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CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS BASED ON A GENERALIZED RAO-RUBIN CONDITION

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SUMMARY. This paper presents a simple solution to the problem of characterizing the distribution of a non-degenerate, non-negative, integral valued Z, subjected to a destructive process when the damage model is binomial and when the Generalized Rao-Rubin condition holds true, namely, $P_r(X = r \mid \pi_1) = P_r(X = r \mid Y = 0, \pi_2)$, $0 < \pi_1, \pi_2 < 1$, $Z = X+Y$, X and Y are independent r.v.s.

1. Introduction

In an early paper, Rao (1965) considered a damage model in which a discrete random variable (r.v.) Z is subjected to a destructive process which reduces Z to $X \leqslant Z$. Let $Y = Z-X$ and denote $s(r \mid n) = P(X = r \mid Z = n)$. When the damage distribution is binomial, that is,

$$s(r \mid n) = \binom{n}{r} \pi^r \phi^{n-r}, \quad 0 < \pi < 1, \qquad \dots \quad (1.1)$$

where $\phi = 1-\pi$, Rao and Rubin (1964) showed that the condition [which will be referred to as the RR condition.]

$$P(X = r \mid \pi) = P(X = r \mid Y = 0, \pi), \; r = 0, 1, \dots \qquad \dots \quad (1.2)$$

for a fixed value of $\pi \in (0, 1)$ is necessary and sufficient for Z to have a Poisson distribution.

Srivastava (1975) showed that under the binomial damage model (1.1), the following conditions are also sufficient for Z to be Poisson :

$$\text{(i)}\quad E(X\mid\pi) = E(X\mid Y=0,\pi)\ \forall\ \pi\in(0,1). \qquad \dots\ (1.3)$$

$$\text{(ii)}\quad P(X=r\mid\pi) = P(X=r\mid Y=k,\pi),\ r=0,1,\dots \qquad \dots\ (1.4)$$

for a fixed $k>0$ and $\forall\ \pi\in(0,1)$.

While using damage models on data from toxicological experiments, Talwalker (1975) observed that for suitable choices of π_1 and π_2 the condition [which we shall refer to as GRR, generalized Rao-Rubin condition]

$$P(X=r\mid\pi_1) = P(X=r\mid Y=0,\pi_2),\ r=0,1,\dots \qquad \dots\ (1.5)$$

holds when Z has binomial, Poisson or negative binomial distributions. In the present paper we solve the converse problem of characterizing the distribution of Z by the GRR condition.

2. Preliminary results

Let $p_n = P(Z=n)$ and $G(s) = \sum_{n=0}^{\infty} p_n s^n$, the probability generating function (p.g.f.) of Z. Throughout this paper, we assume that Z is a non-degenerate r.v. The GRR condition (1.5) yields

$$\sum_{n=0}^{\infty} p_{n+r}\,\frac{(n+r)!}{n!\,r!}\,\pi_1^r\phi_1^n = \frac{p_r\pi_2^r}{G(\pi_2)},\ r=0,1,\dots \qquad \dots\ (2.1)$$

Multiplying both sides of (2.1) by s^r and summing over r, we have the basic functional equation for $G(\cdot)$

$$G(\pi_1 s+\phi_1)G(\pi_2) = G(\pi_2 s),\quad -1\leqslant s\leqslant 1. \qquad \dots\ (2.2)$$

From (2.2), we deduce the following preliminary results.

Lemma 2.1 : *If (2.1) holds, $p_r = 0$ for $r>N$ (suitably chosen positive integer) when $\pi_1>\pi_2$ and $p_r>0$ for all non-negative integral r when $\pi_1<\pi_2$.*

Proof : From (2.1), taking only the first term from the left hand side, we have

$$p_r\pi_1 \leqslant \frac{p_r\pi_2^r}{G(\pi_2)}$$

or,

$$p_r\left[G(\pi_2)-\left(\frac{\pi_2}{\pi_1}\right)^r\right]\leqslant 0,\quad \forall\ r=0,1,\dots \qquad \dots\ (2.3)$$

which is impossible if $\pi_2 < \pi_1$ unless p_r vanishes for all r exceeding a certain positive integer N.

Now, suppose $\pi_1 \leqslant \pi_2$ and $p_k \neq 0$ but $p_{k+1} = 0$ for some k. But $p_{k+1} = 0 \Longrightarrow p_r = 0$ for $r \geqslant k+1$; then the equality in (2.3) holds for $r = k$, that is,

$$G(\pi_2) = \left(\frac{\pi_2}{\pi_1}\right)^k \geqslant 1. \qquad \ldots \quad (2.4)$$

But $G(\pi_2) < 1$, which leads to contradiction. Hence $p_r > 0$ for all $r = 0, 1, \ldots$

Lemma 2.2 : *If* (2.1) *holds, then moments of all orders of the r.v.* Z *are finite.*

We omit the proof of this lemma. It can be proved along the lines of lemma 1.1 of Srivastava and Singh (1975).

Lemma 2.3 : *Let* $Z = X+Y$, *where* X *and* Y *are obtained by the binomial damage model* (1.1). *Let* $f(\cdot)$ *be a function with range* (0, 1) *such that*

$$E\, E(X \mid Z, f(\pi)) = E[X \mid Y = 0, \pi], \forall\, \pi \in (0, 1), \qquad \ldots \quad (2.5)$$

where the left hand term gives $E(X)$ *under the condition that* X *is obtained from* Z *using the binomial probability* $f(\pi)$.

Then the p.g.f. of Z *is of the form*

$$G(s) = e^{-h(s)} \qquad \ldots \quad (2.6)$$

where

$$h(s) = c \int_s^1 \frac{f(\pi)}{\pi}\, d\pi, \quad c = E(Z).$$

Proof: It is easily seen that

$$E(X \mid Y = 0, \pi) = \pi \frac{G^{(1)}(\pi)}{G(\pi)}$$

and

$$E(X \mid f(\pi)) = cf(\pi), \qquad \ldots \quad (2.7)$$

where $c = E(Z)$ which is finite in view of (2.5) and $G^{(1)}(\cdot)$ denotes the first derivative of $G(\cdot)$. Hence (2.5) leads to the equation

$$\frac{G^{(1)}(\pi)}{G(\pi)} = \frac{cf(\pi)}{\pi}, \qquad \ldots \quad (2.8)$$

and the result (2.6) follows from (2.8).

Note 1 : If $f(\pi) = \dfrac{\pi}{1-p(1-\pi)}$, where p is a fixed constant in (0, 1), then

$$G(s) = (q+ps)^{c/p}, \quad q = 1-p. \qquad \ldots \quad (2.9)$$

Note that $f(\pi) = \dfrac{\pi}{1-p(1-\pi)} > \pi$, hence from lemma (1.1), $p_n = 0$ for $n > N$, where N is a fixed positive integer. But in this case $G(s)$ is a polynomial of degree N. Therefore $\dfrac{c}{p}$ is a positive integer and $G(s) = (q+ps)^{c/p}$ is the p.g.f. of a binomial distribution.

Note 2 : If $f(\pi) = \pi$, then $G(s) = e^{-c(1-s)}$ which is the p.g.f. of a Poisson distribution.

Note 3 : If $f(\pi) = \dfrac{p\pi}{1-q\pi}$, $q = 1-p$, then

$$G(s) = \left(\frac{p}{1-qs}\right)^{cp/q} \qquad \ldots \quad (2.10)$$

which is the p.g.f. of a negative binomial distribution.

3. The main result

In this section, we consider the problem of characterizing the class of probability distributions satisfying the GRR condition (1.4). Talwalker (1979) independently studied this problem and obtained the solution. The solution presented here is essentially the same, but simpler and complete. The solution of the functional equation (1.5) is presented in the Appendix.

Theorem : *Let* $Z = X+Y$ *as denfined in* (1.2). *If the* GRR *condition*

$$P(X = r \mid \pi_1) = P(X = r \mid Y = 0, \pi_2)$$

is satisfied for fixed values of π_1, π_2 *in* (0, 1), *then the following hold for the distribution of* Z :

(a) *If* $\pi_1 > \pi_2$, *then* Z *has a binomial distribution with parameter* $p = \dfrac{(\pi_1-\pi_2)}{\pi^1(1-\pi^2)}$

(b) *If* $\pi_1 = \pi_2$, *then* Z *has a Poisson distribution.*

(c) *If* $\pi_1 < \pi_2$, *then the p.g.f. of* Z *is of the form*

$$G(s) = \int_0^\infty e^{-(s_0-s)t}\, d\mu(t), \quad s < s_0 \qquad \ldots \quad (3.1)$$

where $s_0 = \dfrac{\pi_2(1-\pi_1)}{(\pi_2-\pi_1)}$ *and* $\mu(t)$ *is any (possibly infinite) measure such that*

$$c d\mu(t) = d\mu\left(\frac{\pi_2}{\pi_1} t\right) \qquad \dots \quad (3.2)$$

where
$$c = \frac{1}{G(\pi_2)} \cdot$$

Proof: (a) If $\pi_1 > \pi_2$, it is shown in lemma 2.1 that $p_r = 0$ for $r > N$ (a suitably chosen integer). In this case (2.1) becomes

$$\sum_{n=0}^{N-r} p_{n+r} \frac{(n+r)!}{n!} \cdot \phi_1^n = c \cdot r!\, p_r\, \delta^r,\ r = 0, 1, \dots, N \qquad \dots \quad (3.3)$$

where $c = [G(\pi_2)]^{-1}$ and $\delta = \dfrac{\pi_2}{\pi_1}$. It follows from (3.3) (for $r = N$) that $G(\pi_2) = \delta^N$. Therefore in (3.3) we have only a finite number of linear equations with an upper diagonal matrix on the left hand side. The solution for $p_0, \dots, p_N$ with the condition $\sum_{r=0}^{N} p_r = 1$ is therefore unique. Obviously

$$p_r = \binom{N}{r} p^r(1-p)^{N-r},\ r = 0, \dots, N \qquad \dots \quad (3.4)$$

where $p = \dfrac{\pi_1-\pi_2}{\pi_1(1-\pi_2)}$, is the only solution, that is, the distribution Z is binomial.

(b) If $\pi_1 = \pi_2$, it is shown by Rao and Rubin (1964) that Z has a Poisson distribution.

(c) If $\pi_1 < \pi_2$, it is shown in lemma 2.1 that $p_n > 0$ for all $n = 0, 1, \dots$ and the p.g.f. of Z satisfies the functional equation $G(\pi_1 s+\phi_1)G(\pi_2) = G(\pi_2 s)$, $-1 \leqslant s \leqslant 1$. The solution of this equation is presented in the Appendix and is of the form (3.1).

Note 1 : If $\pi_1 < \pi_2$, then

$$G(s) = \int_0^\infty e^{-(s_0-s)t}\, d\mu(t),\ s < s_0$$

and the measure $\mu(t)$ satisfies (3.2). There are many measures which satisfy (3.2). Two of these deserve special mention. The first is,

$$d\mu(t) = At^{\alpha-1}dt$$

where A is a constant. It is easy to see that it satisfies (3.2). In this case

$$G(s) = \frac{A\Gamma(\alpha)}{(s_0-s)^{\alpha}}.$$

Utilizing the condition $G(1) = 1$, we find

$$G(s) = \left(1-\frac{1}{s_0}\right)^{\alpha}\left(1-\frac{s}{s_0}\right)^{-\alpha}$$

which is the p.g.f. of a negative binomial distribution.

Another measure is obtained as follows. Let λ be any finite measure on $[1, \delta]$ and define

$$d\mu(t) = \sum_{k=-\infty}^{\infty} c^k d\lambda(t\delta^{-k}).$$

It is easy to see that the measure $\mu(\cdot)$ satisfies (3.2). As a special case consider a measure λ which puts a unit mass at r. Then

$$\mu = \sum_{k=-\infty}^{\infty} c^k \lambda_\delta k$$

and

$$G(s_0-s) = \int_0^{\infty} e^{-st}\, d\mu(t) = \sum_{-\infty}^{\infty} e^{-s\delta^k} c^k.$$

In this case,

$$p_r = \frac{A}{r\,!} \sum_{k=-\infty}^{\infty} c^k\, \delta^{rk}\, e^{-s_0\delta^k}, \quad r = 0, 1, 2, \ldots$$

where A is chosen so that $\sum_{r=0}^{\infty} p_r = 1$.

Note 2 : If GRR condition is satisfied for two sets of values, $\pi_1 < \pi_2$ and $\pi_1' < \pi_2'$ such that $(\log \pi_1-\log \pi_2)/(\log \pi_1'-\log \pi_2')$ is irrational, then $G(s)$ is the p.g.f. of a negative binomial distribution.

To prove this, we observe that

$$G(\pi_1 s+\phi_1)G(\pi_2) = G(\pi_2 s) \qquad \ldots \quad (3.5)$$

$$G(\pi_1' s+\phi_1')G(\pi_2') = G(\pi_2' s). \qquad \ldots \quad (3.6)$$

But since the point s_0 with $\lim_{s \to s_0^-} G(s) = \infty$, is fixwed, we have

$$\frac{\pi_2(1-\pi_1)}{\pi_2-\pi_1} = s_0 = \frac{\pi_2'(1-\pi_2')}{\pi_2'-\pi_1'}.$$

Define $\bar{G}(s) = G(s_0-s)$, $s > 0$. Then equations (3.5) and (3.6) yield

$$c_1\bar{G}(s) = \bar{G}\left(\frac{\pi_1}{\pi_2}\, s\right), \quad c_1 = \frac{1}{G(\pi_2)} > 1$$

$$c_2\bar{G}(s) = \bar{G}\left(\frac{\pi_1'}{\pi_2'}\, s\right), \quad c_2 = \frac{1}{G(\pi_2')} > 1.$$

So for integers k_1 and k_2, we have

$$c_1^{k_1}c_2^{k_2}\,\bar{G}(s) = \bar{G}\left(\left(\frac{\pi_1}{\pi_2}\right)^{k_1}\left(\frac{\pi_1'}{\pi_2'}\right)^{k_2}\right).$$

If we choose k_1 and k_2 such that

$$\frac{k_1}{k_2} \simeq \frac{\log \frac{\pi_1'}{\pi_2'}}{\log \frac{\pi_1}{\pi_2}},$$

then

$$\left(\frac{\pi_1}{\pi_2}\right)^{k_1}\left(\frac{\pi_1'}{\pi_2'}\right)^{k_2} = 1.$$

Here $\simeq$ denote approximate equality.

By continuity of $\bar{G}(s)$, we must have

$$c_1^{k_1}c_2^{k_2} \simeq 1$$

or

$$\frac{k_1}{k_2} \simeq -\frac{\log c_2}{\log c_1}$$

or

$$\frac{\log c_1}{\log \frac{\pi_1}{\pi_2}} \simeq \frac{\log c_2}{\log \frac{\pi_1'}{\pi_2'}} = -\alpha \text{ (say)}.$$

Now $\dfrac{\log c_1}{\log c_2} = \dfrac{\log G(\pi_1)}{\log G(\pi_1')}$ is irrational For any numbers $s > 0$, choose integers k_1 and k_2 such that

$$k_1 \log c_1 + k_2 \log c_2 \simeq -\alpha \log s.$$

Then

$$k_1 \log \frac{\pi_1}{\pi_2} + k_2 \log \frac{\pi_1'}{\pi_2'} \simeq \log s$$

and

$$\overline{G}(s) = \overline{G}\left(\left(\frac{\pi_1}{\pi_2}\right)^{k_1}\left(\frac{\pi_1'}{\pi_2'}\right)^{k_2}\right) = c_1^{k_1}c_2^{k_2}\,\overline{G}(1) \simeq s^{-\alpha}G(1).$$

Thus $G(s) = \overline{G}(s_0-s) = \overline{G}(1)(s_0-s)^{-\alpha}$ for all $s < s_0$.

Appendix

SOLUTION TO A FUNCTIONAL EQUATION

Now, we give a solution of the function equation

$$G(\pi_1 s + \phi_1) = cG(\pi_2 s), \ -1 \leqslant s \leqslant 1 \qquad \dots \quad (1)$$

where $\pi_1 < \pi_2$ are fixed values in $(0, 1)$, $\phi_1 = 1-\pi_1$ and $c = [G(\pi_2)]^{-1}$.

Equation (1) can be written as :

$$cG(s) = G\left(\frac{\pi_1}{\pi_2}s + \phi_1\right)$$

$$= \frac{1}{c}G\left(\left(\frac{\pi_1}{\pi_2}\right)^2 s + \frac{\pi_1}{\pi_2}\phi_1 + \phi_1\right). \qquad \dots \quad (2)$$

Applying this repeatedly, $G(s)$ can be extended up to the point $s_0 = \dfrac{\pi_2(1-\pi_1)}{\pi_2-\pi_1}$. Note that $s_0 > 1$.

Also (1) can be written as

$$G(s) = cG\left(\frac{\pi_2}{\pi_1}s - \frac{\pi_2(1-\pi_1)}{\pi_1}\right). \qquad \dots \quad (3)$$

Applying this repeatedly $G(s)$ can be extended for all negative values of s.

Thus $G(s)$ is defined for $(-\infty, s_0)$. Also $G^{(r)}(s) > 0$ for $s \in (0, s_0)$, so from the functional equation we get $G^{(s)}(s) < 0$ for all $s \in (-\infty, s_0)$. Let

$$\bar{G}(u) = G(s_0 - u). \qquad \ldots \quad (4)$$

$\bar{G}(u)$ is completely monotonic in $(0, \infty)$. By Bernstein's theorem (see Widder 1952, p. 162), there exists a (possible infinite) measure on $[0, \infty]$ such that

$$G(s_0 - u) = \bar{G}(u) = \int_0^\infty e^{-ut} d\alpha(t), \; u > 0 \qquad \ldots \quad (5)$$

From (1), we have

$$cG(s_0 - u) = G\left(s_0 \frac{\pi_1}{\pi_2} u\right), \; u > 0$$

or

$$c \int_0^\infty e^{-ut} d\alpha(t) = \int_0^\infty e^{-ut \frac{\pi_1}{\pi_2}} d\alpha(t)$$

$$= \int_0^\infty e^{-ut} d\alpha\left(\frac{\pi_2}{\pi_1} t\right), \qquad \ldots \quad (6)$$

By uniqueness of Bernstein's theorem, we have from (6)

$$c d\alpha(t) = d\alpha(\delta t), \quad \text{where } \delta = \frac{\pi_2}{\pi_1} > 1. \qquad (7)$$

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