Sankhyā : The Indian Journal of Statistics
1989, Volume 51, Series B, Pt. 3, pp. 362–374.

INDICES OF GROWTH

By MANORANJAN PAL
Indian Statistical Institute

SUMMARY. In this paper an attempt is made to construct an index of growth from a series of observations on an economic variable. Focus is given to the axiomatic approach based on some norms and properties—called axioms—which a good index of growth should satisfy. The statistical approach based on regression principles is also discussed and the derived formulae are compared with those obtained from the axiomatic approach. Further, the case of non-equidistant observations is also tackled through the axiomatic approach and a generalized formula for the index of growth has been derived by invoking an axiom concerning time factor.

1. Introduction

One of the common measurement problems in economic analysis is that of rate of the growth of an economic variable over a period of time. For example, one may be interested in examining whether one state is in a better position at a certain point of time compared to another state so far as the growth of agricultural production, say, is concerned, considering the agricultural production data for the past ten years (say) of the two states. To answer this type of questions what we need is a summary measure of growth rate. The growth measures enable us to compare between two time series data of different regions or of different time periods.

The purpose of this paper is to focus on the necessity of a rigorous analysis towards construction of such a measure and to discuss different ways of arriving at indices of growth which fulfil certain basic properties.

Let $x = \{x_0, x_1, ..., x_n\}$ denote a series of observations on a particular economic variable at time points $0, 1, ..., n$ respectively, considered in a backward direction (i.e., x_0 is the observation at the present time point, x_1 at the previous time point and so on, the time points being assumed equidistant for the time being).

If we forget the intermediate values $x_1, x_2, ..., x_{n-1}$, then we can think of an index

$$r = (x_0/x_n)^{1/n} - 1. \qquad \dots \quad (1)$$

which is easily established assuming that the growth rate is uniform during this period, and then one gets

$$x_0 = x_n(1+r)^n \qquad \dots \quad (2)$$

AMS (1980) *subject classification :* 90A19.
Key words and phrases : Index of Growth, Axiomatic Approach.

where r is the rate of growth. This is a highly simplistic way of finding the rate of growth which does not consider the intermediate values. For example, the sequences

$$\{8, 4, 2, 1\} \text{ and } \{8, 8, 8, 1\}$$

should not be considered same so far as the implicit growth pattern is concerned. This is because the first sequence shows a steady growth pattern compared to the second one. The magnitude of growth rate, however, depends on how one gives weights to the recent time points.

In what is discussed above we have used words like steady growth pattern etc., which are very difficult to define and hence call for a rigorous analysis. The basis of calculating rate of growth through ratios of consecutive values is, however, unambiguous i.e., for two periods 0 and 1, we have no doubt that the rate of growth will be

$$r = x_0/x_1 - 1. \quad \dots \quad (3)$$

Once we accept this as our basis, we must first make ourselves sure that $x_0, x_1, \dots, x_n$ are all positive and second, that the index lies in the interval $(-1, \infty)$.

In the next section we discuss the case of equidistant time points and derive an index of growth by utilizing some desirable axioms. Next two sections extend this formula to non-equidistant time points. Econometric estimates of growth index have also been found and compared with those derived from the axiomatic approach.

2. The case of equidistant time points

Let $$x = \{x_0, x_1, \dots, x_n\}$$

where $n \in N$, the set of natural numbers, x is a typical element in R^{n+1}_{++} where R^{n+1}_{++} is the strictly positive orthant of the Euclidean $(n+1)$-space R^{n+1}. Consider

$$D = \bigcup_{m \in N - \{1\}} R^{m}_{++}.$$

For any function $F : D \to R$ we denote the restriction of F to R^{n+1}_{++} by F^{n+1}. An index of aggregative growth is a function $I : D \to (-1, \infty)$.

We shall now state a number of properties which may be regarded as intrinsic to the concept of a growth index. These properties seem to be appealing within a quite general framework.

(i) *Normalisation axiom* : For all $(n+1) \in N - \{1\}$,

$$I^{n+1}\{c.1^{n+1}\} = 0 \quad \dots \quad (4)$$

where I^{n+1} is the $(n+1)$-coordinated vector of ones and $c > 0$ is a scalar, i.e., if the value of the economic variable concerned remains same over the time period, then we have a zero growth rate.

(ii) *Homogeneity of degree zero or dimensionality axiom* : For all $(n+1) \in N-\{1\}$, for all $x \in R^{n+1}_{++}$ and for all scalar $\lambda > 0$,

$$I^{n+1}(\lambda x) = I^{n+1}(x). \qquad \dots \quad (5)$$

i.e., if the values are multiplied by a positive constant then the index remains unchanged. In other words, the index is invariant under change of unit. This axiom essentially leads the growth index to a function of ratios of observations.

Since the growth index is a function of observation ratios as implied by the dimensionality axiom, we can think of the index as a function of basic indices. That is,

$$I^{n+1}\{x_0, x_1, \dots, x_n\} = I^*\{I_1, I_2, \dots, I_n\}, \qquad \dots \quad (6)$$

where $I_j = I\{x_{j-1}, x_j\} = x_{j-1}/x_j - 1$. If these values are same for all the periods, (say c), then the overall index should remain same as $c-1$. In other words

(iii) *Identity axiom* : For any $c > 0$ and for all $(n+1) \in N-\{1\}$,

$$I^{n+1}\{c^n, c^{n-1}, \dots c, 1\} = c-1, \qquad \dots \quad (7)$$

i.e., if the relative change in the values over time takes place along a ray of constant proportion, then the index value is equal to the proportionality factor minus one.

We shall introduce another axiom which involves multiplication of two series.

(iv) *Multiplicative identity axiom*: For all $(n+1) \in N-\{1\}$ and $x, x' \in R^{n+1}_{++}$,

$$I^{n+1}\{x_0x_0', x_1x_1', \dots, x_nx_n'\} = (1+\lambda_1)(1+\lambda_2)-1, \qquad \dots \quad (8)$$

where $\lambda_1 = I^{n+1}\{x_0, x_1 \dots, x_n\}$ and $\lambda_2 = I^{n+1}\{x_0', x_1', \dots, x_n'\}$. The motivation behind this is simple. Consider only two points of time 0 and 1. Define

$$\lambda_1 = I\{x_0, x_1\} = x_0/x_1 - 1$$

and

$$\lambda_2 = I\{x_0', x_1'\} = x_0'/x_1' - 1,$$

then

$$\lambda = I\{x_0x_0', x_1'x_1\}$$

$$= \frac{x_0x_0'}{x_1x_1'} - 1$$

$$= (1+\lambda_1)(1+\lambda_2) - 1.$$

(or ...

$$1+\lambda = (1+\lambda_1)(1+\lambda_2) \qquad \dots \quad (9)$$

If we call $1+\lambda$ the expansion ratio, then expansion ratio of the combined series is the product of the expansion ratios of the individual series. The idea becomes clear if we consider the following example :

Suppose the population in an economy increases by an expansion ratio e_1 and the consumption expenditure of each person increases by an expansion ratio e_2, then the expansion ratio of total consumption expenditure in the economy is the product of the two expansion ratios.

The following theorem gives the growth index formula which is a direct consequence of axiom (iv).

Theorem 1 : *The only non-trivial*[1] *growth index that satisfies multiplicative identity axiom is given by*

$$I^{n+1}\{x\} = x_0^{a_0} \prod_{i=1}^{n} (x_{i-1}/x_i)^{a_i} - 1$$

$$= x_0^{a_0} \prod_{i=1}^{n} (1+\lambda_i)^{a_i} - 1, \qquad \ldots \quad (10)$$

where $\lambda_i = x_{i-1}/x_i - 1$ *and* a_i*'s are given constants.*

Proof : Let, for a fixed n

$$f(p) = I^{n+1}\{p, p, \ldots, p\}, \quad p > 0. \qquad \ldots \quad (11)$$

Now, by multiplicative identity, we have

$$1+f(p_1p_2) = (1+f(p_1))(1+f(p_2)). \qquad \ldots \quad (12)$$

Defining $1+f(p)$ as $g(p)$, we have

$$g(p_1p_2) = g(p_1)\, g(p_2). \qquad \ldots \quad (13)$$

This is a well-known Cauchy functional equation whose solution is given by

$$g(p) = p^{m_0} \qquad \ldots \quad (14)$$

for any constant m_0.

Thus, $$I^{n+1}\{p, p, \ldots, p\} = p^{m_0} - 1. \qquad (15)$$

Again, define $$f_1(p) = I^{n+1}\{1, p, \ldots, p\}$$

and $$g_1(p) = 1+f_1(p).$$

---

[1] There are two trivial growth indices satisfying axiom (iv). These are $I \equiv 0$ or $I \equiv -1$. $I \equiv 0$ is, however, a special case of (10) which can be verified by putting all a_i values to zero.

We apply the same argument to get

$$I^{n+1}\{1, p, \ldots, p\} = p^{m_0}-1. \qquad \ldots \quad (16)$$

By repeating the same argument, we have,

$$I^{n+1}\{1, 1, p, \ldots, p\} = p^{m_2}-1$$

$$I^{n+1}\{1, 1, \ldots, 1, p\} = p^{m_n}-1.$$

Now $\quad 1+I^{n+1}\{(x_0, x_1, \ldots, x_n\}$

$$= 1+I^{n+1}\{x_0, x_0 x_1/x_0, \ldots, x_0 x_n/x_0\}$$

$$= [1+I^{n+1}\{x_0, x_0, \ldots, x_0\}][1+I^{n+1}\{1, x_1/x_0, \ldots, x_n/x_0\}]$$

$$= x_0^{m_0}\left[1+I^{n+1}\left\{1, x_1/x_0, \frac{x_2/x_0}{x_1/x_0}\cdot x_1/x_0, \ldots, \frac{x_n/x_0}{x_1/x_0}\cdot x_1/x_0\right\}\right]$$

$$= x_0^{m_0}[1+I^{n+1}\{1, x_1/x_0, \ldots, x_1/x_0\}][1+I^{n+1}\{1, 1, x_2/x_1, \ldots, x_n/x_1\}]$$

$$= x_0^{m_0}(x_1/x_0)^{m_1}[1+I^{n+1}\{1, 1, x_2/x_1, \ldots, x_n/x_1\}]$$

by repeating arguments

$$= x_0^{m_0}(x_1/x_0)^{m_1} \ldots (x_n/x_{n-1})^{m_n}.$$

Hence $\quad I^{n+1}\{x_0, x_1, \ldots, x_n\}$

$$= x_0^{m_0} \prod_{i=1}^{n} (x_i/x_{i-1})^{m_i}-1$$

$$= x_0^{a_0} \prod_{i=1}^{n} (x_{i-1}/x_i)^{a_i}-1, \qquad \ldots \quad (17)$$

where $\quad a_0 = m_0, a_i = -m_i \ \forall\ i > 0.$ Q.E.D.

Corollary 1 : *Axiom (iv) together with normalisation axiom (i) implies that* $a_0 = 0$.

Proof : $1+I^{n+1}\{x, x, \ldots, x\} = x^{a_0}(x/x)^{a_1} \ldots (x/x)^{a_n}$. Hence, by axiom (i), $a_0 = 0$. Q.E.D.

Corollary 2 : *Axiom (iv), identity axiom (iii) and normalisation axiom together imply that*

$$\sum_{i=1}^{n} a_i = 1.$$

Proof : $I^{n+1}\{c^n, c^{n-1}, \ldots, c, 1\}$

$$= c^{na_0}(c^n/c^{n-1})^{a_1}(c^{n-1}/c^{n-2})^{a_2} \ldots (c/1)^{a_{n-1}}$$

$$= c^{na_0} . c^{\Sigma a_i} - 1.$$

Hence, by axiom (iii) and (i),

$$\Sigma a_i = 1. \qquad \text{Q.E.D.}$$

Corollary 3 : *Axiom (iv) and axiom (i) imply axiom (ii).*

Proof : Since axiom (iv) and axiom (i) imlply that $a_0 = 0$, we have

$$I\{cx_0, \ldots, cx_n\} = (cx_0/cx_1)^{a_1} \ldots (cx_{n-1}/c\, x_n)^{a_n}$$

$$= (x_0/x_1)^{a_1} \ldots (x_{n-1}/x_n)^{a_n}$$

$$= I\{x_0, x_1, \ldots, x_n\}. \qquad \text{Q.E.D.}$$

This can also be proved in a direct manner.

$$1+I\{cx_0, cx_1, \ldots, cx_n\} = [1+I\{c, c, \ldots, c\}][1+I\{x_0, x_1, \ldots, x_n\}]$$

$$= 1+I\{x_0, x_1, \ldots, x_n\}. \quad \text{Q.E.D.}$$

Let us now see what happens if we try to tackle the problem from a statistical or rather an econometric angle. In the statistical approach we may assume the simple relation

$$x_{i-1} = (1+\lambda)\, x_i + \varepsilon_i,\; i = 1, \ldots, n \qquad \ldots \quad (18)$$

where λ is the rate of growth to be estimated. This may be viewed as an autoregressive model as described in Kendall and Stuart (1966) and Muth (1960) if we assume $E(x_i) = 0 \;\forall\; i$. In other words, they consider a time series which can not be explained by trend, seasonal or any other systematic factors. We have, instead, considered the case where there is a clear trend in the time series, what we call growth. Though there are different ways to handle this type of models, we shall restrict ourselves to the least squares (LS) and the weighted least squares (WLS) solutions only. This in fact, serves our purpose as illustrations.

The LS solution of λ of the equation (18) is

$$\hat{\lambda} = \Sigma\, \lambda_i\, w_i, \qquad \ldots \quad (19)$$

$w_i = x_i^2/\Sigma\, x_j^2$. Hence λ is the weighted arithmetic average of the basic indices, weights being proportional to the square of the value of the variable. Naturally, higher weight is given to the basic indices in the period with a higher value of the variable. This is not at all desirable so far as estimation of rate of growth is concerned. According to the economic viewpoint, more weight should be given to the recent output. The weights should gradually decrease towards past values. We can use WLS solution to solve this problem, in which case, we get,

$$\hat{\lambda} = \frac{\Sigma \lambda_i a_i x_i^2}{\Sigma a_i x_i^2}$$

$$= \Sigma \lambda_i b_i / \Sigma b_i \qquad \ldots \quad (20)$$

where $b_i = a_i x_i^2 / \Sigma a_j x_j^2$.

Since the definition of growth rate involves ratio of two consecutive values, a more realistic approach would be to take a multiplicative error model of type

$$x_{i-1} = (1+\lambda)\, x_i e_i \qquad \ldots \quad (21)$$

and to take logarithms on both sides to get the WLS solution

$$\hat{\lambda} = \{\Pi\, (1+\lambda_i)^{a_i}\}^{1/\Sigma a_i} - 1. \qquad \ldots \quad (22)$$

Observe that, this index is same as the index given in (10) since $a_0 = 0$ and $\Sigma\, a_i = 1$ in equation (10).

If we take the relation

$$x_0 = (1+\lambda)^i x_i \delta_i \qquad \ldots \quad (23)$$

which is a different way of expressing (21), we get the WLS solution after taking logarithms on both sides as

$$\hat{\lambda} = \{\Pi (x_0/x_i)^{ib_i}\}^{1/\Sigma i^2 b_i} - 1. \qquad \ldots \quad (24)$$

Theorem 2 : *The estimators given by* (22) *and* (24) *are equivalent forms if and only if* a_i *is a monotonic non-increasing function of* i. *i.e.* $a_i \geqslant a_{i+1}$ *for all* i.

Proof : $\hat{\lambda}$ in (24) can be written as

$$\left\{\Pi\, (x_{j-1}/x_j)^{\sum\limits_{i=j}^{n} i\, b_i}\right\}^{1/\Sigma i^2 b_i} - 1.$$

Hence $$a_j = \sum_{i=j}^{n} i b_i \text{ and } b_j = (a_j - a_{j+1})/j \qquad \ldots \quad (25)$$

assuming that $a_{n+1} = 0$. Naturally $b_i \geqslant 0 \;\forall\; i$ imply $a_i \geqslant 0 \;\forall\; i$. We must have the restriction $(a_i - a_{i+1})/i \geqslant 0 \;\forall\; i$, or $a_i \geqslant a_{i+1} \;\forall\; i$. It only remains to show that

$$\Sigma i^2\, b_i = \Sigma a_i.$$

Now, $\Sigma\, i^2\, b_i = \Sigma (a_i - a_{i+1}) i$ from (25)

$$= (a_1 - a_2) + 2(a_2 - a_3) + \ldots + (n-1)(a_{n-1} - a_n) + n(a_n - a_{n+1})$$

$$= \Sigma a_i \text{ since } a_{n+1} = 0.$$ Q.E.D.

3. The case of non-equidistant time points

To tackle the situation of varying time points, we must revise our formula accordingly. Let us assume that the two series are

$$\{x_0, x_1, \ldots, x_{n_1}\} \text{ and } \{y_0, y_1, \ldots, y_{n_2}\}$$

taken at time points

$$\{t_0, t_1, \ldots, t_{n_1}\} \text{ and } \{t'_0, t'_1, \ldots, t'_{n_2}\}$$

respectively. The primary requirement for comparison is that the time span for the two series are the same. i.e.,

$$t_{n_1} - t_0 = t'_{n_2} - t'_0.$$

Otherwise comparison becomes meaningless. The second requirement is the ceteris paribus requirement along with conformity, i.e., the situation prevailing in the first place must be same as that of the second, so that we can place t_0 and t'_0 as comparable starting points and can put t_0 and t'_0 as zero. Naturally $t_{n_1} = t'_{n_2}$. The consecutive points in each series however, need not be equidistant.

Assuming that the relation between x_k and x_{k-1} is

$$x_{k-1} = x_k (1+\lambda)^{t_k - t_{k-1}}$$

in the ideal case, where λ is the rate of growth ; we have the WLS solution, after taking logarithms on both sides, as

$$\lambda = \{\Pi (x_{k-1}/x_k)^{(t_k - t_{k-1}) a_k}\}^{1/\Sigma a_k (t_k - t_{k-1})^2} - 1. \qquad \ldots \quad (26)$$

This is a generalisation of our formula in theorem (1) together with the imposed constraints $a_0 = 0$ and $\Sigma a_t = 1$ as implied by Corollary 1 and Corollary 2. It becomes clear if we take some special cases as discussed below :

Case 1 : If $t_k - t_{k-1} = 1 \ \forall \ k = 1, 2, \ldots, n$ then

$$\lambda = \{\Pi (x_{k-1}/x_k)^{a_k}\}^{1/\Sigma a_k} - 1$$

which reduces to our original formula.

Case 2 : If $t_k - t_{k-1} = r \ \forall \ k = 1, \ldots, n$, then

$$\lambda = \{\Pi (x_{k-1}/x_k)^{a_k}\}^{1/r \Sigma a_k} - 1$$
$$= \{1 + \lambda^*\}^{1/r} - 1.$$

where λ^* is the growth index that one would obtain if the length of time intervals are ignored.

Comment 1 : The index given by theorem (1) is an asymmetric function of x_i's. It becomes symmetric when a_i values are all equal. In such a case the index ignores all the intermediate values and becomes a function of extreme two values only (see Rudra, 1982, 317-320).

Comment 2 : The theorem does not also say anything regarding the choice of the weights a_i's. There are various weights which have appeared in the econometric literature on distributed lag models and general forecasting. Since a planner will naturally put more stress on the recent periods an obvious choice is to take the sequence $\{a_i\}$ as a decreasing function of i. If it is linear, one may assume

$$a_i = 2(n-i+1)/(n(n+1)), \quad i = 1, 2, \dots, n. \qquad \dots \quad (28)$$

There can be other choices among non-linear functions of i for specific situations. Exponential weight is one of them.

In the other model where

$$x_0 = (1+\lambda)^{t_j} x_j e_j. \qquad \dots \quad (29)$$

We have the WLS solution after taking logarithms on both sides, as

$$\lambda = \{\prod_j (x_0/x_j)^{b_j t_j}\}^{1/\Sigma b_i t_i^2} - 1 \qquad \dots \quad (30)$$

Theorem 3 : *The formula* (26) *and* (30) *are equivalent if and only if*

$$(t_j - t_{j-1}) a_j \geqslant (t_{j+1} - t_j)\, a_{j+1}$$

for all $j = 1, 2, \dots, n$, *where* $t_0 = 0$ *and* $a_{n+1} = 0$.

Proof : The proof goes in a similar way as given in the proof of Theorem 2. Here the relations between a_i's and b_i's are as follows :

$$(t_j - t_{j-1}) a_j = \sum_{i=j}^{n} b_i t_i, \quad j = 1, 2, \dots, n ;$$

and

$$b_j = \{(t_j - t_{j-1})\, a_j - (t_{j+1} - t_j)\, a_{j+1}\}/t_j, \quad j = 1, 2, \dots, n.$$

Also,

$$\begin{aligned} \Sigma b_j t_j^2 &= \sum_{j=1}^{n} \{t_j(t_j - t_{j-1}) a_j - t_j(t_{j+1} - t_j) a_{j+1}\} \\ &= t_1(t_1 - t_0) a_1 - t_1(t_2 - t_1) a_2 \\ &\quad + t_2(t_2 - t_1) a_2 - t_2(t_3 - t_2) a_3 \\ &\quad + \dots\dots\dots\dots\dots\dots \\ &\quad + t_n(t_n - t_{n-1}) a_n - t_n(t_{n+1} - t_n) a_{n+1} \\ &= \sum_{i=1}^{n} (t_i - t_{i-1})^2 a_i \end{aligned}$$

since, $t_0 = 0$ and $a_{n+1} = 0$. Q.E.D.

4. Choice of weight functions

The choice of weight functions for a_i's and b_i's are quite subjective except that b_i's and a_i's must be positive and

$$(t_j - t_{j-1})a_j \geqslant (t_{j+1} - t_j)a_{j+1}$$

for all j. We can improve the weight function to make it more deterministic by introducing an axiom. In fact we get an explicit form of b_i or rather b_t as a function of t. The function however, involves a parameter which introduces some amount of subjectivity. For this purpose we reformulate our model as follows :

In our new formulation we view the weight function b_j as a function of t_j and not as a function of j only and write it as $f(t_j)$. Similarly, instead of x_j, we write x_{t_j}.

Hence

$$\log(1+\lambda) = \frac{\sum_{j=1}^{n} t_j f(t_j) \log(x_0/x_{t_j})}{\sum_{j=1}^{n} t_j^2 f(t_j)}. \quad \ldots \quad (31)$$

Suppose after time c we get a value x_{-c}. Now, this becomes our recent value. Hence, we have

$$\log(1+\lambda^*) = \frac{\sum_{j=0}^{n} (t_j+c) f(t_j+c) \log(x_{-c}/x_{t_j})}{\sum_{j=0}^{n} (t_j+c)^2 f(t_j+c)}. \quad \ldots \quad (32)$$

Axiom (v) : $\log(1+\lambda^*) = \log(1+\lambda)$, if $x_{-c} = (1+\lambda)^c x_0$.

Theorem 4 : *Axiom* (v) *implies that*

$$f(t) \propto \frac{1}{t} e^{-at}$$

where a is a constant.

Proof : See Appendix.

By virtue of Theorem 4, we can write

$$\log(1+\lambda) = \frac{\Sigma e^{-at_j} \log(x_0/x_{t_j})}{\Sigma t_j e^{-at_j}}. \quad \ldots \quad (33)$$

Further, imposing the restriction that $f(t)$ is a monotonic non-increasing function of t, we get $a \geqslant 0$. The restriction

$$(t_j - t_{j-1})a_j - (t_{j+1} - t_j)a_{j+1} \geqslant 0,$$

to prove the equivalence, is satisfied for any non-negative 'a', once we define

$$(t_j - t_{j-1})\, a_j = \sum_{i=j}^{n} f(t_i) t_i, \; j = 1, 2, \ldots, n.$$

Case 3 : $a = 0$, Here, $a_j = \dfrac{n-j+1}{t_j - t_{j-1}}$ and $\log (1+\lambda)$ can equivalently be written as

$$\log (1+\lambda) = \frac{\log \Pi \, (x_{j-1}/x_j)^{n-j+1}}{\Sigma(n-j+1)(t_j - t_{j-1})}.$$

If, further, $t_j - t_{j-1} = 1$, we have

$$\log (1+\lambda) = \log \Pi \, (x_{j-1}/x_j)^{2(n-j+1)/n(n+1)}$$

which is same as the growth rate formula defined for the equidistant case when the weight function is chosen as in (28).

Case 4 : When $a \to \infty$, $\log (1+\lambda) \to \dfrac{1}{t_1} \log \;(x_0/x_{t_1})$, i.e., it ignores all the previous observations except the last.

5. Concluding remarks

A few remarks on the axiomatic index of growth that we have developed here is in order. First, it may be noted the index of growth as expressed in its final form by formula (33) is not deterministic in the sense that the value of the index depends on the subjective choice of the weight function implicit in the measurement. We have, however, succeeded in restricting the weight function in such a manner that it depends on a single parameter, viz., 'a'. Although the range of 'a' is wide (and hence it is difficult to choose a specific value of a without inviting criticism), it gives applied economists a freedom of being anywhere between most conservative (i.e., when $a = 0$ is chosen) to most sceptical about the past values beyond the immediate past (i.e., when $a = \infty$). Use of more than one observation in the computation of the index is unavoidable as is evident from the definition itself. Surprisingly, the conventional measure of rate of growth, viz., $(x_0/x_n)^{1/n} - 1$ is not consistent as this does not correspond to any non-negative choice of 'a'. and this is due to the assumption that a_i's are non-increasing.

Next, it is possible to have measure of rate of growth by our proposed index in cases of missing observations as well. In the case of equidistant observations, one can either treat the missing observations, say, x_{r_1}, x_{r_2}, ..., x_{r_k} as unknown parameters and minimize the WLS function with respect to λ and $x_{r_1}, \ldots, x_{r_k}$, or else we can use the general set up in which consecutive observations need not be equidistant. In either case, at can be demonstrated, that the resulting estimator $\hat{\lambda}$ will be identical, namely

$$\log(1+\hat{\lambda}) = \frac{\sum\limits_{\substack{j=1 \\ j\neq r_1, r_2, \ldots, r_k}}^{n} b_j t_j \log(x_0/x_j)}{\sum\limits_{\substack{j=1 \\ j\neq r_1, r_2, \ldots, r_k}}^{n} b_j t_j^2} \quad \ldots \quad (34)$$

Finally, it may be noted that in developing the index of growth here, we have considered a series of observations on an economic 'flow' variable comprising the periodwise values of the variable measured at the end of each period. In case of economic 'stock' variables like the population of an area or capital stock, say, the values measured at the end of each period would be the value of the stock up to that time and not the incremental value for the period. In such a case, the interpretation of the index of growth would be somewhat different.

Appendix

Proof of *Theorem* 4 :

$$\log(1+\hat{\lambda}^*) = \sum_{j=0}^{n} (t_j+c)f(t_j+c)\log\{(1+\lambda)^c x_0/x_{t_j}\} / \sum_{0}^{n} (t_j+c)^2 f(t_j+c)$$

$$= \frac{\sum_{1}^{n} (t_j+c)f(t_j+c)\log(x_0/x_{t_j}) + \sum_{0}^{n} c(t_j+c)f(t_j+c)\log(1+\hat{\lambda})}{\sum_{0}^{n} (t_j+c)^2 f(t_j+c)}$$

$= \log(1+\hat{\lambda})$ by axiom (v).

Hence,
$$\log(1+\hat{\lambda})\left\{\sum_{0}^{n} (t_j+c)^2 f(t_j+c) - \sum_{0}^{n} c(t_j+c)f(t_j+c)\right\}$$

$$= \sum_{1}^{n} (t_j+c)f(t_j+c)\log(x_0/x_{t_j})$$

or,
$$\log(1+\hat{\lambda}) = \frac{\sum_{1}^{n} (t_j+c)f(t_j+c)\log(x_0/x_{t_j})}{\sum_{1}^{n} t_j(t_j+c)f(t_j+c)}.$$

or,
$$\frac{\Sigma t_j f(t_j)\log(x_0/x_{t_j})}{\Sigma t_j^2 f(t_j)} = \frac{\Sigma(t_j+c)f(t_j+c)\log(x_0/x_{t_j})}{\Sigma t_j(t_j+c)f(t_j+c)}.$$

This is true for any $x_0, x_{t_1}, \ldots, x_{t_n}$. Hence, we can put $x_{t_i} \neq x_0$ $x_{t_j} = x_0$ for $j = 1, \ldots, n$ and $j \neq i$.

or, $$\frac{t_i f(t_i) \log (x_0/x_{t_i})}{\Sigma\, t_j^2 f(t_j)} = \frac{(t_i+c)f(t_i+c)\log(x_0/x_{t_i})}{\Sigma\, t_j(t_j+c)f(t_j+c)}$$

or, $$\frac{t_i f(t_i)}{\Sigma\, t_j^2 f(t_j)} = \frac{(t_i+c)f(t_i+c)}{\Sigma\, t_j(t_j+c)f(t_j+c)}$$

or, $$\frac{\Sigma\, t_j^2 f(t_j)}{t_i f(t_i)} = \frac{\Sigma\, t_j(t_j+c)f(t_j+c)}{(t_i+c)f(t_i+c)}.$$

Subtracting t_i from both sides and inverting, we get,

$$\frac{t_i\, f(t_i)}{\sum\limits_{t_j \neq t_i} t_j^2 f(t_j)} = \frac{(t_i+c)f(t_i+c)}{\sum\limits_{t_j \neq t_i} t_j(t_j+c)f(t_j+c)}$$

or, $$\frac{g(t_i)}{\sum\limits_{j \neq i} t_j g(t_j)} = \frac{g(t_i+c)}{\sum\limits_{j \neq i} t_j g(t_j+c)}, \text{ where } g(t) = tf(t)$$

or, $$\frac{g(t_i+c)}{g(t_i)} = \frac{\sum\limits_{j \neq i} t_j g(t_j+c)}{\sum\limits_{j \neq i} t_j\, g(t_j)}.$$

Since, RHS does not involve t_i, we can write it as a function of c only,

or $$g(t_i+c) = g(t_i)\, h(c). \qquad \ldots \quad (35)$$

The solution of (35) is well-known as

$$g(t) = e^{pt} = e^{-at}. \quad \text{(say)}$$

Hence, $$f(t) = \frac{1}{t} e^{-at} \qquad \ldots \quad (36)$$

where 'a' is a constant. Q.E.D.

Acknowledgements. The author is thankful to Dr. Satya R. Chakravarty of Indian Statistical Institute, for his valuable and constructive suggestions made to the earlier version of this paper.

REFERENCES

KENDALL, M. G. and STUART A. (1966) : *The Advanced Theory of Statistics*, 3, Charles Griffin and Co. Ltd., London.

MUTH, J. F. (1960) : Optimal properties of exponentially weighted forecasts. *Jour. Amer. Statist Assoc.* 55, 299-306,

RUDRA, A. (1982) : *Indian Agricultural Economics : Myths and Realities*, Allied Publishers Private Ltd., New Delhi, Calcutta.

Paper received : *June*, 1987.

Revised : *December*, 1988.