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SINGLE SAMPLING LTPD PLANS AS A STOCHASTIC PROGRAMMING PROBLEM

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SUMMARY. In the lot tolerance percent defective (LTPD) single sampling attribute plan proposed by Dodge-Romig (1929) the process average and the LTPD cannot be known precisely. These two parameters are assumed random variables so that the Dodge-Romig problem becomes a stochastic programming problem. Two chance constraint programming models are developed. Solution methods with numerical examples are provided.

1. Introduction

We consider a producer's final inspection of a series of lots of size N, under the production process where each lot retains its identity such as lots of electronic equipment for a large computer or a missile. In designing a single sampling attribute plan (SSP) for acceptance inspection, it is assumed that the producer knows his process average p_1, under normal manufacturing conditions and that he occasionally produces lots of bad quality. He may then select lot tolerance fraction defective (LTPD), p_2 say $p_2 > p_1$ and a risk $P(p_2) = \beta$ of accepting the lots of this quality where $P(p)$ is the operating characteristic of the SSP. The Dodge-Romig (1929) LTPD SSP with total inspection of rejected lots is to find the sample size n and the acceptance number c which

$$\text{minimize} \qquad I(p_1, n, c) = n+(N-n)\,(1-P(p_1)) \qquad \dots \quad (1)$$

$$\text{subject to} \qquad P(p_2, n, c) = \beta \qquad \dots \quad (2)$$

$$\text{and} \qquad n, c \geqslant 0, \text{ integer}. \qquad \dots \quad (3)$$

We note that the above optimization problem (1) through (3) is a nonlinear integer programming (NLIP) problem.

In the above problem, the decision maker (DM) assumes p_1 and p_2 are deterministic. However, in practical applications, p_1 and p_2 cannot be known *precisely* but can be stated only in close range from the experience of the DM.

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So these two parameters are assumed random variables such that the Dodge-Romig problem becomes a stochastic programming (SP) problem. We shall consider two types of the stochastic version of the objective function (1) and develop solution procedures for both the type of the problem. To the best of our knowledge, this approach to design SSP has not appeared in the literature. However, for applications of programming techniques in designing SSPs, see Chakraborty (1986, 1989, 1990).

We shall restrict our discussions under Poisson conditions, see Hald (1981).

2. Stochastic programming problem

Dodge-Romig LTPD SSP when the parameters p_1 and p_2 are assumed random variables is a NLISP problem and may be formulated, under Poisson conditions, as

$$\text{optimize} \quad I(p_1, n, c) = n+(N-n)(1-G(c, np_1)) \quad \dots (4)$$

$$\text{subject to} \quad Pr\{P(p_2) \leqslant \beta\} \geqslant 1-\epsilon \quad \dots (5)$$

$$n, c \geqslant 0, \quad \text{int.} \quad \dots (6)$$

where $0 < \epsilon < 1$.

The objective function is usually assumed to be the mean of the stochastic objective function. The constraint (5) is known in the literature as the chance constraint which was first formulated by Charnes, Cooper and Symonds (1958) and by Charnes and Cooper (1959).

Following Kataoka (1963), we shall consider a second version of the Dodge-Romig problem with the objective function (for a real numbers k and $0 < \alpha < 1$)

$$\text{minimize} \quad \{k \mid Pr\{I(p_1, n, c) \leqslant k\} \geqslant \alpha\} \quad \dots (7)$$

subject to (5) and (6).

2.1. *Deterministic equivalent of the chance constraint.* It is easy to see that the constraint (5) is equivalent to

$$Pr\{p_2 \geqslant m_\beta(c)/n\} \geqslant 1-\epsilon. \quad \dots (8)$$

We assume p_2 follows the distribution $F_{p_2}(z)$ and let p_2^* be the greatest value such that $F_{p_2}(p_2^*) = \epsilon$, so that from (8) we have

Theorem 2.1. *The chance constraint* (5) *is equivalent to the deterministic constraint*

$$p_2^* \geqslant m_\beta(c)/n \quad \dots (9)$$

2.2. *Expectation of the chance objective function: Normal distribution case.* We assume p_1 follows $N(\mu, \sigma^2)$.

Lemma 2.1. $E(g(p_1)) = \frac{n^r}{r!} e^{-\left(n\mu - \frac{n^2\sigma^2}{2}\right)} E^r_{p_1} N(\mu - n\sigma^2, \sigma^2)$ ... (10)

Proof. $E(g(p_1))$

$$= \int_{-\infty}^{\infty} e^{-np_1} \frac{(np_1)^r}{r!} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(p_1-\mu)^2}{2\sigma^2}} dp_1$$

$$= \frac{n^r}{r!} \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} p_1^r\, e^{-\frac{(p_1^2 - 2p_1\mu + \mu^2 + 2np_1\sigma^2)}{2\sigma^2}} dp_1$$

$$= \frac{n^r}{r!} \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} p_1^r$$

$$e^{-\frac{1}{2\sigma^2}\left[p_1^2 - 2p_1(\mu - n\sigma^2) + \mu^2 - 2n\mu\sigma^2 + n^2\sigma^4 + 2n\mu\sigma^2 - n^2\sigma^4\right]} dp_1$$

$$= \frac{n^r}{r!} e^{-\left(n\mu - \frac{n^2\sigma^2}{2}\right)} \left[\int_{-\infty}^{\infty} p_1^r \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(p_1 - (\mu - n\sigma^2))^2} dp_1\right]$$

$$= \frac{n^r}{r!} e^{-\left(n\mu - \frac{n^2\sigma^2}{2}\right)} E^r_{p_1} N(\mu - n\sigma^2, \sigma^2).$$

Theorem 2.2. *The expected value of the chance objective function* (4) *under normal distribution of* p_1 *is equal to*

$$N - (N - n) \sum_{r=0}^{c} \frac{n^r}{r!} e^{-\left(n\mu - \frac{n^2\sigma^2}{2}\right)} E^r_{p_1} N(\mu - n\sigma^2, \sigma^2) \quad \ldots \quad (11)$$

Proof. Follows from Lemma 2.1.

Remarks. The r-th moment μ_r' of $N(\mu, \sigma^2)$ can be obtained from Bain (1969)

$$\mu'_{2r-1} = \sigma^{2r-1} \sum_{i=1}^{r} \frac{(2r-1)!\,(\mu)^{2i-1}}{(2i-1)!\,(r-i)!\,2^{r-i}\sigma^{2i-1}}, \quad r = 1, 2, 3\ldots \quad \ldots \quad (12)$$

$$\mu'_{2r} = \sigma^{2r} \sum_{i=0}^{r} \frac{(2r)!\,\mu^{2i}}{(2i)!\,(r-i)!\,2^{r-i}\sigma^{2i}}, \quad r = 1, 2, 3\ldots \quad \ldots \quad (13)$$

Beta distribution case. We assume p_1 follows Beta distribution $Be\,(p_1, s, t)$. Also $x^{[r]} = x(x+1)\ldots(x+r-1)$.

Lemma 2.2. $E(g(p_1)) = \frac{n^r}{r!} \sum_{x=0}^{\infty} (-1)^x \frac{n^x}{x!} \frac{s^{[r+x]}}{(s+t)^{[r+x]}}$ ... (14)

Proof. $E(g(p_1)) = \int_0^1 e^{-np_1} \frac{(np_1)^r}{r!} p_1^{s-1} \frac{(1-p_1)^{t-1}}{\beta(s,t)} dp_1$

$$= \frac{n^r}{r!\,\beta(s,t)} \int_0^1 e^{-np_1}\ p^{s+r-1} (1-p_1)^{t-1} dp_1$$

$$= \frac{n^r}{r!\,\beta(s,t)} \int_0^1 \left(1-np_1+\frac{n^2p_1^2}{2!}-\ldots\right) p_1^{s+r-1}(1-p_1)^{t-1} dp_1.$$

Theorem 2.3. *The expected value of the chance objective function* (4) *under Beta distribution of* p_1 *is equal to*

$$N-(N-n) \sum_{r=0}^{c} \sum_{x=0}^{\infty} (-1)^x \frac{n^{r+x}}{r!\,x!} \frac{s^{[r+x]}}{(s+t)^{[r+x]}}. \qquad \ldots (15)$$

Proof. Follows from Lemma 2.2.

2.3 *Deterministic equivalent to Kataoka objective function.* We note that.

$$I(p_1, n, c) \leqslant k$$

$$\Leftrightarrow n+(N-n)(1-G(c, np_1)) \leqslant k$$

$$\Leftrightarrow 1-G(c, np_1) \leqslant \frac{k-n}{N-n} = \delta \text{ (say)}$$

$$\Leftrightarrow p_1 \leqslant m_{1-\delta}(c)/n. \qquad \ldots (16)$$

We assume p_1 follows distribution $F_{p_1}(z)$ and let p_1^* be the least value such that $F_{p_1}(p_1^*) = \alpha$, so that from (16) we have

Theorem 2.4. *The probabilistic statement* $Pr\{I(p_1, n, c) \leqslant k\} \geqslant \alpha$ *is equivalent to the deterministic statement*

$$p_1^* \leqslant m_{1-\delta}(c)/n. \qquad \ldots (17)$$

3. Solution methods

3.1. *Expected value objective function case.* When p_1 is assumed $N(\mu_1, \sigma_1^2)$ and p_2 is assumed $N(\mu_2, \sigma_2^2)$ the deterministic equivalent of the Dodge Romig problem is to minimize (11), subject to (9) and (6).

Given μ_2, σ_2 and ϵ, one can find p_2^* from standard normal table. Now for $c = 0, 1, 2, \ldots$, the values of $m_\beta(c)$ can be obtained from Table 1 of Hald (1981) and hence for each $c = 0, 1, 2, \ldots$, the corresponding value n is obtained from (9).

Now for each pair of (c, n), the function (11) is evaluated and by enumeration the optimum pair (c, n) is found out.

Example 3.1. We consider the probabilistic version of the example given in Hald (1981, p. 101)

$$N = 2{,}000, \quad \beta = 0.10, \quad \mu_1 = 0.02, \quad \sigma_1 = 0.001,$$

$$\mu_2 = 0.10, \quad \sigma_2 = 0.002 \quad \text{and} \quad \epsilon = 0.05.$$

Solution. $p_2^* = \mu_2 - 1.645 \times \sigma_2 = 0.0967$. From Table 1 of Hald (1981), we obtain the pairs (c, n). For $c = 2$, $n \geqslant \frac{5.322}{0.0967} = 55.04$, so integer $n = 56$. For this pair we evaluate (11) and obtain 255.24. The values are tabulated in Table 1 below.

TABLE 1. OPTIMAL STOCHASTIC LTPD PLAN WHEN PARAMETERS FOLLOW NORMAL DISTRIBUTIONS

c	n	$E(G(c, np_1^*)$	$E(I(p_1^*))$
2	56	0.89751	255.24
3	70	0.94513	175.90
5	96	0.98692	122.81
*6	109	0.99271	122.79
7	122	0.99525	130.92

The obtimal solution is $n = 109$ and $c = 6$ with $E(I(p_1)) = 122.79$. The deterministic problem has the corresponding solution $n = 93$, $c = 5$ and $I(p_1) = 116$.

When p_1 is assumed $Be(p, s_1, t_1)$ and p_2 is assumed $Be(p, s_2, t_2)$, the solution procedure is modified accordingly. However, for practical problems, the shape parameter s is required to be greater than 20 and t is required to be greater than 200. In this case p_2^* can be found by approximate formulas accurate enough, but evaluating (15) is very difficult since the function converges very slowly.

3.2. *Kataoka-type objective function case*. When p_1 is assumed $N(\mu_1, \sigma_1^2)$ and p_2 is assumed $N(\mu_2, \sigma_2^2)$ the deterministic equivalent of the Dodge-Romig problem is to minimize k, subject to (17), (9) and (6).

The solution can be obtained by adapting the procedure of subsection 3.1.

Example 3.2. Same as Example 3.1 with an additional parameter $\alpha = 0.05$.

Solution. As in Example 3.1, we obtain pairs of (c, n) satisfying (9) and (6). The value of p_1^* is obtained as 0.0216 and for each pair of (c, n) we find the minimum δ satisfying (17) and hence calculate minimum $k = (N-n)\delta+n$ for the pair and hence obtain the optimal (n, c). The values are given in Table 2.

TABLE 2. STOCHASTIC LTPD PLAN OF KATAOKA TYPE OBJECTIVE FUNCTION

c	n	minimum δ	minimum k
2	66	0.1226	294.38
3	70	0.0672	199.70
5	96	0.0194	132.94
6	109	0.0105	128.91
7	122	0.0058	132.82

The optimal solution is $n = 109$, $c = 6$.

4. Effect of Variability and Concluding Remarks

In a production process, the parameters p_1 and p_2 cannot be known precisely and it is more appropriate to assume them as random variables. If the variability of the parameters is also taken into account in the model, then the DM will be more confident that his ultimate objective of sending very few bad lots to the market will be fulfilled. The effect of variability of the parameters on the expected average inspection is presented in Table 3.

TABLE 3. EXPECTED AVERAGE INSPECTION FOR DIFFERENT VALUES OF σ_1 and σ_2; $N = 2{,}000$, $\mu_1 = 0.02$, $\mu_2 = 0.10$, $\beta = 0.10$, $\alpha = 0.05$

σ_1	0.000			0.001			0.002			0.003		
σ_2	n	c	$E(I)$	n	c	$E(I)$	n	c	$E(I)$	n	c	$E(I)$
0.000	93	5	115.9	93	5	116.3	93	5	117.5	93	5	119.5
0.001	95	5	120.2	95	5	120.6	95	5	121.9	95	5	124.0
0.002	96	5	122.4	109	6	122.8	109	6	123.8	109	6	125.4
0.005	115	6	132.6	115	6	133.1	115	6	134.3	115.	6	136.3

From Table 3 it is seen that for small variability of p_1 and p_2, the increase in the expected average inspection is insignificant for practical applications. We may note that the classical Dodge-Romig model is robust in the sense that even for stating the values of p_1 and p_2 reasonably produces near optimal plans. Since it is well known that the deterministic Dodge-Romig optimal and its neighbouring plans have nearly same average inspection, it will be advantageous to apply the next higher neighbouring plan which will take care the possible variability of the parameters p_1 and p_2. For example, in the case of the example considered, (see Hald (1981), p. 101) the optimal plan for the deterministic case is $n = 93, c = 5, I(p_1) = 116$ and its next higher neighbouring plan is $n = 106, c = 6, I(p_1) = 117$. It is seen that it would be advantageous to apply SSP $n = 106$, $c = 6$ in the situation where it is required to assume small variability of p_1 and p_2.

The procedure developed is simple and the required plans can be designed easily for any type of distribution appropriate to an environment such as triangular, uniform, Beta etc. However, from experience it seems that assumption of normal distribution is realistic in most of the industrial applications.

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