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EXTENSIONS OF RAJCHMAN'S STRONG LAW OF LARGE NUMBERS

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SUMMARY. Chandra (1989) introduced a new condition called the Cesàro uniform integrability of a sequence of random variables. Using this idea, this note extends the strong law of large numbers of Rajchman (1932).

Introduction and main results

We shall first prove *in an elementary way* the following *general result* on the strong law of large numbers (SLLN).

Theorem 1. *Let $\{X_n\}$ be a sequence of random variables defined on the same probability space and with finite $E(X_n^2)$. Assume that (i) $E(X_m+\ldots+X_{m+n})^2 \leqslant C_1[\sigma_m^2+\ldots+\sigma_{m+n}^2]$ for each integer $m, n \geqslant 1$ where C_1 is a constant free from m and n and $\sigma_n = (E(X_n^2))^{1/2}$, $n \geqslant 1$; and (ii) $\sum_n n^{1/2}\sigma_n^2(f(n))^{-2} < \infty$ for some increasing sequence $f(n) > 0$ such that $n^{-\alpha} f(n) \leqslant C_2 m^{-\alpha} f(m)$ for each $n \leqslant m$ where $\alpha > 1/4$ and C_2 is some constant. Then $S(n)/f(n) \to 0$ almost surely as $n \to \infty$ where $S(n) = X_1+\ldots+X_n$, $n \geqslant 1$.*

In particular, $n^{-\alpha}S(n) \to 0$ almost surely if Assumption (i) holds and $\sum n^{-2\alpha+1/2}\sigma_n^2 < \infty$ for some $\alpha > 1/4$.

The most important feature of the above theorem is that nothing is assumed about the independence of the underlying random variables. There are two well-known results, namely, the Kolmogorov SLLN and the Rademacher-Menshov SLLN (see page 114 of Rao (1973)), which resemble the above theorem ; they, however, require stronger assumption of 'mutual independence' or 'orthogonality'. This is the reason why Theorem 1 does not follow from and standard result on SLLN.

The above theorem includes the following two well-known results as special cases.

Theorem 2 (The extended Borel SLLN ; see page 18 of Loève (1977)).

Let $\{X_n\}$ be a sequence of uniformly bounded uncorrelated random variables, then $n^{-1}[S(n)-E(S(n))] \to 0$ almost surely as $n \to \infty$.

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Theorem 3 (The Rajchman SLLN ; see Rajchman (1932) and page 103 of Chung (1974)). *Let $\{X_n\}$ be a sequence of uncorrelated random variables such that $\{var(X_n)\}$ is bounded. Then $n^{-1}[S(n)-E(S(n)]\to 0$ almost surely as $n\to 0$.*

It may be noted that in Theorem 3, $\{X_n-E(X_n)\}$ is uniformly integrable. Before starting the proof of Theorem 1, let us note the following lemma which gives a simple sufficient condition for Assumption (i) of Theorem 1.

Lemma 1. *Let $\{X_n\}$ be a sequence of random variables such that $E(X_iX_j) \leqslant r(|i-j|)\sigma_i\sigma_j$ for each integer $i,\ j\geqslant 1$ where $r(k)\geqslant 0$ for each $k\geqslant 1$ $(r(0)=1)$ and $\sum_{n=1}^{\infty} r(n)<\infty$. Then Assumption (i) of Theorem 1 holds with $C_1=1+2\sum_{n=1}^{\infty} r(n)$.*

Proof of Lemma 1. Note that

$$E(X_m+\ldots+X_{m+n})^2\leqslant(\sigma_m^2+\ldots+\sigma_{m+n}^2)+2\sum r(|i-j|)\sigma_i\sigma_j.$$

The second term on the right side is

$$2\sum_{j=1}^{n} r(j)\sum_{i=m}^{m+n-j}\sigma_i\sigma_{i+j}\quad\sum_{j=1}^{n} r(j)\sum_{i=m}^{m+n-j}(\sigma_i^2+\sigma_{i+j}^2)$$

$$\leqslant 2\sum_{j=1}^{n} r(j)(\sigma_m^2+\ldots+\sigma_{m+n}^2)\leqslant 2\sum_{j=1}^{\infty} r(j)(\sigma_m^2+\ldots+\sigma_{m+n}^2).$$

To prove Theorem 1, we use only the Chebyshev inequality and the following lemma.

Lemma 2 (see page 18 of Loeve, 1977). *If $\{X_n\}$ is a sequence of random variables defined on the same probability space and $\sum_n P(|X_n|\geqslant 1/m)<\infty$ for each $m\geqslant 1$, then $X_n\to 0$ almost surely as $n\to\infty$.*

Below C stands for a generic constant, not necessarily the same at each appearance.

Proof of Theorem 1. 1^0 : We first show that

$$S(n^2)/f(n^2)\to 0 \text{ almost surely as } n\to\infty. \qquad \ldots (1)$$

Let $\epsilon>0$. Then

$$\epsilon^2\sum_n P(|S(n^2)|>f(n^2)\epsilon)\leqslant\sum_n E(S(n^2))^2(f(n^2))^{-2}$$

$$\leqslant C\sum_{n=1}^{\infty}\cdot\sum_{j=1}^{n^2}\sigma_j^2(f(n^2))^{-2}=C\sum_{j=1}^{\infty}\sigma_j^2\sum_{n=k_j}^{\infty}(f(n^2))^{-2},$$

where $k_j = \inf\{n \geqslant 1 : n^2 \geqslant j\}, j \geqslant 1$. Let $j \geqslant 4$; then

$$\sum_{n=k_j}^{\infty} n^{-4\alpha} \leqslant \sum_{n=k_j}^{\infty} \int_{n-1}^{n} x^{-4\alpha}\, dx = \int_{k_j-1}^{\infty} x^{-4\alpha}\, dx$$

$$= (4\alpha-1)^{-1}(k_j-1)^{-4\alpha+1} \leqslant (4\alpha-1)^{-1}(j^{1/2}-1)^{-4\alpha+1}.$$

Thus

$$\sum_{j=4}^{\infty} \sigma_j^2 \sum_{n=k_j}^{\infty} (f(n^2))^{-2} \leqslant C \sum_{j=4}^{\infty} j^{2\alpha}(f(j))^{-2}\sigma_j^2 \sum_{k_j}^{\infty} n^{-4\alpha} < \infty$$

by Assumption (ii). As $\sum_{n=k_j}^{\infty} n^{-4\alpha} < \infty$ (since $\alpha > 1/4$) for each $j \geqslant 1$, Lemma 2 establishes (1).

2° : Define $m_n = [n^{1/2}]$, the greatest integer $\leqslant n^{1/2}$ $(n \geqslant 1)$. Note that $|S(n)| \leqslant |S(n)-S(m_n^2)| + |S(m_n^2)|$. By 1°, it sufficies to show that

$$n^{-\alpha}|S(n)-S(m_n^2)| \to 0 \text{ almost surely as } n \to \infty. \qquad \ldots (2)$$

As before we use Lemma 2. Thus (2) will follow if we show that $\Sigma' P(|S(n)-S(m_n^2)| > f(n)\epsilon) < \infty$, *the sum being taken over all $n \geqslant 1$ such that $n^{1/2}$ is not an integer*, *i.e.*, $m_n < n^{1/2} < m_n+1$.

By the Chebyshev inequality, it suffices to show that

$$\Sigma' E(S(n)-S(m_n^2))^2 (f(n))^{-2} < \infty. \qquad \ldots (3)$$

But the last sum is bounded above by

$$C\,\Sigma' \sum_{j=m_n^2+1}^{n} \sigma_j^2 (f(n))^{-2} \leqslant C \sum_{j=1}^{\infty} \sigma_j^2\, \Sigma''\, (f(n))^{-2}$$

where Σ'' denotes the summation over all n such that $j \leqslant n \leqslant j+2j^{1/2}$; here we have interchanged the order of summation and used the following fact:

$$m_n^2+1 \leqslant j \Longrightarrow n^{1/2}-1 < m_n \leqslant (j-1)^{1/2}$$

$$\Longrightarrow n \leqslant j+2j^{1/2}.$$

Thus the sum in (3) is bounded above by

$$C \sum_{j=1}^{\infty} \sigma_j^2 \Sigma'' (f(j))^{-2} = 2C \sum_{j=1}^{\infty} \sigma_j^2 j^{1/2} (f(j))^{-2} < \infty.$$

Remark. In the above proof, we have used only the nonnegativity of the σ_n but we have not used that σ_n^2 is in fact $E(X_n^2)$.

We finally prove a significant and novel extension of the Rajchman SLLN using the key idea of Chandra (1989).

Theorem 4. *Let* $\{X_n\}$ *be a sequence of random variables with finite* $E(X_n^2)$, *and put* $\alpha_n = n^{-1} \sum_{i=1}^{n} E(X_i^2)$, $n \geqslant 1$. *If* $\alpha_n \leqslant K n^{\theta}$ *for each* $n \geqslant 1$ *where* K *is a constant free from* n *and* $0 \leqslant \theta < 2\alpha - 3/2$, *and Assumption* (*i*) *of Theorem* 1 *holds, then* $n^{-\alpha} S_n \to 0$ *almost surely as* $n \to \infty$.

It may be noted that if the constant θ of Theorem 4 is zero, i.e., if $\{\alpha_n\}$ is bounded, then $\{X_n\}$ is Cesàro uniformly integrable as defined in Chandra (1989).

Proof of Theorem 4. We shall use Theorem 1 (with $f(n) = n^{\alpha}$) to prove this result. Let $N \geqslant 2$. Below we use the formula of summation by parts (see page 194 of Apostol, (1974). Put $a_n = n^{-\alpha+1/2}$, $n \geqslant 1$. Note, with $\alpha_0 = 0$, that

$$\sum_{n=1}^{N} E(X_n^2) a_n = \sum_{n=1}^{N} [n\alpha_n - (n-1)\alpha_{n-1}]\, a_n$$

$$= N\alpha_N a_N + \sum_{n=1}^{N-1} [n\, \alpha_n (a_n - a_{n+1}), \text{ using summation by parts}$$

$$\leqslant K[N^{1+\theta}\; a_n + \sum_{n=n}^{N-1} n^{1+\theta}\, (a_n - a_{n+1})].$$

Clearly, $a_n - a_{n+1} \leqslant (2\alpha - 1/2)\, n^{-2\alpha-1/2}$. Letting $N \to \infty$ and noting that $2\alpha - 1/2 - \theta > 1$, we get Assumption (ii) of Theorem 1.

For other applications of Cesàro uniform integrability in the context of large numbers, see Chandra and Goswami (1991).

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