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ON A DECOMPOSITION OF THE LIMIT DISTRIBUTION OF A SEQUENCE OF ESTIMATORS

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SUMMARY. When the likelihood function is locally asymptotically normal it is shown that the (possibily sub-stochastic) limiting kernel of a convergent sub-sequence of a sequence of estimators can be decomposed as a convolution for all points of the parameter space except perhaps for a subset of Lebesgue measure zero. An analogous conditional convolution result is obtained when the likelihood function is locally asymptotically mixed normal. By applying these results, certain global asymptotic lower bounds for risk functions of estimators are obtained. A result on the invariance of limit distributions is also obtained.

1. Introduction

When the log likelihood function is locally asymptotically normal (LAN), Hajek (1970) has established a basic result that limit distribution of a regular sequence of estimators can be decomposed as a convolution for all points of the parameter space. Independently, this result was also obtained by Inagaki (1970) under more restrictive assumptions. LeCam (1972) has extended this result to a much more general family than the LAN family. In all the above works results were obtained under a crucial restriction that the sequence of estimators are asymptotically invariant in some sense. Usual examples (see e.g. LeCam, 1953) show that this type of invariance restriction cannot be relaxed if one tries to establish such a decomposition for all points of the parameter space. LeCam (1973) discusses the above mentioned invariance restriction and obtains some deep results on the invariance of the limits of experiments; further he remarks that, as a consequence of his invariance results that the convolution result of Hajek (1970) and LeCam (1972) can be obtained, without the invariance restriction, for almost all points of the parameter space. Our aim in this paper is to state and prove a simple invariance lemma and, as a consequence of this invariance lemma, to prove that the limiting kernel of any convergent subsequence of an arbitrary sequence of estimators can be decomposed as a convolution for all points of the parameter space except perhaps for a subset of Lebesgue measure zero.

*In a book under preparation LeCam gives a very detailed discussion on the invariance of the possible limits of experiments and the limits of distributions.

When the observations are dependent, it has been recently observed (see e.g. Heyde, 1978) that the log-likelihood function is, in general, locally asymptotically mixed normal. We show that in such cases a conditional convolution result holds; under the usual invariance restriction this convolution result has been established in Jeganathan (1979a).

In Section 2 we present notations and the convolution result; proof of the convolution result is presented in Section 3. In Section 3 we prove a result on the invariance of limit distributions. The conditional convolution result is stated and a proof is briefly indicated in Section 4. As an application of those results, in Section 5 we obtain certain global asymptotic lower bounds for risk functions of estimators.

2. NOTATIONS AND THE CONVOLUTION RESULT

Let (M_n, F_n), $n \geqslant 1$, be a sequence of measurable spaces and let Θ be an open subset of R^k. Assume that $\{P_{\theta,n} | F_n : \theta \in \Theta\}$, $n \geqslant 1$, is a sequence of families of p-measures (probability measures) such that the functions $\theta \to P_{\theta,n}(A)$. $A \in F_n$, $n \geqslant 1$, are Borel measurable.

If $P | F$ and $Q | F$ are p-measures on a measurable space (M, F) then dP/dQ denotes the Radon-Nykodym derivative of the Q-continuous part of P with respect to Q. If $X : (M, F) \to (R^q, H^q)$, $q \geqslant 1$, is a Borel measurable function, where H^q being the Borel σ-field on R^q, then $\mathcal{L}(X | P)$ denotes the image of P induced by X on H^q. $N(a, B)$ denotes the normal distribution with mean $a \in R$ and covariance matrix B. For a vector $h \in R^q$, h' denotes the transpose of h and $|h|$ denotes the Euclidean norm. For a matrix B, $\|B\|$ denotes the usual norm. $\mu^k | H^k$ denotes Lebesgue measure.

We shall now state the definition of the local asymptotic mixed normality (LAMN) condition in μ^k-measure.

Definition : The sequence $\{P_{\theta,n} : \theta \in \Theta\}$, $n \geqslant 1$, satisfies the local asymptotic mixed normality (LAMN)-condition in μ^k-measure if there exist

(1) a sequence of positive definite matrices δ_n, $n \geqslant 1$, such that $\|\delta_n\| \to 0$,

(2) measurable functions $T_n(\theta)$, $n \geqslant 1$, mapping $\Theta \times M_n$ to the set of $k \times k$ symmetric matrices such that

$$P_{\theta,n}(T_n(\theta) \text{ is positive definite}) = 1 \text{ ofor every } n \geqslant 1 \text{ and } \theta \in \Theta,$$

(3) measurable functions $W_n : \Theta \times M_n \to R^k$, $n \geqslant 1$, such that, for μ^k—almost all $\theta \in \Theta$, $(T_n(\theta), W_n(\theta))$ converges weakly to $(T(\theta), W)$, where $T(\theta)$ is, for each $\theta \in \Theta$, an almost surely positive definite matrix and W is a copy of $N(0, I)$ independent of $T(\theta)$,

(4) measurable functions $Z_n(.,h) : \Theta \times M_n \rightarrow R$, $n \geqslant 1$, $h \in R$, satisfying the following condition :

For μ^k-almost all $\theta \in \Theta$, $Z_n(\theta, h)$ tends to zero in $P_{\theta,n}$-probability for every $h \in R^k$ such that

$$\frac{dP_{\theta+\delta_n h, n}}{dP_{\theta,n}} = \exp(h'T_n^{1/2}(\theta)W_n(\theta) - \frac{1}{2}h'T_n(\theta)h + Z_n(\theta, h))$$

for $n \geqslant 1$.

In the special case when $T_n(\theta) = T(\theta) =$ a constant matrix for all $\theta \in \Theta$ and $n \geqslant 1$ we say that the sequence $\{P_{\theta,n} : \theta \in \Theta\}$, $n \geqslant 1$, satisfies the local asymptotic normality (LAN)—condition in μ^k-measure.

Remarks (1): See e.g. LeCam (1970), Ibragimov and Khasminshii (1975) and Roussas (1979) for sufficient conditions for the LAN-condition as defined above.

(2) A detailed study of the general LAMN-condition is presented in Jeganathan (1979a and 1979b).

Let $G(H^q)$ be the space of all sub-stochastic measures on H^q, H^q being the Borel σ-field on R^q. Let (S, E, v) be a σ-finite measure space. Consider the (sub-stochastic) kernels $P : S \rightarrow G(H^q)$. Let C_∞ be the space of continuous functions on R^q vanishing outside compacts. Define the $C_\infty(R^q) \otimes L_1(v)$ topology of the set of all kernels to be the smallest topology such that all functions

$$P \rightarrow \iint f(x)P(t)(dx)g(t)v(dt),$$

$f \in C_\infty$, $g \in L_1(v)$, are continuous; this topology was introduced in LeCam (1973). It is known that the set of all kernels endowed with this topology is mortrizable and compact.

Let $V_n : (M_n, F_n) \rightarrow (R^k, H^k)$, $n \geqslant 1$, be a sequence of estimators. Let $\{m\} \subseteq \{n\}$ be a sub-sequence such that $\mathcal{L}(\delta_m^{-1}(V_m - \theta) \mid P_{m,\theta})$ is $C_\infty(R^k) \otimes L_1(\mu^k)$ convergent to a kernel Q_θ. We now state the convolution result.

Theorem 1 : *Suppose that the sequence $\{P_{\theta n} : \theta \in \Theta\}$, $n \geqslant 1$ satisfies the LAN-condition in μ^k-measure. Let Q_θ be as above. Then there exists a kernel K_θ on H^k and a Lebesgue null set N such that*

$$Q_\theta = N(0, T^{-1}(\theta)) * K_\theta$$

for every $\theta \in \Theta - N$, where $*$ denotes convolution.

3. Proof of theorem 1

First we shall prove the following lemma

(Invariance) Lemma 1 : *Let* $\{F_{\theta,n}\}$, $n \geqslant 1$, *be a sequence of kernels* $F_{\theta,n} : R^k \to G(H^q)$, $q, k \geqslant 1$, *being* $C_\infty(R^q) \otimes L_1(\mu^k)$ *convergent to a kernel* F_θ. *Let* δ_n, $n \geqslant 1$, *be a sequence of positive definite matrices such that* $\|\delta_n\| \to 0$ *as* $n \to \infty$. *Then the sequence* $\{F_{\theta+\delta_n h, n}\}$, $n \geqslant 1$ *is also* $C_\infty(R^q) \otimes L_1(\mu^k)$ *convergent to* F_θ *for every* $h \in R^k$.

Proof: For every subsequence $\{m\} \subseteq \{n\}$ there exists a further subsequence $\{r\} \subseteq \{m\}$ and a kernel $F_{\theta,h}$ such that the sequence $\{F_{\theta+\delta_r h_r, r}\}$ is $C_\infty(R^q) \otimes L_1(\mu^k)$ convergent to $F_{\theta,h}$. For simplicity assume that $k = 1$ and $q = 1$. We then have

$$\int_{t_1}^{t_2} \int_R f(x) F_{\theta,r}(dx) d\theta \to \int_{t_1}^{t_2} \int_R f(x) F_\theta(dx) d\theta$$

and

$$\int_{t_1}^{t_2} \int_R f(x) F_{\theta+\delta_r h, r}(dx) d\theta \to \int_{t_1}^{t_2} \int_R f(x) F_{\theta,h}(dx) d\theta$$

as $r \to \infty$ or every $t_1, t_2 \in R$ and $f \in C_\infty$. Now

$$\int_{t_1}^{t_2} \int_R f(x) F_{\theta+\delta_r h, r}(dx) d\theta = \int_{t_1+\delta_r h}^{t_2+\delta_r h} \int_R f(x) F_{\theta,r}(dx) d\theta.$$

Hence,

$$\left| \int_{t_1}^{t_2} \int_R f(x) F_{\theta+\delta_r h, r}(dx) d\theta - \int_{t_1}^{t_2} \int_R f(x) F_{\theta,r}(dx) d\theta \right|$$

$$\leqslant C\mu([t_1, t_2] \Delta ([t_1, t_2] + \delta_r h)) \text{ (for some } C > 0)$$

$$\leqslant C 2\delta_r |h| \to 0 \text{ as } r \to \infty.$$

Thus we have

$$\int_{t_1}^{t_2} \int_R f(x) F_\theta(dx) d\theta = \int_{t_1}^{t_2} \int_R f(x) F_{\theta,h}(dx) d\theta$$

for every $t_1, t_2 \in R$ and $f \in C_\infty$. This implies

$$\int f(x) F_\theta(dx) = \int_R f(x) F_{\theta,h}(dx) \text{ a.s. [Lebesgue]}$$

for every $f \in C_\infty$. This completes the proof.

We shall present the proof of the convolution result for $\Theta = R^k$; since $\Theta \subseteq R^k$ is open, the proof for the general case is essentially the same. At this point we further note that the LAMN-condition in μ^k-measure implies that the sequence $\{P_{\theta+\delta_n h, n}\}$ and $\{P_{\theta,n}\}$, $n \geqslant 1$, $h \in R^k$, are contiguous for μ^k-almost all $\theta \in \Theta$. Hence, without loss of generality, we can assume in what follows, that $P_{\theta+\delta_n h, n}$ is absolutely continuous with respect to $P_{\theta,n}$, $n \geqslant 1$, $h \in R^k$, for μ^k-almost all $\theta \in \Theta$; see e.g. Roussas (1972), Chapter 1. The following proof is based on the ideas of Bickel's simple short proof of Hajek's convolution theorem; see Roussas (1972) for a published version of Bickel's proof.

Proof of Theorem 1: Let $\{m\} \subseteq \{n\}$ be a sub-sequence such that $\mathcal{L}(\delta_m^{-1}(V_m - \theta),\ W_m(\theta) \mid P_{\theta,m})$ is $C_\infty(R^{2k}) \otimes L_1(\mu^k)$ convergent to a kernel Q'_θ. Let Q_θ be a kernel such that $\mathcal{L}(\delta_m^{-1}(V_m - \theta) \mid P_{\theta,m})$ is $C_\infty(R^k) \oplus L_1(\mu^k)$ convergent to Q_θ. Then, according to lemma 1, $\mathcal{L}(\delta_m^{-1}(V_m - \theta - \delta_m h) \mid P_{\theta+\delta_m h, m})$ is also $C_\infty(R^k) \otimes L_1(\mu^k)$ convergent to Q_θ. For simplicity we shall assume that the parameter space is of one-dimension. Let $f(u, x) = (e^{iux} - 1)/ix$, $u, x \in R$. Note that $f(u, x) \xrightarrow{|x| \to \infty} 0$ for every $u \in R$. Hence we have, setting $\theta_m = \theta + \delta_m h$.

$$\int_R \int_{M_m} f(u, \delta_m^{-1}(V_m - \theta_m)) dP_{\theta_m, m}\, g(\theta) d\theta \to \int_R \int_R f(u, x) Q_\theta(dx) g(\theta) d\theta, \quad \ldots \quad (3.1)$$

for every $g(\theta) \in L_1(\mu)$ and $u, h \in R$. On the other hand the left hand side of (3.1) can be written as

$$\int_R \int_{M_m} f(u, \delta^{-1}(V_m - \theta_m)) \frac{dP_{\theta_m, m}}{dP_{\theta, m}} dP_{\theta, m}\, g(\theta) d\theta$$

$$= \int_R \int_{M_m} f(u, \delta^{-1}(V_m - \theta_m)) \exp(h T^{1/2}(\theta) W_m(\theta) - \frac{h^2}{2} T(\theta) + Z_m(\theta, h)) dP_{\theta, m}\, g(\theta) d\theta$$

and it is not difficult to see that this converges to

$$\int_R \int_{R^2} f(u, (x-h)) \exp(h T^{1/2}(\theta) w - \frac{h^2}{2} T(\theta)) Q'(dx, dw) g(\theta) d\theta$$

for every $g(\theta) \in L_1(\mu)$ and $u, h \in R$. Hence we see that for every $(u, h) \in R^2$, there exists a Lebesgue null set $N_{(u, h)}$, possibly depending on (u, h), such that

$$\int_R f(u, x) Q\,(dx) = \int_{R^2} f(u, (x-h)) \exp\left(h T^{1/2}(\theta) w - \frac{h^2}{2} T(\theta)\right) Q'_\theta(dx, dw)$$

for every $\theta \in \Theta - N_{(u,h)}$. Let D be the set of all rationals in R^2 and let $N = \bigcup_{(u,h)\in D} N_{(u,h)}$. Then whenever $\theta \in \Theta - N$,

$$\int_R f(u, x) Q_\theta(dx) = \int_{R^2} f(u, (x-h)) \exp\left(hT^{1/2}(\theta)w - \frac{h^2}{2} T(\theta)\right) Q_\theta^*(dx, dw) \quad \dots (3.2)$$

for every $(u, h) \in D$. Now for every $(u, h) \in R^2$, there exists a sequence $(u_r, h_r) \in D$, $r \geqslant 1$, such that $(u_r, h_r) \to (u, h)$ as $r \to \infty$. Clearly, for every $\theta \in \Theta - N$.

$$\int f(u_r, x) Q_\theta(dx) \to \int f(u, x) Q_\theta(dx).$$

We shall now show that

$$\int_{R^2} f(u_r, (x-h_r)) \exp\left(h_r T^{1/2}(\theta) w - \frac{h_r^2}{2} T(\theta)\right) Q_\theta^*(dx, dw)$$

converges for every $\theta \in \Theta - N$, to

$$\int_{R^2} f(u, (x-h)) \exp\left(hT^{1/2}(\theta)w - \frac{h^2}{2} T(\theta)\right) Q_\theta^*(dx, dw) \text{ as } r \to \infty.$$

Since

$$\left| f(u_r, (x-h_r)) \exp\left(h_r T^{1/2}(\theta)w - \frac{h_r^2}{2} T(\theta)\right)\right| \leqslant C \exp\left(h_r T^{1/2}(\theta)w - \frac{h_r^2}{2} T(\theta)\right)$$

(for some $C > 0$),

it is enough to show that for every $\theta \in \Theta - N$,

$$\exp\left(h_r T^{1/2}(\theta) w - \frac{h_r^2}{2} T(\theta)\right)$$

converges in the first mean to

$$\exp\left(hT^{1/2}(\theta)w - \frac{h^2}{2} T(\theta)\right) \text{ as } r \to \infty.$$

This follows at once from the fact that

$$\exp\left(h_r T^{1/2}(\theta) w - \frac{h_r^2}{2} T(\theta)\right) \to \exp\left(hT^{1/2}(\theta)w - \frac{h^2}{2} T(\theta)\right)$$

as $r \to \infty$ for every $w \in R$ and $\theta \in \Theta$, and

$$\int_R \exp\left(h_r T^{1/2}(\theta) w - \frac{h_r^2}{2} T(\theta)\right) \bar{Q}'_\theta(\bar{R}, dw) = 1$$

for all $r \geqslant 1$ and $\theta \in \Theta - N$, where $\bar{Q}'_\theta$ is the probability measure on the compactification $\bar{R}^2$, induced by Q'_θ. (Note that $\bar{Q}'(\bar{R}, .) = N(0, 1)$ for μ-almost all $\theta \in \Theta$). Thus we see that whenever $\theta \in \Theta - N$,

$$\int_R f(u, x) Q_\theta(dx) = \lim_{r \to \infty} \int_R f(u_r, x) Q_\theta(dx)$$

$$= \lim_{r \to \infty} \int_{R^2} f(u_r, (x - h_r)) \exp\left(h_r T^{1/2}(\theta) w - \frac{h_r^2}{2} T(\theta)\right) Q'_\theta(dx, dw)$$

$$= \int_{R^2} f(u, (x-h) \exp\left(h T^{1/2}(\theta) w - \frac{h^2}{2} T(\theta)\right) Q'_\theta(dx, dw) \quad \ldots (3.3)$$

for every $(u, h) \in R^2$. Now (3.3) implies (cf. Loeve (1963), p. 189) that whenever $\theta \in \Theta - N$.

$$\int_R \exp(iux) Q_\theta(dx) = \int_{R^2} \exp(iu(x-h) + hT^{1/2}(\theta) w - \frac{h^2}{2} T(\theta)) Q'_\theta(dx, dw) \quad \ldots (3.4)$$

for every $(u, h) \in R^2$. It can be shown that the right hand side of (3.4) is analytic in h for every $\theta \in \Theta - N$. Hence replacing h by ih in (3.4) we have, whenever, $\theta \in \Theta - N$,

$$\int_R \exp(iux) Q_\theta(dx) = \exp(uh) \int_{R^2} \exp(iux + ihT^{1/2}(\theta) w + \frac{h^2}{2} T(\theta) . Q'_\theta(dx, dw) \quad \ldots (3.5)$$

for every $(u, h) \in R^2$. Setting $h = -T^{-1}(\theta) u$ in (3.5), we have

$$\int_R \exp(iux) Q_\theta(dx) = \exp(-\tfrac{1}{2} T^{-1}(\theta) u^2) \int_{R^2} \exp(iu(x - T^{-1/2} w)) Q'_\theta(dx, dw) \quad \ldots (3.6)$$

for every $u \in R$ and $\theta \in \Theta - N$. This proves the result.

4. Conditional convolution result

Let $V_n : (M_n, F_n) \to (R^k, H^k)$, $n \geqslant 1$, be a sequence of estimators. Let $\{m\} \subseteq \{n\}$ be a subsequence and Q_θ be a kernel such that

$$\{\mathcal{L}(\delta_m^{-1}(V_m - \theta), T_m(\theta) \mid P_{\theta, m})\}$$

is $C_\infty(R^{k+k^2}) \otimes L_1(\mu_k)$-convergent to Q_θ. Let $\overline{Q}_\theta$ be the probability measure on $\overline{R} \times R^{k^2}$ induced by Q_θ, where $\overline{R}^k$ is the compactification of R^k. Let $\overline{Q}_\theta^T$ be a kernel (on $\overline{R}^k$) such that

$$\overline{Q}_\theta(C) = \int I(C)\overline{Q}_\theta^t(dx)\, \mathcal{L}_\theta(dt)$$

for every Borel set $C \subseteq \overline{R}^k \times R^{k^2}$ where $\mathcal{L}_\theta$ is the law of $T(\theta)$. Let $\{r\} \subseteq \{m\}$ be a further subsequence and Q_θ' be a kernel such that

$$\{\mathcal{L}(\delta_r^{-1}(V_r-\theta),\, W_r(\theta),\, T_r(\theta) \mid P_{\theta,r})\}$$

is $C_\infty(R^{2k+k^2}) \otimes L_1(\mu_k)$-convergent to Q_θ'. Note that Q_θ and the law of $(W, T(\theta))$ are the marginals of $\overline{Q}_\theta'$ for the μ^k-almost all θ.

Theorem 2 : *Suppose that the sequence $\{P_{\theta,n} : \theta \in (h)\}$, $n \geqslant 1$, satisfies the LAMN-condition in μ^k-measure. Let $\overline{Q}_\theta^T$ be as above and let Q_θ^T be the restriction of $\overline{Q}_\theta^T$ to R^k. Then there exists a Lebesgue null set $N \subseteq \Theta$ and a kernel K_θ such that*

$$Q_\theta^T = N(0, T^{-1}(\theta)) * K_\theta$$

for every $\theta \in \Theta - N$.

The following Lemma is implicit in LeCam (1974, Ch. 12).

Lemma 2 : *Suppose that the sequence $\{P_{\theta,n} : \theta \in \Theta\}$, $n \geqslant 1$, satisfies the LAMN-condition in μ^k-measure. Then there are measurable functions $T_n^*(\theta)$, $n \geqslant 1$, mapping $\Theta \times M_n$ to the set of $k \times k$ symmetric matrices such that, for μ^k-almost all $\theta \in \Theta$,*

(i) *The difference $T_n^*(\theta) - T_n(\theta)$ tends to zero in $P_{\theta+\delta_n h, n}$ probability for every $h \in R^k$, and*

(ii) *the difference $T_n^*(\theta+\delta_n h) - T_n^*(\theta)$ tends to zero in $P_{\theta,n}$ probability for every $h \in R^k$.*

Proof : For simplicity assume that the parameter space is of one-dimension. First note that

$$T_n(\theta) = -4[\Lambda_n^*(\theta, 1) - 2\Lambda_n^*(\theta, 1/2)]$$

where

$$\Lambda_n^*(\theta, h) = hT_n^{1/2}(\theta)W_n(\theta) - \frac{h^2}{2}T_n(\theta).$$

Now set

$$T_n^*(\theta) = -4[\Lambda_n(\theta, 1) - 2\Lambda_n(\theta, 1/2)]$$

where

$$\Lambda_n(\theta, h) = \log \frac{dP_{\theta+\delta_n h, n}}{dP_{\theta, n}}$$

The first part of the lemma follows from the fact that the difference $\Lambda_n(\theta, h) - \Lambda_n^*(\theta, h)$ tends to zero in $P_{\theta+\delta_n s, n}$ probability for every $s, h \in R$.

Next observe that

$$\Lambda_n(\theta+\delta_n h, s) = \Lambda_n(\theta, h+s) - \Lambda_n(\theta, h).$$

Hence the second part of the lemma follows from the fact that

$$\left[\Lambda_n^*(\theta, h+1) - \Lambda_n^*(\theta, h) - 2\Lambda_n^*\left(\theta, h+\frac{1}{2}\right) + 2\Lambda_n^*(\theta, h)\right] = -\frac{1}{4} T_n(\theta).$$

This proves the lemma.

Proof of the Theorem 2 : For simplicity we assume that the parameter space is of one-dimension. In view of lemma 1 and lemma 2 it easily follows that the sequences

$$\mathcal{L}(T_m^*(\theta), \delta_m^{-1}(V_m - \theta) \mid P_{\theta, m}) \text{ and } \mathcal{L}(T_m^*(\theta), \delta_m^{-1}(V_m - \theta_m) \mid P_{\theta_m, m})$$

are $C_\infty(R^2) \otimes \Gamma_1(\mu)$ convergent to the same kernel Q_θ. Hence proceeding as in the proof of Theorem 1, it is easily seen that there exists a Lebesgue null set N, such that

$$\int_{R^2} \exp(iux+ivt)\bar{Q}_\theta'(dx, \bar{R}, dt)$$

$$= \int_{R^3} \exp\left(iu(x-h)+ivt+ht^{1/2}w-\frac{h^2}{2}t\right) \bar{Q}_\theta'(dx\ dw, dt)$$

for every $\theta \in \Theta - N$ and $h, u, v \in R$. This implies that the corresponding conditional characteristics are also equal almost surely i.e.

$$\int_R \exp(iux)\bar{Q}_\theta^T(dx) = \int_{R^2} \exp\left(iu(x-h)+hT^{1/2}(\theta)w-\frac{h^2}{2}T(\theta)\right) \bar{Q}_\theta'^T(dx, dw)$$

for every $\theta \in \Theta - N$ and $u, h \in R$, where $\bar{Q}_\theta'^T$ denotes a regular conditional probability measure of $\bar{Q}_\theta'$ given $T(\theta)$. Now using a simple continuity argument and then proceeding as in the proof of the Theorem 1, we obtain the desired conclusion.

5. Some applications

Let L be the class of all loss functions $l : R^k \to R$ of the form $l(0) = 0$, $l(y) = (|y|)$ and $l(y) \leqslant l(z)$ if $|y| \leqslant |z|$. Let λ be a σ-dfinite measure on (R^k, H^k) such that $\lambda \ll \mu$, and let $L^+(\lambda)$ be the class of all positive integrable functions on (R^k, H^k, λ). The distribution of $T^{-1}(\theta)W$ will be denoted by ϕ_θ.

Proposition 1 : Suppose that the sequence $\{P_{\theta, n} : \theta \in \Theta\}$, $n \geqslant 1$, satisfies the LAN condition in μ^k-measure. Let $\{V_n\}$, $n \geqslant 1$, be a sequence of estimators. Then

$$\liminf_{n \to \infty} \int_{\Theta} \int_{M_n} l(\delta_n^{-1}(V_n-\theta))b(\theta)dP_{\theta,n}\lambda(d\theta) \geqslant \int_{\Theta} \int_{R_k} l(x)b(\theta)\phi_\theta(dx)\lambda(d\theta) \quad \ldots \quad (5.1)$$

for every $l \in L$ and $b \in L_1^+(\lambda)$.

Proof : The proof is an easy consequence of Theorem 1.

Proposition 2 : Suppose that the sequence $\{P_{\theta, n} : \theta \in \Theta\}$, $n \geqslant 1$, satisfies the LAMN-condition in μ^k-measure. Let $\{V_n\}$, $n \geqslant 1$, be a sequence of estimators. Then

$$\liminf_{n \to \infty} \int_{\Theta} \int_{M_n} l(\delta_n^{-1}(V_n-\theta))b(\theta)dP_{\theta,n}\lambda(d\theta) \geqslant \int_{\Theta} \int_{R_k} l(x)b(\theta)\phi_\theta(dx)\lambda(d\theta) \quad \ldots \quad (5.2)$$

for every $l \in L$ and $b \in L_1^+(\lambda)$.

Proof : The proof is an easy consequence of Theorem 2.

Remarks (1) : Proposition 1 occur explicitly in the form given here in Strasser (1978) for the LAN case; this result seems to be implicit for the LAN-case in, for example, LeCam (1973) also since Theorem 1 was essentially mentioned in this paper for the LAN case.

(2) In connection with the results of the present paper, it should be mentioned here that when the sequence of estimators are asymptotically normal, LeCam (1953) and Bahadur (1964) have shown that the variance of the limit distribution is greater than or equal to the reciprocal of the Fisher information number for almost all points of the parameter space; this result can be easily deduced from Theorem 1. A more general result of Pfanzagl (1970, Theorem 2) can also be deduced from Theorem 1; an analogous result under the LAMN-condition can be deduced from Theorem 2.

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