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ON SUFFICIENCY AND PAIRWISE SUFFICIENCY IN STANDARD BOREL SPACES

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SUMMARY. Equivalence of pairwise sufficiency and sufficiency is established for countably generated σ-fields when the underlying spaces are Standard Borel and the probabilities are discrete. Further, some investigations are also made on the existence of minimal sufficient subfields.

1. Introduction

This note is essentially a continuation of Roy and Ramamoorthi (1979). We proved (Roy and Ramamoorthi 1979), that for the statistical structure $(\mathscr{X}, \mathscr{A}, \mathscr{P}_\theta: \theta \epsilon \Theta;\ (\Theta, \mathscr{C}))$ where $(\mathscr{X}, \mathscr{A}), (\Theta, \mathscr{C})$ are Standard Borel and P_θ's discrete then under an assumption of weak coherence, a countably generated sub σ-algebra $\mathscr{B}$ of $\mathscr{A}$ is pairwise sufficient iff it is sufficient. In this note we remove the assumption of weak coherence by showing that when the underlying spaces are Standard Borel and P_θ's discrete then the statistical structure is necessarily weakly coherent. Thus the conjecture of Blackwell on the equivalence of Bayes sufficiency and sufficiency in the Standard Borel case is settled for discrete probabilities. However, these methods do not extend to the general case.

In the later part of the paper we study some questions related to the existence of minimal sufficient sub σ-algebras. The existence of minimal sufficient σ-algebras is shown to be equivalent to some set theoretic conditions on the minimal pairwise sufficient partition. From this equivalence some sufficient conditions for the existence of minimal sufficient subfields can be derived.

Assumptions : Throughout this note we assume

(i) $(\mathscr{X}, \mathscr{A})$ and $(\Theta, \mathscr{C})$ are Standard Borel.

(ii) For each $\theta \epsilon \Theta$, P_θ is a discrete probability on $(\mathscr{X}, \mathscr{A})$. Further $\theta \to P_\theta$ is a measurable transition function, i.e., for all A in $\mathscr{A}$ $P_\theta(A)$ is as a function of θ, $\mathscr{C}$ measurable.

(iii) $A \epsilon \mathscr{A}$, $P_\theta(A) = 0$ for all $\theta \Longrightarrow A = \phi$.

2. Weak coherence of discrete probabilities when the underlying spaces are standard Borel

Definition : A real valued function $f_\theta(x)$ on $\Theta \times X$ is said to be finitely coherent if for any θ_1, θ_2 in Θ there is an $\mathscr{A}$ measurable function $f_{\theta_1\,\theta_2}(x)$ such that

$$f_{\theta_1,\theta_2} = \begin{cases} f_{\theta_1}[P_{\theta_1}] \\ f_{\theta_2}[P_{\theta_2}]. \end{cases}$$

$f_\theta(x)$ is said to be coherent if there is a $\mathscr{A}$ measurable function f such that

$$f = f_\theta\,[P_\theta] \text{ for all } \theta \in \Theta.$$

Definition : $(\mathscr{X}, \mathscr{A}, \mathscr{P}_\theta : \theta \in \Theta)$ is said to be weakly coherent if any $f_\theta(x)$ jointly measurable in θ and x which is finitely coherent is coherent.

Proposition 1 : *Under the assumptions $P_\theta(x)$ is a transition function iff $P_\theta(x)$ is jointly measurable in θ and x.*

Proof : Suppose $P_\theta(x)$ is a transition function.

Let $$\mathscr{M} = \{A \subset X \times X;\ P_\theta(A^x) \text{ is measurable in } \theta \text{ and } x\}$$

$\mathscr{M}$ is clearly closed under finite disjoint unions and is a monotone class containing rectangles and hence contains the product σ-algebra. Now since D the diagonal in $X \times X$ is in $\mathscr{M}$, $P_\theta(x) = P_\theta(D^x)$ is jointly measurable in θ and x. For the converse note that let $S = \{(\theta, x) : P_\theta(x) > 0\}$. S is a Borel set in $\Theta \times X$. Further θ sections of s are atmost countable. Therefore there is (Kuratowski and Mostowski, 1968) a sequence of measurable functions f_n defined on Θ and taking values in X such that $S = \bigcup_n (\theta, f_n(\theta))$. Define a sequence of functions $\phi_n(\theta)$ as

$$\phi_1(\theta) = P_\theta(f_1(\theta))$$

$$\phi_2(\theta) = \begin{cases} P_\theta(f_2(\theta)) \text{ on } \{\theta : f_1(\theta) \neq f_2(\theta)\} \\ 0 \quad \text{otherwise} \end{cases}$$

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$$\phi_n(\theta) = \begin{cases} P_\theta((f_n(\theta)) \text{ on } \{\theta : f_n(\theta) \neq f_i(\theta) \text{ for any } i = 1, \dots, n-1\}. \\ 0 \quad \text{otherwise.} \end{cases}$$

Then for any Borel set A in $\mathscr{A}$

$$P_\theta(A) = \sum_{n=1}^{\infty} I_A(f_n(\theta))\phi_n(\theta).$$

Since for each n, $\phi_n(\theta)$ and $I_A(f_n(\theta))$ are $\mathcal{C}$ measurable as functions of θ, $P_\theta(A)$ is also $\mathcal{C}$ measurable.

Proposition 2 : $(\mathcal{X}, \mathcal{A}, P_\theta : \theta \in \Theta)$ *is weakly coherent.*

Proof : By Proposition 1, $S = \{(\theta, x) : P_\theta(x) > 0\}$ is a Borel set in $\Theta \times X$.

Let $f_\theta(x)$ be jointly measurable in θ and x. Further let $f_\theta(x)$ be finitely coherent.

Define $f(x) = \sup_{\theta \in S^x} f_\theta(x)$.

Since $f_\theta(x)$ is finitely coherent, for each x, $f_\theta(x)$ is constant on S^x. Thus for each θ in Θ, $f(x) = f_\theta(x)[P_\theta]$. We will complete the proof by showing $f(x)$ is Borel measurable.

$$\{x : f(x) > a\} = \{x : \sup_{S^x} f_\theta(x) > a\} = P_X[\{(\theta, x) : f_\theta(x) > a\} \cap S]$$

where P_X denotes the projection to the X coordinate. Since $f_\theta(x)$ is for each x constant on the x section of S

$$\{x : f(x) \leqslant a\} = \{x : \sup_{S^x} f_\theta(x) \leqslant a\} = P_X[\{(\theta, x) : f_\theta(x) \leqslant a\} \cap S].$$

Thus being projections of Borel sets $\{x : f(x) > a\}$ and its complement $\{x : f(x) \leqslant a\}$ are both analytic and consequently Borel (vide Kuratowski and Mostowski, 1968) Hence $f(x)$ is $\mathcal{A}$ measurable.

Theorem : *A countably generated sub σ-algebra of $\mathcal{A}$ is pairwise sufficient iff it is sufficient.*

Proof : The theorem follows from Proposition 1 and Theorem 4 of Roy and Ramamoorthi (1979). However since the proof is short we give an outline.

Let $$N_\theta = \{A \in \mathcal{A} : P_\theta(A) = 0\}.$$

We first show that $\bigcap_\theta \mathcal{B} \vee N_\theta$ is sufficient.

Given any bounded $\mathcal{A}$ measurable function f, since $\mathcal{B}$ is countably generated there is a function $f_\theta(x)$ jointly measurable in θ and x such that for all θ, $f_\theta(x) = E_\theta(f \mid \mathcal{B})$. Further pairwise sufficiency of $\mathcal{B}$ implies that $f_\theta(x)$ is finitely coherent. By Proposition 1 there is an $\mathcal{A}$ measurable function f satisfying $f = f_\theta(x)$ $[P_\theta]$ for all θ. Consequently f is $\mathcal{B} \vee N_\theta$ measurable for each θ and is also a version of $E_\theta(f \mid \bigcap_\theta \mathcal{B} \vee N_\theta)$. Hence $\bigcap_\theta \mathcal{B} \vee N_\theta$ is sufficient. We will complete the proof by showing $\bigcap_\theta \mathcal{B} \vee N_\theta = \mathcal{B}$. Suppose E is a set in $\bigcap_\theta \mathcal{B} \vee N_\theta$

such that for some atom F of $\mathcal{B}$, $E \cap F \neq \phi$ and also $E^c \cap F \neq \phi$. Then there exist θ_1, θ_2 such that $P_{\theta_1}(E \cap F) > 0$ and $P_{\theta_2}(E^c \cap F) > 0$.

But $$\bigcap_{\theta} \mathcal{B} \vee N_\theta = \bigcap_{\theta_i \theta_j} \mathcal{B} \vee N_{\theta_i \theta_j}, \quad \text{where} \quad N_{\theta_i \theta_j} = N_{\theta_i} \cap N_{\theta_j}.$$

by pairwise sufficiency of $\mathcal{B}$. Therefore given E in $\bigcap_{\theta} \mathcal{B} \vee N_\theta$ there is an E' in $\mathcal{B}$ such that $P_{\theta_1}(E \Delta E') = P_{\theta_2}(E \Delta E') = 0$. But this is not possible. We have thus shown that every set in $\bigcap_{\theta} \mathcal{B} \vee N_\theta$ is a union of $\mathcal{B}$ atoms. Hence $\bigcap_{\theta} \mathcal{B} \vee N_\theta = \mathcal{B}$. (See Blackwell, 1955).

Pairwise sufficiency being weaker than Bayes sufficiency the following corollary is immediate

Corollary : *A countably generated σ-algebra is Bayes sufficient iff it is sufficient.*

Certain other notions of sufficiency in terms of decision rules namely 2 adequacy, finite adequacy, etc. were discussed in Roy and Ramamoorthi (1979). Pairwise sufficiency being the weakest of these it follows that in the set up considered above all these notions are equivalent.

2. Characterization of minimal sufficient sub-σ-field

We now turn to the existence of minimal sufficient sub σ-algebras. Towards this we first define a partition of X. For $\theta \in \Theta$, S_θ will denote the support $\{x : P_\theta(x) > 0\}$ of P_θ.

Define, for $\theta, \theta' \in \Theta$, $\dfrac{P_\theta}{P_\theta + P_{\theta'}}$ as

$$\frac{P_\theta}{P_\theta + P_{\theta'}}(x) = \begin{cases} \dfrac{P_\theta(x)}{P_\theta + P_{\theta'}(x)} & \text{for } x \in S_\theta \\ 0 & \text{for } x \notin S_\theta. \end{cases}$$

Let $\mathcal{B}$ denote the smallest σ-algebra generated by $\dfrac{P_\theta}{P_\theta + P_{\theta'}}$, $\theta, \theta' \in \Theta$. The σ-algebra $\mathcal{B}$ is atomic, in fact each atom is atmost countable; and this gives a partition of X. This partition of X will be denoted by $\hat{\mathcal{B}}$. It is easily seen that $\mathcal{B}$ is pairwise sufficient, contains supports of P_θ and is also minimal. That is if $\mathcal{B}_1$ is any other pairwise sufficient σ-algebra containing S_θ's then $\mathcal{B} \subseteq \mathcal{B}_1$.

Proposition 3 : *The following are equivalent for a sub σ-algebra $\mathcal{C}$ of $\mathcal{A}$*

(i) *$\mathcal{C}$ is minimal sufficient*

(ii) *$\mathcal{C}$ is the smallest countably generated σ-algebra containing $\mathcal{B}$*

(iii) $\bigcap_\theta \mathcal{B} \vee N_\theta$ *is sufficient and* $\bigcap_\theta \mathcal{B} \vee N_\theta = \mathcal{C}$.

Proof : (i) $\Longrightarrow$ (ii). By the theorem of Section 1 any countably generated σ-algebra containing $\mathcal{B}$ is sufficient. Hence $\mathcal{C}$ is contained in any countably generated σ-algebra containing $\mathcal{B}$.

(ii) $\Longrightarrow$ (iii). We will show that under (ii), $\mathcal{C} = \bigcap_\theta \mathcal{B} \vee N_\theta$. Since $\mathcal{B} \subset \mathcal{C}$, $\mathcal{C}$ is pairwise sufficient. An argument similar to the proof of theorem 1 yields $\mathcal{C} = \bigcap_\theta \mathcal{C} \vee N_\theta$

$$\therefore \quad \bigcap_\theta \mathcal{B} \vee N_\theta \subset \bigcap_\theta \mathcal{C} \vee N_\theta. = \mathcal{C}.$$

The other inclusion will follow from the following two facts

(a) Atoms of $\mathcal{C}$ are the same as atoms of $\mathcal{B}$. Suppose not, since $\hat{\mathcal{B}} \subset \mathcal{C}$ let E be an element of $\mathcal{B}$ containing more than one atom of $\mathcal{C}$. Then the σ-algebra $\mathcal{C}' = \mathcal{C} \bigcap E^c \vee E$ is a countably generated σ-algebra containing $\mathcal{B}$ and strictly contained in $\mathcal{C}$. This contradicts the minimality of $\mathcal{C}$ stated in (ii).

(b) $E \epsilon \mathcal{A}$ and E is a union of $\mathcal{B}$ atoms then $E \epsilon \bigcap_\theta \mathcal{B} \vee N_\theta$.

$$I_E(x) = I_E(x) \cdot I_{S_\theta}(x)[P_\theta] \text{ all } \theta \epsilon \Theta.$$

Since for each θ, S_θ is a countable set in $\mathcal{B}$ and E is a union of $\mathcal{B}$ atoms $I_E(x) \,.\, I_{S_\theta}(x)$ is for each θ in Θ $\mathcal{B}$ measurable. Therefore $E \epsilon \bigcap_\theta \mathcal{B} \vee N_\theta$.

(iii) $\Longrightarrow$ (i). $\bigcap_\theta \mathcal{B} \vee N_\theta$ is sufficient, hence $\mathcal{C}$ is countably generated (Burkholder, 1961). That $\mathcal{C}$ is minimal sufficient now follows from minimality of $\mathcal{B}$, and from the fact that every sufficient sub σ-algebra of $\mathcal{A}$ is necessarily countably generated (see Burkholder, 1961).

We need the following before stating the next proposition. Any atomic σ-algebra on X induces an equivalence relation on X, namely $x \sim y$ iff x and y belong to the same atom. We say that an equivalence relation is Borel if the set $\{(x, y) : x \sim y\}$ is a Borel set in the product. A partition is said to be induced by a real valued measurable function f, if f is Borel measurable and $f(x) = f(y)$ iff x and y belong to the same atom.

Proposition 4 : *The following are equivalent.*

(i) $\mathcal{B}$ *is induced by a real valued measurable function.*

(ii) *The relation induced by* $\mathcal{B}$ *is Borel.*

(iii) $\bigcap_\theta \mathcal{B} \vee N_\theta$ *is sufficient.*

Proof : (i) $\Longrightarrow$ (ii) is immediate for if T is a function inducing $\mathcal{B}$ then $x \sim y \Longleftrightarrow T(x) = T(y)$.

(ii) $\Longrightarrow$ (iii). We shall show that if $\mathcal{B}$ induces a Borel relation then given any set $A \in \mathcal{A}$ there is a jointly measurable function $f_\theta(x)$ such that $f_\theta(x) = E_\theta(I_A \mid \mathcal{B})[P_\theta]$ for all $\theta \in \Theta$.

Define
$$f_\theta(x) = \frac{P_\theta(A \cap R^x)}{P_\theta(R^x)} I_{S_\theta}(x)$$
where
$$R = \{(x, y) : x \sim y\}.$$

An argument similar to the proof of proposition 1 shows that $f_\theta(x)$ is jointly measurable in θ and x. Proof of sufficiency of $\bigcap_\theta \mathcal{B} \vee N_\theta$ is in the same lines as that of theorem 1.

(iii) $\Longrightarrow$ (i) is clear since sufficiency of $\bigcap_\theta \mathcal{B} \vee N_\theta$ implies it is countably generated and hence given by a real valued measurable function.

Proposition 4 yields some sufficient conditions for the existence of minimal sufficient sub σ-algebras. The relation induced by $\mathcal{B}$ can be easily characterised as the intersection of the two relations,

(i) $(x, y) \in R_1$ iff $P_\theta(x) > 0 \Longleftrightarrow P_\theta(y) > 0$,

(ii) $(x, y) \in R_2$ iff for all θ_1, θ_2 such that $P_{\theta_1}(x) P_{\theta_1}(y) > 0$, $P_{\theta_2}(x) P_{\theta_2}(y) > 0$

and
$$\frac{P_{\theta_1}(x)}{P_{\theta_1}(y)} = \frac{P_{\theta_2}(x)}{P_{\theta_2}(y)}.$$

R_1^c can be written as
$$P[\{(\theta, x, y) : P_\theta(x) > 0\ P_\theta(y) = 0\} \cup \{(\theta, x, y) : P_\theta(x) = 0\ P_\theta(y) > 0\}]$$
R_2^c can be written as
$$P\left[\{(\theta_1, \theta_2, x, y) : P_{\theta_1}(x) P_{\theta_1}(y) P_{\theta_2}(x)\ P_{\theta_2}(y) > 0\} \cap \left\{(\theta_1, \theta_2, x, y) : \frac{P_{\theta_1}(x)}{P_{\theta_1}(y)} \neq \frac{P_{\theta_2}(x)}{P_{\theta_2}(y)}\right\}\right].$$
where P denotes projection to the $X \times X$ co-ordinate space.

Clearly both R_1^c and R_2^c are analytic or R_1 and R_2 are coanalytic and the intersection $R_1 \cap R_2$ is also in general coanalytic.

In general $\mathcal{B}$ will not induce a Borel relation, equivalently there will not in general exist a minimal sufficient sub σ-algebra even in the discrete case. Below we give an example of such a situation. We note that in the example $\theta_1 \neq \theta_2 \Longrightarrow P_{\theta_1} \neq P_{\theta_2}$.

Example. $\Theta = [0, 2]$ $X = [-1, 2]$.

A is a symmetric non Borel analytic subset of $[-1, 1]$.

g is a measurable map from $[1, 2]$ onto A.

Define ϕ_1 and ϕ_2 two measurable functions on Θ to X as

$$\phi_1(\theta) = \theta$$

$$\phi_2(\theta) = \begin{cases} g(\theta) \text{ on } [1, 2]. \\ -\theta \text{ on } [0, 1] \end{cases}$$

$$P_\theta(E) = \tfrac{1}{2} I_E(\phi_1(\theta) + \tfrac{1}{2} I_E(\phi_2(\theta)).$$

$\mathcal{B}$ then has atoms $(x, -x)$ for x in A^c and singletons otherwise and this is not induced by a Borel function. This example was suggested to us by B. V. Rao.

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