$, $P_2 = P_2(\delta)$, from the conditions of size and local unbiasedness (up to $o(n^{-1})$) it follows that b_1, b_2, c_1, c_2 must satisfy

$$P_1(0) = 0,\ P_1'(0) = 0, \quad \ldots \quad (3.12a)$$

$$P_2(0) = 0,\ P_2'(0) = 0, \quad \ldots \quad (3.12b)$$

where primes denote differentiation with respect to δ. Since (vide Chandra and Joshi, 1983)

$$m_1 = \frac{1}{2}(L_{11}+L_3)I^{-1/2}\delta^2,\ F_1(\xi,\delta) = \frac{1}{2}\delta L_3\xi^2+\frac{1}{6}L_3\xi^3, \quad \ldots \quad (3.13)$$

and therefore,

$$\int_{z-\delta I^{1/2}}^{\infty} F_1(-I^{-1/2}D,\delta)\,\phi(y)dy$$

$$= L_3\left[\frac{1}{2}\delta I^{-1}(z-\delta I^{1/2})+\frac{1}{6}I^{-3/2}\{(z-\delta I^{1/2})^2-1\}\right]\phi(z-\delta I^{1/2}),$$

$$-\int_{-\infty}^{-z-\delta I^{1/2}} F_1(-I^{-1/2}D,\delta)\,\phi(y)dy = L_3\left[\frac{1}{2}\delta I^{-1}(z+\delta I^{1/2})\right.$$

$$\left.-\frac{1}{6}I^{-3/2}\{(z+\delta I^{1/2})^2-1\}\right]\phi(z+\delta I^{1/2}).$$

it follows from (3.10), (3.12a), after some simplification, that

$$b_1 = \frac{1}{6} L_3 I^{-3/2} z^2 + C_1^{(00)}(z),\ b_2 = \frac{1}{6} L_3 I^{-3/2} z^2 + C_2^{(00)}(-z). \qquad \ldots \quad (3.14)$$

Hence,

$$P_1 = \frac{1}{6} [\phi(z - \delta I^{1/2}) \{-L_3 I^{-3/2} + \delta z L_3 I^{-1} + \delta^2(3L_{11} + L_3) I^{-1/2}\}$$
$$+ \phi(z + \delta I^{1/2}) \{L_3 I^{-3/2} + \delta z L_3 I^{-1} - \delta^2(3L_{11} + L_3) I^{-1/2}\}], \qquad \ldots \quad (3.15)$$

which agrees with the corresponding expression in Chandra and Joshi (1983). Observe that the expression for P_1, as in (3.15), does not depend on Q_1, Q_2, Q_3, Q_4. This proves (*ii*) in Theorem 1.

By (3.13),

$$\int_{z-\delta I^{1/2}}^{\infty} DF_1(-I^{-1/2}D, \delta)\, \phi(y) dy$$

$$= -L_3 \left[\frac{1}{2} \delta I^{-1} \{(z - \delta I^{1/2})^2 - 1\} + \frac{1}{6} I^{-3/2} \{(z - \delta I^{1/2})^3 - 3(z - \delta I^{1/2})\} \right] \phi(z - \delta I^{1/2})$$
$$\ldots \quad (3.16a)$$

$$\int_{-\infty}^{-z-\delta I^{1/2}} DF_1(-I^{-1/2}D, \delta)\, \phi(y) dy$$

$$= L_3 \left[\frac{1}{2} \delta I^{-1} \{(z + \delta I^{1/2})^2 - 1\} - \frac{1}{6} I^{-3/2} \{(z + \delta I^{1/2})^3 - 3(z + \delta I^{1/2})\} \right] \phi(z + \delta I^{1/2}),$$
$$\ldots \quad (3.16b)$$

Further, since for each v, $C_v(.)$ is a polynomial and hence the left-hand member of (3.7) is a finite sum ,one obtains, by term-by-term differentiation, from (3.7) the recursion relation

$$C_v^{(sj)}(u) = \left\{ \frac{d^j}{d\delta^j} \sum_r C_v^{(r)}(\delta) I^{-r/2} J_{r+s}(u - \delta I^{1/2})/r\,! \right\}_{\delta=0} /j\,!$$
$$= s j^{-1} I^{1/2} C_v^{(s-1, j-1)}(u), \qquad \ldots \quad (3.17)$$

where $1 \leqslant v \leqslant 12$,, $1 \leqslant j \leqslant s$ and $u = z$ or $-z$.

From (3.11), (3.12b), making use of (3.13), (3.14), (3.16a, b), and proceeding as in the derivation of (3.14), detailed expressions for c_1, c_2 may be obtained. These expressions are analogous to (3.14) but more involved (and will not be used explicitly in the rest of the work) and hence not shown here. Now, with c_1, c_2 so determined, (iii) in Theorem 1 follows from (3.11), (3.13), (3.14), (3.16a, b), (3.17) after some algebra. Note that $V(\delta)$, as in Theorem 1, satisfies $V(0) = 0$, $V'(0) = 0$.

4. Further reduction

In this section, Theorem 2 will be proved. For $v = 1, 3$, let Σ_v denote the dispersion matrix of $(q_{v1}(X), \ldots, q_{vr_v}(X))'$ under θ_0. Then by (2.3b), (2.5), Assumption 1 and the definition of $C_v(\delta)$ (see also the proof of Lemma 1),

$$C_v(\delta) = E[g_v(Z^*_{v1}+\delta\beta'_{v1}(\theta_0), \ldots, Z^*_{vr_v}+\delta\beta'_{vr_v}(\theta_0))],$$

where $Z^*_{v1}, \ldots, Z^*_{vr_v}$ are jointly normal with null mean vector and dispersion matrix $\Sigma_v(v = 1, 3)$. Considering a multivariate Taylor's expansion for the polynomial $g_v(.)$, it follows that

$$C_v^{(j)}(0) = E\left[\left\{\sum_{s=1}^{r_v} \beta'_{vs}(\theta_0)D_{vs}\right\}^j g_v(Z^*_{v1}, \ldots, Z^*_{vr_v})\right] (j = 0, 1, 2, \ldots), \quad \ldots \quad (4.1)$$

where D_{vs} is the operator of partial differentiation with respect to $Z^*_{vs}(1 \leqslant s \leqslant r_v;\ v = 1, 3)$. Note that by (2.7) with $\delta = 0$,

$$C_v^{(00)}(u) = \sum_j C_v^{(j)}(0)\, I^{-j/2}\, J_j(u)/j\,!, \quad \ldots \quad (4.2)$$

for $v = 1, 3$; $u = z$ or $-z$.

Since the summation in the right-hand member of (4.2) is essentially finite, it follows from (4.1) and a well-known result on the generating function of Hermite polynomials that

$$\begin{aligned} C_v^{(00)}(u) &= E\Big[\sum_j (J_j(u)/j\,!)\left\{I^{-1/2}\sum_{s=1}^{r_v}\beta'_{vs}(\theta_0)D_{vs}\right\}^j g_v(Z^*_{v1}, \ldots, Z^*_{vr_v})\Big] \\ &= E\Big[\exp\Big\{I^{-1/2}u\sum_{s=1}^{r_v}\beta'_{vs}(\theta_0)D_{vs} - \frac{1}{2}I^{-1}\Big(\sum_{s=1}^{r_v}\beta'_{vs}(\theta_0)D_{vs}\Big)^2\Big\} \\ &\qquad g_v(Z^*_{v1}, \ldots, Z^*_{vr_v})\Big] \\ &= E\Big[\exp\Big\{-\frac{1}{2}I^{-1}\Big(\sum_{s=1}^{r_v}\beta'_{vs}(\theta_0)D_{vs}\Big)^2\Big\} \\ &\qquad g_v(Z^*_{v1}+I^{-1/2}u\beta'_{v1}(\theta_0), \ldots, Z^*_{vr_v}+I^{-1/2}u\beta'_{vr_v}(\theta_0)\Big], \end{aligned} \quad \ldots \quad (4.3)$$

where $u = z$ or $-z$. Defining $\boldsymbol{\beta}^{(v)}$ as in section 2, it follows from the multi-parameter Rao-Cramer inequality that $\Sigma_v - \boldsymbol{\beta}^{(v)}\boldsymbol{\beta}^{(v)'}$ is non-negative definite. Hence by (4.3) and Lemma 2,

$$C_v^{(00)}(u) = E(g_v(\mathbf{Z}^{(v)} + u\boldsymbol{\beta}^{(v)}))\ (v = 1, 3\ ;\ u = z, -z), \quad \ldots \quad (4.4a)$$

where $\mathbf{Z}^{(v)} = (Z_{v1}, \ldots, Z_{vr_v})'$ is r_v-variate normal with a null mean vector and a dispersion matrix $\Sigma_v - \boldsymbol{\beta}^{(v)}\boldsymbol{\beta}^{(v)'}$. Similarly,

$$C_{v+8}^{(00)}(u) = E(g_v^2(\mathbf{Z}^{(v)} + u\boldsymbol{\beta}^{(v)}))\ (v = 1, 3\ ;\ u = z, -z). \quad \ldots \quad (4.4b)$$

Also, denoting the dispersion matrix, under θ_0, of $\left(\frac{d^2}{d\theta^2} \log f(X, \theta_0), q_{v1}(X), \ldots, q_{vr_v}(X)\right)'$ by

$$\widetilde{\Sigma}_v = \begin{pmatrix} \sigma_{00} & \boldsymbol{\sigma}'_{(1v)} \\ \boldsymbol{\sigma}_{(1v)} & \Sigma_v \end{pmatrix},$$

and assuming a simple regularity condition, it may be seen that

$$C_{v+4}^{(00)}(u) = E[(Z_0 + u\beta_0)g_v(\mathbf{Z}^{(v)} + u\boldsymbol{\beta}^{(v)})]\ (v = 1, 3, u = z, -z), \quad \ldots \quad (4.4c)$$

where $\beta_0 = I^{-1/2}L_{11}$, and $(Z_0, \mathbf{Z}^{(v)'})'$ is distributed as (r_v+1)-variate normal with null mean vector and a dispersion matrix $\widetilde{\Sigma}_v - \widetilde{\boldsymbol{\beta}}^{(v)}\widetilde{\boldsymbol{\beta}}^{(v)'}$, with $\widetilde{\boldsymbol{\beta}}^{(v)} = (\beta_0, \boldsymbol{\beta}^{(v)'})'$. In particular, observe that

$$\text{var}(Z_0) = \sigma_{00} - \beta_0^2 = L_{02} - I^2 - I^{-1}L_{11}^2 = I^{-1}|M_{\theta_0}|, \quad \ldots \quad (4.5)$$

$|M_{\theta_0}|$ being the determinant of the dispersion matrix, under θ_0, of the first two derivatives of the loglikelihood (based on a single observation).

Since by Theorem 1(iii), $P_2 = V(\delta) + \delta^3\phi(z)R + O(\delta^3)$, where $V(\delta)$ is free from Q_1, Q_2, Q_3, Q_4, and

$$R = \tfrac{1}{2}Iz[\{C_1^{(00)}(z)\}^2 + \{C_3^{(00)}(-z)\}^2 - C_9^{(00)}(z) - C_{11}^{(00)}(-z)]$$
$$+ \tfrac{1}{2}[C_5^{(00)}(z) - C_7^{(00)}(-z) - L_{11}I^{-1/2}z\{C_1^{(00)}(z) + C_3^{(00)}(-z)\}],$$

Theorem 2 now follows from (4.4a-c).

5. Performance of the LMPU and Rao's tests

Intuitively, it is felt that the LMPU test should be optimal in the class $\mathscr{F}$ in the sense of maximizing P_2 for small δ. This will now be formally demonstrated by showing that R is maximum, over the family $\mathscr{F}$, for the LMPU test.

From Mukerjee and Chandra (1987), for the LMPU test, $Q_1 = -Q_2 = (2Iz)^{-1}H_2$, and

$$R = (4I^2z)^{-1}|M_{\theta_0}| = R(\text{LMPU}), \quad \dots \quad (5.1)$$

say. Considering the value of R for any other test procedure in $\mathscr{F}$, it follows from (4.5), (5.1) and Theorem 2 that

$$\begin{aligned} R(\text{LMPU})-R &= \tfrac{1}{2}\,[(4Iz)^{-1}\,\text{var}(Z_0)+Iz\,\text{var}(g_1(\mathbf{Z}^{(1)}+z\boldsymbol{\beta}^{(1)})) \\ &\quad -\text{cov}(Z_0, g_1(\mathbf{Z}^{(1)}+z\boldsymbol{\beta}^{(1)}))+(4Iz)^{-1}\text{var}(Z_0) \\ &\quad +Iz\,\text{var}(g_3(\mathbf{Z}^{(3)}-z\boldsymbol{\beta}^{(3)}))+\text{cov}(Z_0, g_3(\mathbf{Z}^{(3)}-z\boldsymbol{\beta}^{(3)}))] \\ &\geqslant \tfrac{1}{2}\,[\{\text{var}(Z_0)\text{var}(g_1(\mathbf{Z}^{(1)}+z\boldsymbol{\beta}^{(1)}))\}^{1/2}-\text{cov}(Z_0, g_1(\mathbf{Z}^{(1)}+z\boldsymbol{\beta}^{(1)})) \\ &\quad +\{\text{var}(Z_0)\text{var}(g_3(\mathbf{Z}^{(3)}-z\boldsymbol{\beta}^{(3)}))\}^{1/2}+\text{cov}(Z_0, g_3(\mathbf{Z}^{(3)}-z\boldsymbol{\beta}^{(3)}))] \geqslant 0, \end{aligned} \quad \dots \quad (5.2)$$

for every z (> 0). The inequality (5.2) demonstrates the optimality of the LMPU test in the family $\mathscr{F}$ under the stated assumptions. Following the line of Chandra and Mukerjee (1985) and using Theorem 2, it is straightforward to derive a detailed expression, in terms of Hermite polynomials, for the deficiency of any test procedure in $\mathscr{F}$ relative to the LMPU test.

In view of the findings of Chandra and Joshi (1983) and Chandra and Mukerjee (1984, 1985) on the optimality of Rao's test, it is interesting to examine the performance of Rao's test as a member of $\mathscr{F}$. Note that for Rao's test, $Q_1 = Q_2 = Q_3 = Q_4 = 0$ and hence by Theorem 2, $R = 0 = R$ (Rao), say.

Let $\mathscr{F}_0$ be a subclass of $\mathscr{F}$ consisting of the test procedures for which R is zero or a polynomial in z. Since by (4.5), (5.1), (5.2),

$$R(\text{LMPU})-R = (4Iz)^{-1}\,\text{var}\,(Z_0)-R \geqslant 0,$$

for every positive z, it follows that for every test in $\mathscr{F}_0$, the coefficient of the highest power of z in R must be non-positive. Thus given any test procedure in $\mathscr{F}_0$, it is possible to find a critical value (depending on the particular test under consideration) such that $R(\text{Rao})-R = -R \geqslant 0$, whenever z exceeds that critical value, and, in this sense, Rao's test is optimal in $\mathscr{F}_0$. By (2.3b) and Theorem 2, the class $\mathscr{F}_0$ includes, in particular, the test procedures for which Q_1 and Q_2 are free from z. This observation extends the earlier results on the optimality of Rao's test (vide Chandra and Joshi, 1983, Chandra and Mukerjee, 1984, 1985) to a much more general setting.

Rao's test, however, is not optimal in the entire class $\mathscr{F}$ since it is dominated by the LMPU test (see e.g., Mukerjee and Chandra, 1984) Also, it is not optimal in the subclass $\mathscr{F}_1$ of $\mathscr{F}$ consisting of the test procedures for which $Q_1 = Q_3$, $Q_2 = Q_4$ (note that the LMPU test does not belong to $\mathscr{F}_1$). The following example serves as an illustration.

Example : Consider a test procedure in $\mathscr{F}_1$ for which

$$Q_1 = Q_3 = (2z^2 I^{3/2})^{-1} H_1 H_2,\ Q_2 = Q_4 = 0.$$

By (4.5) and Theorem 2, for this test procedure,

$$\begin{aligned} R = \frac{1}{2} [&\operatorname{cov} \{Z_0, (2z^2 I^{3/2})^{-1} (Z_1 + z\beta_1)(Z_0 + z\beta_0)\} \\ &- Iz \operatorname{var} \{(2z^2 I^{3/2})^{-1} (Z_1 + z\beta_1)(Z_0 + z\beta_0)\} \\ &- \operatorname{cov} \{Z_0, (2z^2 I^{3/2})^{-1} (Z_1 - z\beta_1)(Z_0 - z\beta_0)\} \\ &- Iz \operatorname{var} \{(2z^2 I^{3/2})^{-1} (Z_1 - z\beta_1)(Z_0 - z\beta_0)\}], \end{aligned}$$

where $\beta_0 = I^{-1/2} L_{11}$, $\beta_1 = I^{1/2}$, and the joint distribution of Z_0, Z_1 is bivariate normal with a null mean vector and a dispersion matrix

$$\begin{pmatrix} I^{-1} |M_{\theta_0}| & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence it may be seen that $R = (4I^2 z)^{-1} |M_{\theta_0}| > R$ (Rao), for every $z (> 0)$ provided $|M_{\theta_0}| > 0$.

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Appendix

Proof of Lemma 1: For notational simplicity, consider the situation $r_v = 1$ (the proof for general r_v is similar but the notation is more involved) and let, without loss of generality, $Q_v = Q_{v1}^r$, where r is a non-negative integer and Q_{v1} is given by (2.3b). Since

$$Q_{v1} = Q_{v1}^* + \delta\beta_{v1}'(\theta_0) + n^{-1/2}\delta^2\beta_{v1}''(\theta_0)/2 + o(n^{-1/2}),$$

where Q_{v1}^* is as in (2.5), one obtains

$$Q_v = \sum_{j=0}^{r} \binom{r}{j} (Q_{v1}^*)^j (\delta\beta_{v1}'(\theta_0))^{r-j}$$

$$+ n^{-1/2} \frac{r\delta^2}{2} \beta_{v1}''(\theta_0) \sum_{j=0}^{r-1} \binom{r-1}{j} (Q_{v1}^*)^j (\delta\beta_{v1}'(\theta_0))^{r-j-1} + o(n^{-1/2}). \quad \ldots \quad (A.1)$$

Note that $E_{\theta_n}(Q_{v1}^*) = 0 = E_{\theta_n}(\Delta_1)$. Also let

$$E_{\theta_n}((Q_{v1}^*)^j \Delta_1^{2-j}) = \eta_{j2-j} + n^{-1/2}\rho_{j2-j} + o(n^{-1/2}) \; (j = 0, 1, 2),$$

$$E_{\theta_n}((Q_{v1}^*)^j \Delta_1^{3-j}) = n^{-1/2}\rho_{j3-j} + o(n^{-1/2}) \; (j = 0, 1, 2, 3).$$

the η's and ρ's being free from n. Then the joint characteristic function of Q_{v1}^* and Δ_1, upto $o(n^{-1/2})$, is given by

$$\left[1 + n^{-1/2}\Sigma\Sigma\rho_{j_1j_2}\xi_1^{j_1}\xi_2^{j_2}/(j_1!\,j_2!)\right] \exp((\eta_{20}\xi_1^2 + 2\eta_{11}\xi_1\xi_2 + \eta_{02}\xi_2^2)/2) + o(n^{-1/2}),$$

where $\xi_1 = it_1$, $\xi_2 = it_2$, $i^2 = -1$, and the summation within squared brackets extends over non-negative integral j_1, j_2 satisfying $j_1 + j_2 = 2$ or 3.

Hence with $\xi = it$,

$$E_{\theta_n}((Q_{v1}^*)^j e^{\xi\Delta_1}) = E(Y_1^j e^{\xi Y_2}) + n^{-1/2}\Big[\Big(\frac{1}{2}\rho_{02}\xi^2 + \frac{1}{6}\rho_{03}\xi^3\Big)E(Y_1^j e^{\xi Y_2})$$

$$+ j\Big(\rho_{11}\xi + \frac{1}{2}\rho_{12}\xi^2\Big) E(Y_1^{j-1}e^{\xi Y_2})$$

$$+ \binom{j}{2}(\rho_{20} + \rho_{21}\xi)E(Y_1^{j-2}e^{\xi Y_2}) + \binom{j}{3}\rho_{30}E(Y_1^{j-3}e^{\xi Y_2})\Big]$$

$$+ o(n^{-1/2}), \quad \ldots \quad (A.2)$$

where the joint distribution of Y_1, Y_2 is bivariate normal with zero means and a dispersion matrix

$$\begin{pmatrix} \eta_{20} & \eta_{11} \\ \eta_{11} & \eta_{02} \end{pmatrix}$$

Since $$E(Y_1^u e^{\xi Y_2}) = E((Y_1+\xi\eta_{11})^u) \exp(\eta_{02}\xi^2/2),$$

for every non-negative integral u, and by standard regularity conditions $\eta_{11} = \beta'_{v1}(\theta_0)$, $\eta_{02} = I$, it follows from (A.1), (A.2), that

$$\left.\begin{aligned} E_{\theta_n}(Q_v e^{\xi A_1}) = \Big[\nu_0+n^{-1/2}\Big\{\Big(\frac{1}{2}\,\delta^2\beta''_{v1}(\theta_0)\nu_1+\frac{1}{2}\,\rho_{20}\nu_2+\frac{1}{6}\,\rho_{30}\nu_3\Big) \\ +\Big(\rho_{11}\nu_1+\frac{1}{2}\,\rho_{21}\nu_2\Big)\xi+\Big(\frac{1}{2}\,\rho_{02}\nu_0+\frac{1}{2}\,\rho_{12}\nu_1\Big)\xi^2 \\ +\frac{1}{6}\rho_{03}\nu_0\xi^3\Big\}\Big]\exp(I\xi^2/2)+o(n^{-1/2}), \end{aligned}\right\} \quad \ldots \quad (A.3)$$

where

$$\nu_u = r(r-1)\ldots(r-u+1)E[\{Y_1+(\delta+\xi)\beta'_{v1}(\theta_0)\}^{r-u}] \; (u = 0, 1, 2, 3)$$

By Assumption 2, the relation (A.3) with $\xi = 0$, yields

$$\left.\begin{aligned} C_v(\delta) = \nu_0^*,\; M_v(\delta) = \frac{1}{2}\,\delta^2\beta''_{v1}(\theta_0)\nu_1^*+\frac{1}{2}\,\rho_{20}\nu_2^*+\frac{1}{6}\,\rho_{30}\nu_3^*, \\ \text{where} \\ \nu_u^* = r(r-1)\ldots(r-u+1)E((Y_1+\delta\beta'_{v1}(\theta_0))^{r-u}) \; (u = 0, 1, 2, 3). \end{aligned}\right\} \quad \ldots \quad (A.4)$$

Similarly, by Assumption 2 and some standard regularity conditions,

$$C_{v+1}(\delta) = \delta L_{11}\nu_0^*+\psi\nu_1^*, \quad \ldots \quad (A.5)$$

where

$$\psi = \text{cov}_{\theta_0}\Big(q_{v1}(X),\; \frac{d^2}{d\theta^2}\,\log f(X, \theta_0)\Big).$$

Let $\lambda_{v1}(\theta) = \text{var}_\theta\;(q_{v1}(X))$. Then using regularity conditions and some findings in Chandra and Joshi (1983),

$$\left.\begin{aligned} \rho_{20} = \delta\lambda_{v1}(\theta_0),\; \rho_{11} = \delta\beta'_{v1}(\theta_0),\; \rho_{02} = -\delta(L_{11}+L_{001}),\; \rho_{21} = \lambda'_{v1}(\theta_0), \\ \rho_{12} = \beta''_{v1}(\theta_0)-\psi,\; \rho_{03} = L_3,\; L_{001}+3L_{11}+L_3 = 0, \end{aligned}\right\} \quad \ldots \quad (A.6)$$

and ρ_{30} is free from δ. The proof of (a) now follows from (A.3)—(A.6). The proofs of (b) and (c) are similar.

Proof of Lemma 2 : For $l_1, l_2, \ldots, l_r \geq 0$, let $g_{l_1 \ldots l_r}(Z^*) = D_1^{l_1} \ldots D_r^{l_r}$ $g(Z^*)$. Also, for any non-negative integer s, let $\mathcal{S}^s$ be the set of all possible choices of non-negative integers $l_1, \ldots, l_r$ such that $l_1+\ldots+l_r = s$. Define

$\widetilde{Z}$ as a standard normal variate distributed independently of Z and observe that Z^* and $Z+\widetilde{Z}\beta$ are identically distributed. Hence

$$E\left[\exp\left\{-\frac{1}{2}\left(\sum_{s=1}^{r}\beta_s D_s\right)^2\right\} g(Z^*+\mu)\right]$$

$$= E\left[\sum_j \frac{1}{j\,!}(-1/2)^j \sum_{l_1,\ldots,l_r \in \mathcal{S}_{2j}} \frac{(2j)\,!}{l_1\,!\cdots l_r\,!}\,\beta_1^{l_1}\ldots\beta_r^{l_r}\, g_{l_1\ldots l_r}(Z+\mu+\widetilde{Z}\beta)\right]$$

$$= E\left[\sum_j \frac{1}{j\,!}(-1/2)^j \sum_{l_1,\ldots,l_r \in \mathcal{S}_{2j}} \frac{(2j)\,!}{l_1\,!\ldots l_r\,!}\,\beta_1^{l_1}\ldots\beta_r^{l_r}\right.$$

$$\left.\times\sum_s \frac{(\widetilde{Z})^s}{s\,!}\sum_{\zeta_1,\ldots,\zeta_r\in\mathcal{S}_s}\frac{s\,!}{\zeta_1\,!\ldots\zeta_r\,!}\,\beta_1^{\zeta_1}\ldots\beta_r^{\zeta_r}\, g_{l_1+\zeta_1\ldots l_r+\zeta_r}(Z+\mu)\right], \qquad \ldots \text{(A.7)}$$

applying a Taylor's expansion for $g_{l_1\ldots l_r}(Z+\mu+\widetilde{Z}\beta)$. All the summations in (A.7), including those over j and s, are finite, since $g(.)$ is a polynomial. Hence noting that Z and $\widetilde{Z}$ are independently distributed and that

$$E(\widetilde{Z}^s) = \frac{s\,!}{(s/2)\,!\,2^{s/2}} \text{ for even } s, = 0 \text{ for odd } s,$$

it follows from (A.7) by a change in the order of summation that

$$E\left[\exp\left\{-\frac{1}{2}\left(\sum_{s=1}^{r}\beta_s D_s\right)^2\right\} g(Z^*+\mu)\right]$$

$$= E\left[\sum_j\sum_s \frac{(-1)^j}{j\,!\,s\,!\,2^{j+s}} \sum_{l_1,\ldots,l_r\in\mathcal{S}_{2j}}\ \sum_{\zeta_1,\ldots,\zeta_r\in\mathcal{S}_{2s}} \frac{(2j)\,!\,(2s)\,!}{l_1\,!\ldots l_r\,!\,\zeta_1\,!\ldots\zeta_r\,!}\right.$$

$$\left.\times\beta_1^{l_1+\zeta_1}\ldots\beta_r^{l_r+\zeta_r} g_{l_1+\zeta_1\ldots l_r+\zeta_r}(Z+\mu)\right]$$

$$= E\left[\sum_{\substack{u_1,\ldots,u_r\\ u_1+\ldots+u_r \text{ even}}} \frac{\beta_1^{u_1}\ldots\beta_r^{u_r}}{2^{(u_1+\cdots+u_r)/2}}\, g_{u_1\ldots u_r}(Z+\mu)\right.$$

$$\times\left\{\sum_{\substack{l_1=0\\ l_1+\ldots+l_r \text{ even}}}^{u_1}\ldots\sum_{l_r=0}^{u_r} \frac{(l_1+\ldots+l_r)\,!\,(u_1+\ldots+u_r-l_1-\ldots-l_r)\,!}{l_1\,!\ldots l_r\,!\,(u_1-l_1)\,!\ldots(u_r-l_r)\,!}\right.$$

$$\left.\left.\times\frac{(-1)^{(l_1+\cdots+l_r)/2}}{\left(\frac{1}{2}(l_1+\ldots+l_r)\right)!\left(\frac{1}{2}(u_1+\ldots+u_r-l_1-\ldots-l_r)\right)!}\right\}\right]. \qquad \ldots \text{(A.8)}$$

Now for fixed $u_1, \ldots, u_r$, such that $u_1+\ldots+u_r = 2w$, say, the term within second brackets in (A.8) equals

$$\sum_{j=0}^{w} \sum_{\substack{l_1,\ldots,l_r \in \mathcal{S}_{2j} \\ l_i \leqslant u_i \forall i}} \frac{(2j)!\,(2w-2j)!}{l_1!\ldots l_r!\,(u_1-l_1)!\ldots(u_r-l_r)!} \; \frac{(-1)^j}{j!\,(w-j)!}$$

$$= \sum_{j=0}^{w} \frac{(2j)!\,(2w-2j)!\,(-1)^j}{j!\,(w-j)!\,u_1!\ldots u_r!} \left\{ \sum_{\substack{l_1,\ldots,l_r \in \mathcal{S}_{2j} \\ l_i \leqslant u_i \forall i}} \binom{u_1}{l_1} \cdots \binom{u_r}{l_r} \right\}$$

$$= \sum_{j=0}^{w} \frac{(2j)!\,(2w-2j)!\,(-1)^j}{j!\,(w-j)!\,u_1!\ldots u_r!} \binom{u_1+\ldots+u_r}{2j}$$

$$= \frac{(2w)!}{u_1!\ldots u_r!} \sum_{j=0}^{w} \frac{(-1)^j}{j!(w-j)!} \qquad \ldots \quad \text{(A.9)}$$

The right-hand member of (A.9) vanishes unless $w = 0$. Hence the lemma follows from (A.8).

Proof of Lemma 3 : By (3.1), (3.2),

$$e^{\xi T_n} = \Big[1+n^{-1/2}\xi(Q_1-b_1+m_1)+n^{-1}\xi(Q_2-c_1+m_2)+\frac{1}{2}\, n^{-1}\xi^2(Q_1-b_1+m_1)^2 \Big]$$
$$\times \exp(\xi\Delta I^{-1/2})+o(n^{-1}). \qquad \ldots \quad \text{(A.10)}$$

Since by a formal expansion $\Delta_2 = H_2-\delta L_{11}+O(n^{-1})$, it follows from (3.2) and Lemma 1 that

$$E_{\theta_n}[Q_1 \exp(\xi\Delta I^{-1/2})] = E_{\theta_n}[Q_1(1-n^{-1/2}\xi I^{-1/2}\,\delta\Delta_2)\exp\,(\xi\Delta_1 I^{-1/2})]+o(n^{-1/2})$$
$$= E_{\theta_n}[Q_1 \exp(\xi\Delta_1 I^{-1/2})]+n^{-1/2}\xi I^{-1/2}\delta E_{\theta_n}[Q_1(\delta L_{11}-H_2)\exp(\xi\Delta_1 I^{-1/2})]+o(n^{-1/2})$$
$$= \Big[C_1(\delta+I^{-1/2}\xi)+n^{-1/2}\Big\{ M_1(\delta+I^{-1/2}\xi)-L_{001}\xi^2\Big(\frac{1}{2}\delta I^{-1}+\frac{1}{6}\, I^{-3/2}\xi\Big)\, C_1(\delta+I^{-1/2}\xi)$$
$$+\delta^2 I^{-1/2}\xi L_{11} C_1(\delta+I^{-1/2}\xi)-\xi\,\Big(\frac{1}{2}\, I^{-1}\xi+I^{-1/2}\delta\Big)\, C_5(\delta+I^{-1/2}\xi)\Big\}\Big] e^{\xi^2/2}$$
$$+o(n^{-1/2}), \qquad \ldots \quad \text{(A.11)}$$

$$E_{\theta_n}[Q_2 \exp(\xi\Delta I^{-1/2})] = C_2(\delta+I^{-1/2}\xi)e^{\xi^2/2}+O(n^{-1/2}), \quad \ldots \quad \text{(A.12)}$$

$$E_{\theta_n}[Q_1^2 \exp(\xi\Delta I^{-1/2})] = C_3(\delta+I^{-1/2}\xi)e^{\xi^2/2}+O(n^{-1/2}). \quad \ldots \quad \text{(A.13)}$$

From (3.3) and (A.10)—(A.13), the lemma follows.

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