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HERMITIAN AND NONNEGATIVE DEFINITE SOLUTIONS OF SOME MATRIX EQUATIONS CONNECTED WITH DISTRIBUTION OF QUADRATIC FORMS

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SUMMARY. In this paper we obtain most general hermitian and nnd solutions X to each of the following matrix equations. (We consider two cases (i) B is hermitian and (ii) B is n. n. d.)

(1) $BXBXB = BXB$

(2) $XBXBX = XBX$

We thus determine (a) the class of all normal distributions $N_p(0, \Sigma)$ of Y under which a given quadratic from $Y'AY$ is distributed as a chi-square. We also determine the class of all quadratic forms $Y'BY$ which are distributed independently of $Y'AY$ and also the class of all $Y'BY$ distributed as a chi-square independently of $Y'BY$ when $Y \sim N_p(0, \Sigma)$.

1. Introduction

Let $\boldsymbol{Y} \sim N_p(\boldsymbol{0}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ is a nonnegative definite (n.n.d.) matrix. It is well known (see Rao, 1972 and Khatri 1963) that $\boldsymbol{Y'AY}$, where $\boldsymbol{A}$ is a real symmetric matrix, has a chi-square distribution if and only if

$$\boldsymbol{\Sigma A \Sigma A \Sigma} = \boldsymbol{\Sigma A \Sigma} \qquad \ldots \quad (1.1)$$

Further, $\boldsymbol{Y'AY}$ and $\boldsymbol{Y'BY}$ are independently distributed if and only if $\boldsymbol{\Sigma A \Sigma B \Sigma} = \boldsymbol{0}$.

The matrix equation

$$\boldsymbol{BXBXB} = \boldsymbol{BXB} \qquad \ldots \quad (1.2)$$

was considered by Mitra (1968) and

$$\boldsymbol{XBXBX} = \boldsymbol{XBX} \qquad \ldots \quad (1.3)$$

by Mitra and Bhimasankaram (1970), where they obtained the most general solutions for an arbitrary complex matrix $\boldsymbol{B}$. Mitra (1968) also obtained the most general hermitian solution to (1.2) when $\boldsymbol{B}$ is hermitan, whereby one obtains the class of all quadratic forms $\boldsymbol{Y'AY}$ which is distributed as a chi-square for a given distribution $N_p(\boldsymbol{0}, \boldsymbol{\Sigma})$ of $\boldsymbol{Y}$.

In this paper we determine the class of all normal distributions $N_p(\boldsymbol{0}, \boldsymbol{\Sigma})$ of $\boldsymbol{Y}$ under which a given quadratic form $\boldsymbol{Y'AY}$ is distributed as a chi-square.

Moreover, we find the class of all quadratic forms $\boldsymbol{Y'BY}$ which are distributed independently of $\boldsymbol{Y'AY}$ and also the class of all $\boldsymbol{Y'BY}$ distributed as a chi-square independently of $\boldsymbol{Y'AY}$, when $\boldsymbol{Y} \sim N_p(\boldsymbol{0}, \boldsymbol{\Sigma})$.

In Section 2, we prove some preliminary results some of which are needed in the later sections and which are also of independent interest. In Sections 3 and 4 we obtain the most general hermitian and n.n.d. solutions to (1.2) and (1.3) respectively, when $\boldsymbol{B}$ is hermitian or n.n.d. In Section 5 we consider the problem of independence of two quadratic forms.

We consider matrices over the field of complex numbers and follow the same notations as in Rao and Mitra ((1971). Further for a matrix $\boldsymbol{A}$, $\boldsymbol{A}_s^-$, $\boldsymbol{A}_h^-$ and $\boldsymbol{A}_n^-$ denote respectively symmetric, hermitian and n.n.d g-inverses of $\boldsymbol{A}$, and $\boldsymbol{A}_{rh}^-$, $\boldsymbol{A}_{rn}^-$ denote reflexive hermitian and reflexive n.n.d g-inverses of $\boldsymbol{A}$. Whenever we consider the statistical aspects we deal with matrices over the real field. Throughout the paper orthogonal projection operators are used with reference to unitary inner product.

The inspiration for the present paper came from Khatri and Mitra (1976).

2. Preliminary results

We state the following well-known lemma (see Rao and Mitra, 1971, page 17) which we need in the sequel.

Lemma 2.1 : *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices of same order. Then $\boldsymbol{A^*A} = \boldsymbol{B^*B}$ if and only if $\boldsymbol{A} = \boldsymbol{UB}$ where $\boldsymbol{U}$ is a unitary matrix.*

We prove

Lemma 2.2 : *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices of orders $n_1 \times m$ and $n_2 \times m$ respectively. Then $\boldsymbol{A^*A} = \boldsymbol{B^*B}$ if and only if $\boldsymbol{A} = \boldsymbol{UB}$ where $\boldsymbol{U}$ is a semiunitary matrix such that*

$$\mathcal{M}(\boldsymbol{B}) \subset \mathcal{M}(\boldsymbol{U^*}).$$

Proof: If $n_1 \geqslant n_2$ the proof is trivial. Let $n_1 < n_2$. If part is trivial because $\mu(\boldsymbol{B}) \subset \mu(\boldsymbol{U^*}) \Longrightarrow \boldsymbol{U^*UB} = \boldsymbol{B}$ (since $\boldsymbol{U}$ is semiunitary, $\boldsymbol{U^*} = \boldsymbol{U^+}$). To prove the only if part, first observe that in view of Lemma 2.1, there exists a semiunitary matrix $\boldsymbol{U}$ such that $\boldsymbol{A} = \boldsymbol{UB}$ and $\boldsymbol{UU^*} = \boldsymbol{I}$. Now,

$$\boldsymbol{A^*A} = \boldsymbol{B^*B} \Longrightarrow \boldsymbol{B^*U^*}\ \boldsymbol{UB} = \boldsymbol{B^*B}.$$

$$\Longrightarrow \boldsymbol{B^*(I-U^*U)}\ \ \boldsymbol{B} = \boldsymbol{0} \Longrightarrow (\boldsymbol{I}-\boldsymbol{U^*U})\ \boldsymbol{B} = \boldsymbol{0}.$$

$$\Longrightarrow \mu(\boldsymbol{B}) \subset \mu(\boldsymbol{U^*}),$$

since $\boldsymbol{I}-\boldsymbol{U^*U}$ is an orthogonal projection operator,
This completes the proof.

Let
$$\boldsymbol{B} = \boldsymbol{P}\begin{pmatrix} \boldsymbol{\Delta}_1 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{\Delta}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\boldsymbol{P}^*$$

where $\boldsymbol{P}$ is nonsingular, $\boldsymbol{\Delta}_1$ of order $r_1 \times r_1$ and $\boldsymbol{\Delta}_2$ of order $r_2 \times r_2$ are diagonal with positive diagonal elements be a hermitian matrix of order n. Let $\boldsymbol{D}$ be any matrix partioned as $\boldsymbol{D} = (\boldsymbol{P}^*)^{-1}\begin{pmatrix} \boldsymbol{L} \\ \boldsymbol{M} \\ \boldsymbol{N} \end{pmatrix}$ where the number of rows in $\boldsymbol{L}$, $\boldsymbol{M}$ and $\boldsymbol{N}$ are r_1, r_2 and $n-r_1-r_2$ respectively. ... (2.1)

Theorem 2.1 : *Consider the set up in* (2.1). *Given* $\boldsymbol{B}$, $\boldsymbol{D}^*\boldsymbol{B}\boldsymbol{D}$ *is n.n.d if and only if*

$\boldsymbol{N}$ *is arbitrary,*

$\boldsymbol{M}$ *is an arbitrary matrix of rank* $\leqslant r_1$, *and*

$\boldsymbol{L} = \boldsymbol{\Delta}_1^{-1/2}\boldsymbol{U}\boldsymbol{Q}$ *where*

$\boldsymbol{Q}^*\boldsymbol{Q} = \boldsymbol{M}^*\boldsymbol{\Delta}_2\boldsymbol{M} + \boldsymbol{K}^*\boldsymbol{K}$,

$\boldsymbol{K}$ *being arbitrary subject to the condition,* $R(\boldsymbol{Q}) \leqslant r_1$ *and*

$\boldsymbol{U}$ *is an arbitrary semiunitary matrix such that* $\mu(\boldsymbol{Q}) \subset \mu(\boldsymbol{U}^*)$.

Proof: Observe that $\boldsymbol{D}^*\boldsymbol{B}\boldsymbol{D} = \boldsymbol{L}^*\boldsymbol{\Delta}_1\boldsymbol{L} - \boldsymbol{M}^*\boldsymbol{\Delta}_2\boldsymbol{M}$. Now, $\boldsymbol{D}^*\boldsymbol{B}\boldsymbol{D}$ is *n.n.d*

$$\iff \boldsymbol{L}^*\boldsymbol{\Delta}_1\boldsymbol{L} - \boldsymbol{M}^*\boldsymbol{\Delta}_2\boldsymbol{M} = \boldsymbol{K}^*\boldsymbol{K} \text{ for some } \boldsymbol{K}$$
$$\iff \boldsymbol{L}^*\boldsymbol{\Delta}_1\boldsymbol{L} = \boldsymbol{M}^*\boldsymbol{\Delta}_2\boldsymbol{M} + \boldsymbol{K}^*\boldsymbol{K} \text{ for some } \boldsymbol{K}.$$

'If part'. If $\boldsymbol{M}$, $\boldsymbol{N}$, $\boldsymbol{K}$, $\boldsymbol{U}$ and $\boldsymbol{L}$ are chosen as in the hypothesis, clearly $\boldsymbol{L}^*\boldsymbol{\Delta}_1\boldsymbol{L} = \boldsymbol{Q}^*\boldsymbol{Q} = \boldsymbol{M}^*\boldsymbol{\Delta}_2\boldsymbol{M} + \boldsymbol{K}^*\boldsymbol{K}$ and hence $\boldsymbol{D}^*\boldsymbol{B}\boldsymbol{D} = \boldsymbol{K}^*\boldsymbol{K}$.

'Only if' part. $\boldsymbol{D}^*\boldsymbol{B}\boldsymbol{D}$ is n.n.d

$$\implies \boldsymbol{L}^*\boldsymbol{\Delta}_1\boldsymbol{L} - \boldsymbol{M}^*\boldsymbol{\Delta}_2\boldsymbol{M} \text{ is n.n.d.}$$

Write
$$\boldsymbol{L}^*\boldsymbol{\Delta}_1\boldsymbol{L} = \boldsymbol{M}^*\boldsymbol{\Delta}_2\boldsymbol{M} + \boldsymbol{K}^*\boldsymbol{K}.$$

Notice that

$$R(\boldsymbol{M}) = R(\boldsymbol{M}^*\boldsymbol{\Delta}_2\boldsymbol{M}) \leqslant R(\boldsymbol{L}^*\boldsymbol{\Delta}_1\boldsymbol{L}_1) \leqslant r_1.$$

Let $\boldsymbol{Q}$ be a matrix such that

$$\boldsymbol{Q}^*\boldsymbol{Q} = \boldsymbol{M}^*\boldsymbol{\Delta}_2\boldsymbol{M} + \boldsymbol{K}^*\boldsymbol{K}$$

Thus $L^*\Delta_1 L = Q^*Q$. Now by Lemma 2.2 it follows that $L = \Delta_1^{-1/2} UQ$ where U is a semiunitary matrix such that $\mu(Q) \subset \mu(U^*)$.

Remark : If $r_1 \geqslant r_2$, then M can be chosen arbitrarily. The rank condition on Q is the most general since $X^*C^*CX = E^*E$ is consistent if and only if $R(E) \leqslant R(C)$.

Theorem 2.2 : *Consider* B *as in* (2.1). *The most general solution to* $X^*BX = 0$ *is*

$$X = (P^*)^{-1} \begin{pmatrix} Y \\ \Delta_2^{-1/2} U \Delta_1^{1/2} Y \\ Z \end{pmatrix}$$

where Z *is arbitrary,* Y *is an arbitrary matrix of rank* $\leqslant r_2$ *and* U *is an arbitrary semiunitary matrix such that*

$$\mu(\Delta_1^{1/2} Y) \subset \mu(U^*).$$

Proof follows from Lemma 2.2 along the lines of Theorem 2.1.

Theorem 2.3 : *Let* B *be a hermitian matrix of order* $n \times n$.

(a) *The most general hermitian solution to* $XBX = X$ *is*

$$X = D(D^*BD)^-_{rh} D^*$$

where D *is an arbitrary matrix with* n *rows.*

(b) *The most general n.n.d solution to* $XBX = X$ *is*

$$X = D(D^*BD)^-_{rn} D^*$$

where D *is any matrix with* n *rows chosen as in Theorem* 2.1.

Proof follows along the same lines as Theorem 3a of Mitra (1968).

Remark : If in the above theorem B is n.n.d. then a general n.n.d solution to $XBX = X$ is

$$X = D(D^*BD)^-_{rn} D^*$$

where D is arbitrary.

3. The equation $BXBXB = BXB$

We consider the cases where B is hermitian and B is n.n.d. For each, we consider hermitian and n.n.d solutions of $BXBXB = BXB$. Thus four cases arise.

Theorem 3.1 (Mitra, 1968) : *Let B be a hermitian matrix. The most general hermitian solution to (1.2) is*

$$X = D(D^*BD)^-_{rh} D^* + E - B^{-*}BEBB^-$$

where E is an arbitrary hermitian matrix and D is arbitrary.

Remark : A general hermitian solution to $BXBXB = BXB$ when B is n.n.d is of the same form as in Theorem 3.1 since every n.n.d matrix is hermitian. However, with no loss in generality, we may write

$$X = D(D^*BD)^-_{h} D^* + E - B^{-*}BEBB^-. \qquad \ldots \quad (3.1)$$

This determines the class of all quadratic forms $Y'XY$ which are central chi-squares when $Y \sim N_n(0, B)$.

Corollary 3.1.1 : *Let K be any given matrix. The class of all hermitian matrices X such that KXK^* is an orthogonal projection operator is given by X as in* (3.1) *where $B = K^*K$.*

Proof follows once it is noticed that for $B = K^*K$

$$BXBXB = BXB \Longleftrightarrow KXK^* \text{ is hermitian and idempotent.}$$

Theorem 3.2 : *Let B be a hermitian matrix. The most general n.n.d solution to* (1.2) *is $X = L^*L$ where*

$$L = UFQ^{*-} + Z(I - Q^*Q^{*-})$$

where $B = Q^*\Delta Q$, Δ being nonsingular,

F is any matrix such that $F^*F = D(D^*\Delta D)^-_{rn} D^*$

D being an arbitrary matrix such that $D^*\Delta D$ is n.n.d,

U is an arbitrary semiunitary matrix such that $\mu(F) \subset \mu(U^*)$ and

Z is arbitrary.

Proof : When $B = Q^*\Delta Q$ where Δ is nonsingular and $X = L^*L$,

$$BXBXB = BXB \Longleftrightarrow QL^*LQ^*\Delta QL^*LQ^* = QL^*LQ^*$$

$$\Longleftrightarrow QL^*LQ^* = D(D^*\Delta D)^-_{rn} D^* = L^*L$$

where $D^*\Delta D$ is n.n.d, by Theorem 2.3

$$\Longleftrightarrow LQ^* = UF$$

where $\mu(F) \subset \mu(U^*)$ by Lemma 2.2

$$\Longleftrightarrow L = \acute{U}FQ^{*-} + Z(I - Q^*Q^{*-})$$

where Z is arbitrary.

Corollary 3.2.1 : *X as specified in Theorem 3.2 is the most general n.n.d solution to $BXBX = BX$ (or equivalently $XBXB = XB$) where B is hermitian.*

Proof follows from Theorem 3.2 once it is observed that $BXBXB = BXB \Longleftrightarrow BXBX = BX$ when X is n.n.d and B is hermitian.

As before, the most general n.n.d solution X to (1.2) where B is n.n.d can be obtained from Theorem 3.2. However, we give below an alternative simple form for the same.

Theorem 3.3 : *Let B be a n.n.d matrix. The most general n.n.d solution to (1.2) is*

$$X = C^-PC^{-*} + (I - C^-C)U(I - C^-C)^*$$

where

P is an arbitrary orthogonal projection operator

C is any matrix such that $C^*C = B$ is a rank factorization of B

C^- is an arbitrary g-inverse of C, and

U is an arbitrary n.n.d matrix.

Proof: Let C be a matrix such that $B = C^*C$ is a rank factorisation of B. Now, $BXBXB = BXB \Longleftrightarrow CXC^*$ is hermitian and idempotent. $\Longleftrightarrow CXC^* = P$ where P is an arbitrary orthogonal projection operator.

By Lemma 2.1 of Khatri and Mitra (1976) the most general solution X is given by

$$X = C^-PC^{-*} + (I - C^-C)U(I - C^-C)^*$$

where

C^- is an arbitrary g-inverse of C, and

U is an arbitrary n.n.d matrix.

4. The equation $XBXBX = XBX$

Theorem 4.1 : *Let* B *be a hermitian matrix. The most general hermitian solution to* (1.3) *is given by*

$$X = D(D^*BD)_{rh}^- D^* + Z$$

where D *is arbitrary and*

$$Z = P\Delta P^*$$

where Δ *is an arbitrary real diagonal matrix and*

$$P = (I - D(D^*BD)_{rh}^- D^*B)Y$$

where Y *is an arbitrary solution of*

$$Y^*(B - BD(D^*BD)_{rh}^- D^*B)\, Y = 0$$

Proof: It is easy to observe along the lines of Lemma 4.3 of Mitra and Bhimasankaram (1970) that the most general hermitian solution to (1.3) is given by

$$X = W + Z$$

where W is a general hermitian solution to $WBW = W$ and Z is a general hermitian solution to

$$WBZ = 0$$

and

$$ZBZ = 0.$$

Now from Theorem 2.3(a) the general solution W to $WBW = W$ is given by

$$W = D(D^*BD)_{rh}^- D^*$$

where D is arbitrary.

Let Y be a general solution of

$$Y^*(B - BD(D^*BD)_{rh}^- D^*B)Y = 0.$$

Write

$$D(D^*BD)_{rh}^- D^* = W.$$

Now observe that, $B - BWB = (I - WB)^* B(I - WB)$. Hence Y is the general solution to $Y^* (I - WB)^* B(I - WB)Y = 0$ Thus $P = (I - WB)Y$ is the general solution to $WBP = 0$ and $P^*BP = 0$. Hence for any real diagonal matrix Δ, $WBP\Delta P^* = 0$, $P\Delta P^*BP\Delta P^* = 0$ and $P\Delta P^*$ is hermitian.

Conversely, let $\boldsymbol{Z}$ be a hermitian solution to $\boldsymbol{WBZ} = \boldsymbol{0}$ and $\boldsymbol{ZBZ} = \boldsymbol{0}$. Write $\boldsymbol{Z} = \boldsymbol{P\Delta P^*}$ where $\boldsymbol{\Delta}$ is nonsingular diagonal. Clearly $\boldsymbol{WBP} = \boldsymbol{0}$ and $\boldsymbol{P^*BP} = \boldsymbol{0}$. Hence $\boldsymbol{P} = (\boldsymbol{I}-\boldsymbol{WB})\boldsymbol{Y}$ where $\boldsymbol{Y}$ satisfies $\boldsymbol{Y}^*(\boldsymbol{I}-\boldsymbol{WB})^*\boldsymbol{B}(\boldsymbol{I}-\boldsymbol{WB})\boldsymbol{Y} = \boldsymbol{0}$, which is the same as $\boldsymbol{Y}^*(\boldsymbol{B}-\boldsymbol{BWB})\boldsymbol{Y} = \boldsymbol{0}$.

Remark : The general solution to $\boldsymbol{Y}^*(\boldsymbol{B}-\boldsymbol{BWB})\boldsymbol{Y} = \boldsymbol{0}$ is obtained using Theorem 2.2.

Theorem 4.2 : *Consider the same set up as in* (2.1). *The most general n.n.d solution to* (1.3) *is given by*

$$\boldsymbol{X} = \boldsymbol{DD}^*$$

where

$\boldsymbol{N}$ *is arbitrary,*

$\boldsymbol{M}$ *is an arbitrary matrix of rank* $\leqslant r_1$, *and*

$$\boldsymbol{L} = \boldsymbol{\Delta}_1^{-1/2}\,\boldsymbol{UQ}$$

where

$$\boldsymbol{Q}^*\boldsymbol{Q} = \boldsymbol{M}^*\boldsymbol{\Delta}_2\boldsymbol{M}+\boldsymbol{K}^*\boldsymbol{K},$$

$\boldsymbol{K}$ *being an arbitrary semiunitary matrix such that* $R(\boldsymbol{Q}) \leqslant r_1$ *and* $\boldsymbol{U}$ *is a semiunitary matrix such that* $\mu(\boldsymbol{Q}) \subset \mu(\boldsymbol{U}^*)$.

Proof : Write $\boldsymbol{X} = \boldsymbol{DD}^*$.

First observe that $\boldsymbol{XBXBX} = \boldsymbol{XBX} \Longleftrightarrow \boldsymbol{D}^*\boldsymbol{BD}$ is an orthogonal projection operator

$\Longleftrightarrow \boldsymbol{D}^*\boldsymbol{BD} = \boldsymbol{K}^*\boldsymbol{K}$ where $\boldsymbol{KK}^* = \boldsymbol{I}$ or where $\boldsymbol{K}$ is a semiunitary matrix.

Now the proof follows in the same lines as in Theorem 2.1.

Remark : Let $\boldsymbol{Y}$ have a p variate normal distribution with null mean vector. Theorem 4.2 determines the class of all dispersion matrices $\boldsymbol{X}$, and hence the distributions of $\boldsymbol{Y}$ such that $\boldsymbol{Y}'\boldsymbol{BY}$ has a central chi-square distribution.

The most general hermitian solution to (1.3) when $\boldsymbol{B}$ is n.n.d was obtained by Mitra (1968). Only one needs to observe the equivalence of (1.3) and $\boldsymbol{BXBX} = \boldsymbol{BX}$ in this case. For completeness we quote below Mitra's theorem.

Theorem 4.3 (Mitra) : *If* $\boldsymbol{B}$ *is n.n.d, the general hermitian solution to* $\boldsymbol{BXBX} = \boldsymbol{BX}$ (*or equivalently* $\boldsymbol{XBXBX} = \boldsymbol{XBX}$) *is given by* $\boldsymbol{X} = \boldsymbol{C}^*(\boldsymbol{CBC}^*)^-\,\boldsymbol{CBC}^*(\boldsymbol{CBC}^*)^{-*}\boldsymbol{C}+\boldsymbol{FDF}^*$ *where* $\boldsymbol{C}$ *and* $\boldsymbol{F}$ *are arbitrary except that* $\boldsymbol{BF} = \boldsymbol{0}$ *and* $\boldsymbol{D}$ *is an arbitrary diagonal matrix with real elements.*

Let $\boldsymbol{B}$ be a n.n.d matrix. Let $\boldsymbol{X}$ be n.n.d. $\boldsymbol{XBXBX} = \boldsymbol{XBX} \Longleftrightarrow \boldsymbol{XBXB} = \boldsymbol{XB} \Longleftrightarrow \boldsymbol{BXBXB} = \boldsymbol{BXB}$. Hence the general n.n.d solution to $\boldsymbol{XBXBX} = \boldsymbol{XBX}$ where $\boldsymbol{B}$ is n.n.d is given by Theorem 3.3.

Notice that when $\boldsymbol{B}$ and $\boldsymbol{X}$ are n.n.d, $\boldsymbol{XBXBX} = \boldsymbol{XBX} \Longleftrightarrow \boldsymbol{KBK}^*$ is an orthogonal projection operator where $\boldsymbol{K}$ is any matrix such that $\boldsymbol{X} = \boldsymbol{K}^*\boldsymbol{K}$. Thus we determine the class of all $\boldsymbol{K}$ such that $\boldsymbol{KBK}^*$ is an orthogonal projection operator. Mitra and Bhimasankaram (1970) obtained an expression for the same (see their Lemma 3.3). We deduce below yet another alternative expression which is simpler than the above.

Theorem 4.4 : *The most general form of* $\boldsymbol{K}$ *such that* $\boldsymbol{KBK}^*$ *is an orthogonal projection operator where* $\boldsymbol{B} = \boldsymbol{CC}^*$ *is any n.n.d matrix is*

$$\boldsymbol{K} = \boldsymbol{QUC}^- + \boldsymbol{Z}(\boldsymbol{I} - \boldsymbol{CC}^-)$$

where $\boldsymbol{Q}$ *and* $\boldsymbol{U}$ *are arbitrary semiunitary matrices such that*

$$R(\boldsymbol{Q}) \leqslant R(\boldsymbol{B}),$$

$$\mu(\boldsymbol{Q}^*) \subset \mu(\boldsymbol{U})$$

and

$$\boldsymbol{Z} \text{ is arbitrary.}$$

Proof: $\boldsymbol{KBK}^*$ is an orthogonal projection operator $\Longleftrightarrow \boldsymbol{KCC}^*\boldsymbol{K}^* = \boldsymbol{QQ}^*$ where $\boldsymbol{Q}^*\boldsymbol{Q} = \boldsymbol{I}$ and $R(\boldsymbol{Q}) \leqslant R(\boldsymbol{B})$. Now the proof follows from Lemma 2.2.

5. Independence of quadratic forms in normal variables

Let $\boldsymbol{Y} \sim N_p(\boldsymbol{0}, \boldsymbol{\Sigma})$. It is well known (see Rao and Mitra, 1971) that $\boldsymbol{Y}'\boldsymbol{\Delta Y}$ and $\boldsymbol{Y}'\boldsymbol{BY}$ are independently distributed if and only if $\boldsymbol{\Sigma A \Sigma B \Sigma} = \boldsymbol{0}$. In this section we determine the class of all $\boldsymbol{B}$ such that given $\boldsymbol{A}$ and $\boldsymbol{\Sigma}$, $\boldsymbol{Y}'\boldsymbol{BY}$ is (i) independently distributed of $\boldsymbol{Y}'\boldsymbol{AY}$ and (ii) distributed as central χ^2 independently of $\boldsymbol{Y}'\boldsymbol{AY}$.

Theorem 5.1 : *Let* $\boldsymbol{Y} \sim N_p(\boldsymbol{0}, \boldsymbol{\Sigma})$. *Let* $\boldsymbol{A}$ *be a given real symmetric matrix. Then the class of all real symmetric matrices* $\boldsymbol{B}$ *such that* $\boldsymbol{Y}'\boldsymbol{BY}$ *is independently distributed of* $\boldsymbol{Y}'\boldsymbol{AY}$ *is given by*

$$\boldsymbol{B} = (\boldsymbol{C}+\boldsymbol{\Sigma})^-(\boldsymbol{Y}+\boldsymbol{Z})(\boldsymbol{C}+\boldsymbol{\Sigma})^{-\prime}$$

$$+\boldsymbol{U} - (\boldsymbol{C}+\boldsymbol{\Sigma})^-(\boldsymbol{C}+\boldsymbol{\Sigma})\boldsymbol{U}(\boldsymbol{C}+\boldsymbol{\Sigma})(\boldsymbol{C}+\boldsymbol{\Sigma})^{-\prime}$$

where

$C = \boldsymbol{\Sigma A \Sigma A \Sigma}$

$\boldsymbol{U}$ is an arbitrary real symmetric matrix, and

$\boldsymbol{Y}$ and $\boldsymbol{Z}$ are arbitrary real symmetric solutions of the equations

$$\boldsymbol{Y}(\boldsymbol{C}+\boldsymbol{\Sigma})^{-}\boldsymbol{\Sigma} = \mathbf{0}$$

$$\boldsymbol{C}(\boldsymbol{C}+\boldsymbol{\Sigma})^{-}\boldsymbol{Z} = \mathbf{0}.$$

Proof: $\quad \boldsymbol{\Sigma A \Sigma B \Sigma} = \mathbf{0} \Longleftrightarrow \boldsymbol{\Sigma A \Sigma A \Sigma B \Sigma} = \mathbf{0}.$

A general solution to $\boldsymbol{B}$ of the second equation above is given by Theorem 2.4 of Khatri and Mitra (1976).

Theorem 5.2 : *Consider the set up as in Theorem 5.1. The class of all $\boldsymbol{B}$ such that $\boldsymbol{Y'BY}$ is distributed as central χ^2 independently of $\boldsymbol{Y'AY}$ is*

$$\boldsymbol{B} = \boldsymbol{D}(\boldsymbol{D}'\boldsymbol{\Sigma D})_s^{-}\boldsymbol{D}'+\boldsymbol{E}-\boldsymbol{\Sigma}^{-}\boldsymbol{\Sigma E \Sigma \Sigma}^{-\prime}$$

where

$\boldsymbol{E}$ is an arbitrary real symmetric matrix, and

$\boldsymbol{D}$ is an arbitrary solution of $\boldsymbol{\Sigma A \Sigma D} = \mathbf{0}$.

Proof: $\boldsymbol{Y'BY}$ has a central χ^2

$$\Longleftrightarrow \boldsymbol{B} = \boldsymbol{D}(\boldsymbol{D}'\boldsymbol{\Sigma D})_s^{-}\boldsymbol{D}'+\boldsymbol{E}-\boldsymbol{\Sigma}^{-}\boldsymbol{\Sigma E \Sigma \Sigma}^{-\prime}$$

where

$\boldsymbol{D}$ is arbitrary and $\boldsymbol{E}$ is an arbitrary real symmetric matrix (by (3.1)).

In view of the above,

$$\boldsymbol{\Sigma A \Sigma B \Sigma} = \mathbf{0}$$

$$\Longleftrightarrow \boldsymbol{\Sigma A \Sigma D}(\boldsymbol{D}'\boldsymbol{\Sigma D})_s^{-} = \mathbf{0}$$

$$\Longleftrightarrow \boldsymbol{\Sigma A \Sigma D} = \mathbf{0}.$$

Since this paper was written, it has come to our attention that Baksalary, Hanke and Kala (1980) have obtained expressions for the general solutions in Theorem 2.3(b) (with no restrictions on $\boldsymbol{B}$) and Theorem 4.2. Their forms are different from those given in this paper. Moreover, they have obtained general n.n.d $\boldsymbol{\Sigma}$ whih satisfiees $\boldsymbol{\Sigma A \Sigma B \Sigma} = \mathbf{0}$, given A and $\boldsymbol{B}$, thus determining the class of all distributions under which two given quadratic forms are independent.

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