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# A ZERO-ONE DICHOTOMY THEOREM FOR $r$-SEMI-STABLE LAWS ON INFINITE DIMENSIONAL LINEAR SPACES

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*SUMMARY*. Let $\mu$ be an $r$-semistable probability measure on a real linear space $E$. It is shown that the $\mu$-measure of any translate of an arbitrary measurable linear subspace over certain countable subfield of reals is 0 or 1. This result yields immediately the 0-1 laws for tables measures of Dudley-Kanter (1974) and also a more recent 0-1 law of Fernique for quasi-stable measures which is included in his ISI lectures of September, 1978. It is also shown that $r$-semi-stable measures—like stable ones—are continuous, i.e., they assign zero mass to singletons.

## 1. Introduction

Let $(E, \mathcal{F})$ be a measurable vector space in the sense of Dudley and Kanter and $\mu$ a stable probability measure (p.m.) on $E$. Recently, Dudley-Kanter (1974) have shown that the $\mu$-measure of certain measurable subspaces of $E$ is 0 or 1. More recently Fernique exhibited a similar $0-1$ law for what he calls quasi-stable p.m. A natural and nontrivial generalization of stable p.m's is the class of $r$-semistable p.m's, which was first introduced and studied on the real line $R$ by Lévy (1937). Later Kruglov (1972a) obtained a quite explicit form of the characteristic function of $r$-semistable p.m's on $R$ and showed that this class has many properties similar to those exhibited by stable probability measures. (This in Hilbert space setting is also shown in Kruglov (1972b) and Kumar (1976). Partly motivated from these papers we raised and completely answered the question whether $r$-semistable p.m's share with stable measures the $0-1$ dichotomy results obtained by Dudley and Kanter (1974). Explicitly we prove that if $(E, \mathcal{F})$ is a measurable vector space over $R$, $\mu$ a $r$-semistable p.m. (see Section 2) on $(E, \mathcal{F})$ and $G$ a measurable subspace over the field $Q(c)$, the smallest subfield containing

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$Q$, the rationals, and $c = c(r)$, then $\mu(G-z) = 0$ or $1$, for every $z \in E$ (Theorem 3.1). This result includes and, in fact, extends the 0—1 theorems for stable p.m's obtained in Dudley and Kanter (1974) (Corollary 3.2); also the method of proof of the result includes a recent 0—1 dichotomy theorem of Fernique (ISI, Calcutta, Lectures 1978) for quasi-stable p.m's (Corollary 3.3). Further, we also show that, like stable p.m's non-degenerate $r$-semistable p.m's are continuous; that is, they assign zero mass to singletons (Corollary 3.4). Our proof of the 0—1 dichotomy theorem seems new as well as simpler than those in Dudley and Kanter (1974) (we use only the definition of convolution and Fubini's theorem); in particular, we do not require any number theory results which was not the case in the proofs of Dudley and Kanter (1974).

## 2. Preliminaries

Let $(E, \mathcal{F})$ be a measurable vector space and $\mu$ be a p.m. on $\mathcal{F}$. Let $r \in (0, 1)$; then $\mu$ is called *r-semistable* if there is a constant $c(r) = c$ with $0 < c \neq 1$ and a semigroup $\{\mu^s; s > 0\}$ of p.m's on $\mathcal{F}$ and a sequence $\{x(m)\}$ in $E$ such that the following hold

$$\mu^1 = \mu \qquad \ldots \quad (2.1)$$

$$\mu^{r^m} = T_{c^m}\mu * \delta_{x(m)}, \qquad \ldots \quad (2.2)$$

for each $m = 1, 2, \ldots$, where for $a > 0$, $T_a\mu$ denotes the measure $T_a\mu(B) = \mu(a^{-1}B)$, for every $B \in \mathcal{F}$ and $*$ denotes the usual convolution.

The above definition is motivated from a characterization of a class of measures also called $r$-semistable on locally convex topological vector spaces (LCTVS) obtained in Chung, Rajput and Tortrat (1979). It follows from there that our results are applicable for $r$-semistable (and hence stable and Gaussian) measures studied in Chung, Rajput and Tortrat (1979).

## 3. 0-1 Dichotomy theorem for $r$-semistable measures

The main result we propose to prove is the following :

Theorem 3.1 : *Let $\mu$ be a r-semistable p.m. on a measurable vector space $(E, \mathcal{F})$ over $R$ and let $G$ be a subspace over the subfield $Q(c)$ such that $G \in \mathcal{F}$ ($c$ is the constant appearing in (2.2)). Then $\mu(G-z) = 0$ or $1$, for all $z \in E$.*

*Proof* : Let $z_1 \in E$ and assume that $\mu(G-z_1) > 0$. We will show that $\mu(G-z_1) = 1$. Choose an integer $n_r$ so that $0 < 1/n_r < 1-r$.

Let

$$\mathcal{H} = \{G-x \mid \mu(G-x) > 0 \text{ or } \mu^{1/n_r}(G-x) > 0\} \subseteq E/G,$$

$$<\mathcal{H}> = \text{linear span of } \mathcal{H} \text{ in } E/G \text{ over the field } Q(c),$$

and

$G_0 =$ inverse image of $<\mathcal{H}>$ under natural projection $= \bigcup <\mathcal{H}>$. Then $G_0$ is a vector subspace of $E$ over $Q(c)$ and clearly, $G_0 \in \mathcal{J}$. since $G_0$ is a countable union of sets in $\mathcal{G}$.

For the sake of clarity, the remainder of the proof will be divided into seven parts.

(i) $\mu^{1-r} * \delta_{x(1)}(G_0) = 1$.

*Proof of* (i) : Observe that $\mu(G_0-c^{-1}y) = 0$, for all $y \in G_0^c$, and that $\mu = \mu^r * \mu^{1-r} = T_c\mu * \mu^{1-r} * \delta_{x(1)}$.

Thus

$$0 < \mu(G_0) = \int_E T_c\mu(G_0-y)\mu^{1-r} * \delta_{x(1)}(dy)$$

$$= \int_{G_0} \mu(G_0-c^{-1}y)\mu^{1-r} * \delta_{x(1)}(dy)$$

$$= \mu(G_0)\mu^{1-r} * \delta_{x(1)}(G_0).$$

Consequently, $\mu^{1-r} * \delta_{x(1)}(G_0) = 1$.

(ii) $\mu^{1/n_r}(G_0) = 1$.

*Proof of* (ii) : Since

$$\mu = \mu^{1/n_r} * (\mu^{1/n_r})^{*(n_r-1)},$$

we have

$$0 < \mu(G-z_1) = \int_E \mu^{1/n_r}(G-z_1-y)(\mu^{1/n_r})^{*(n_r-1)}(dy).$$

Thus there exists $y \in E$ so that $\mu^{1/n_r}(G-z_1-y) > 0$, and hence $\mu^{1/n_r}(G_0) > 0$.

Now

$$\mu^{1-r} * \delta_{x(1)} = \mu^{1/n_r} * \mu^{1-r-1/n_r} * \delta_{x(1)},$$

and so, from (i),

$$1 = \mu^{1-r} * \delta_{x(1)}(G_0) = \int_E \mu^{1-r-1/n_r} * \delta_{x(1)}(G_0-y)\mu^{1/n_r}(dy),$$

which implies that

$$\mu^{1-r-1/n_r} * \delta_{x(1)}(G_0-y) = 1 \text{ a.s. } [\mu^{1/n_r}].$$

Since $\mu^{1/n_r}(G_0) > 0$, it follows that

$$\mu^{1-r-1/n_r} * \delta_{x(1)}(G_0) = 1.$$

Consequently,

$$1 = \mu^{1-r} * \delta_{x(1)}(G_0) = \int_{G_0} \mu^{1/n_r}(G_0-y)\mu^{1-r-1/n_r} * \delta_{x(1)}(dy)$$

$$= \mu^{1/n_r}(G_0)\mu^{1-r-1/n_r} * \delta_{x(1)}(G_0)$$

$$= \mu^{1/n_r}(G_0).$$

(iii) $\mu(G_0) = 1$.

*Proof of* (iii) : It follows from (ii) that

$$\mu(G_0) = \int_{G_0} (\mu^{1/n_r})^{*(n_r-1)}(G_0-y)\mu^{1/n_r}(dy)$$

$$= (\mu^{1/n_r})^{*(n_r-1)}(G_0)\mu^{1/n_r}(G_0)$$

$$= (\mu^{1/n_r})^{*(n_r-1)}(G_0)$$

$$= (\mu^{1/n_r}(G_0))^{n_r-1}$$

$$= 1.$$

We will use the fact that $\mu(G_0) = 1$ to conclude that $\mu(G-z_1) = 1$ (see (vii)). To this end, we proceed.

Recall that $G_0$ is a countable (possibly finite) union of disjoint cosets of $G$. Let $\{x_1, x_2, \ldots\}$ be a sequence of distinct points in $E$ so that $G_0 = \sum_k G-x_k$, where $\sum$ denotes disjoint union. Clearly, we may assume, without loss of generality, that $\mu(G-x_1) \geqslant \mu(G-x_2) \geqslant \ldots$ . Let $N_1$ be the largest integer so that $\mu(G-x_1) = \mu(G-x_{N_1})$. For the sake of simplicity of notation, let $t = t(m) = c^m$, $m = 1, 2, \ldots$ and let $\nu_t = \mu^{1-r^m} * \delta_{x(m)}$. Then $\mu = T_t\,\mu * \nu_t$, for any $t$.

(iv) For each $t$, $\nu_t\left(\sum_{k=1}^{N_1} G-x_n+tx_k\right)=1$, for $1 < n < N_1$.

*Proof of* (iv) : Observe that if $y \in G-x_k$, then $G-x_n-ty=G-x_n+tx_k$, for all $n$ and $k$.

Thus

$$\begin{aligned}\mu(G-x_n) &= \int_{G_0} \nu_t(G-x_n-ty)\,\mu(du)\\ &= \sum_k \nu_t(G-x_n+tx_k)\mu\,(G-x_k), \qquad \dots \quad (3.1)\end{aligned}$$

for $n=1, 2, \dots$ .

Now, for $1 < n < N_1$, we have

$$\begin{aligned}\mu(G-x_n) &= \sum_k \nu_t(G-x_n+tx_k)\,\mu(G-x_k)\\ &\leqslant \mu(G-x_n)\,\sum_k\,\nu_t(G-x_n+tx_k)\\ &= \mu(G-x_n)\nu_t\left(\sum_k G-x_n+tx_k\right)\\ &\leqslant \mu(G-x_n).\end{aligned}$$

Thus

$$\mu(G-x_n)\,\nu_t(G-x_n+tx_k)=\mu(G-x_k)\,\nu_t(G-x_n+tx_k),$$

for $1 \leqslant n \leqslant N_1$ and any $k$, which implies that $\nu_t(G-x_n+tx_k)=0$ for $1 \leqslant n \leqslant N_1$ and $k > N_1$.

Thus

$$\begin{aligned}\mu(G-x_n) &= \sum_{k=1}^{N_1} \nu_t(G-x_n+tx_k)\mu(G-x_k)\\ &= \mu(G-x_n)\sum_{k=1}^{N_1} \nu_t(G-x_n+tx_k)\\ &= \mu(G-x_n)\nu_t\sum_{k=1}^{N_1}\,(G-x_n+tx_k),\end{aligned}$$

for $1 \leqslant n \leqslant N_1$.

Hence

$$1 = \nu_t\left(\sum_{k=1}^{N_1} G - x_n + tx_k\right),$$

for $1 \leqslant n \leqslant N_1$, since $\mu(G - x_n) = \mu(G - x_1) > 0$ for $1 \leqslant n \leqslant N_1$.

(v) $N_1 = 1$ or, equivalently, $(G - x_1) > \mu(G - x_k)$, for all $k > 1$.

*Proof of* (v) : Suppose $N_1 \geqslant 2$ and consider the $2 \times N_1$ array $M_1$ :

$$\begin{matrix} G - x_1 + tx_1 & G - x_1 + tx_2 & G - x_1 + tx_3 \ldots G - x_1 + tx_{N_1} \\ G - x_2 + tx_1 & G - x_2 + tx_2 & G - x_2 + tx_3 \ldots G - x_2 + tx_{N_1} \end{matrix}.$$

By (iv), the $\nu_t$-measure of row 1 of $M_1$ is 1. Thus there is an integer $k_1$, $1 \leqslant k_1 \leqslant N_1$, so that $\nu_t(G - x_1 + tx_{k_1}) > 0$, for infinitely many values of $t$. Now, the $\nu_t$-measure of row 2 is also 1 (by (iv) again), which implies that $G - x_1 + tx_{k_1}$ intersects row 2, for infinitely many values of $t$. Thus there is an integer $k_2$, $1 \leqslant k_2 \leqslant N_1$, so that $G - x_1 + tx_{k_1} = G - x_2 + tx_{k_2}$, for infinitely many values of $t$. Consequently, there are integers $k_1$ and $k_2$, $1 \leqslant k_1 \leqslant N_1$, $1 \leqslant k_2 \leqslant N_1$, so that

$$G - x_1 + x_2 = G - t(x_{k_2} - x_{k_1}), \qquad \ldots (3.2)$$

for infinitely many values of $t$. In particular, there exist $t_1$ and $t_2$, $t_1 \neq t_2$, so that $G - t_1(x_{k_2} - x_{k_1}) = G - t_2(x_{k_2} - x_{k_1})$ which implies that $G = G + (t_1 - t_2)(x_{k_2} - x_{k_1})$ and so, $(t_1 - t_2)(x_{k_2} - x_{k_1}) \in G$ from which it follows that $G - x_{k_1} = G - x_{k_2}$. Consequently, since $G_0$ is a disjoint union, we have $k_1 = k_2$ which implies, from (3.2), that $G - x_1 = G - x_2$. But $G - x_1 \neq G - x_2$. Hence (v) follows.

(vi) For each $t$, $\nu_t(G - x_1 + tx_1) = 1$.

*Proof of* (vi) : This is immediate from (iv) and (v).

(vii) $\mu(G - z_1) = \mu(G_0)$.

*Proof of* (vii) : Suppose $\mu(G - z_1) < \mu(G_0)$. Then $\mu(G - x_2) > 0$. Let $N_2$ be the largest integer so that $\mu(G - x_2) = \mu(G - x_{N_2})$. Observe that, by (vi) we have that for each $t$, $\nu_t(G - x_n + tx_1) = 0$, for all $n \geqslant 2$; otherwise, we get $G - x_n = G - x_1$, for some $n \geqslant 2$.

Thus by (3.1), for $2 \leqslant n \leqslant N_2$,

$$\mu(G-x_n) = \nu_t(G-x_n+tx_1)+ \sum_{k\geqslant 2} \nu_t(G-x_n+tx_k)\mu(G-x_k)$$

$$= \sum_{k\geqslant 2} \nu_t(G-x_n+tx_k)\mu\,(G-x_k)$$

$$\leqslant \mu(G-x_n) \sum_{k\geqslant 2} \nu_t(G-x_n+tx_k)$$

$$= \mu(G-x_n)\,\nu_t\left(\sum_{k\geqslant 2} G-x_n+tx_k\right)$$

$$\leqslant \mu(G-x_n).$$

It follows that

$$\mu(G-x_n)\nu_t\,(G-x_n+tx_k) = \mu(G-x_k)\nu_t\,(G-x_n+tx_k),$$

for $2 \leqslant n \leqslant N_2$ and any $k \geqslant 2$, which implies that $\nu_t(G-x_n+tx_k) = 0$, for $2 \leqslant n \leqslant N_2$ and $k > N_2$.

Consequently,

$$\mu(G-x_n) = \sum_{k=2}^{N_2} \nu_t(G-x_n+tx_k)\mu(G-x_k)$$

$$= \mu(G-x_n) \sum_{k=2}^{N_2} \nu_t(G-x_n+tx_k)$$

$$= \mu(G-x_n)\,\nu_t\left(\sum_{k=2}^{N_2} G-x_n+tx_k\right),$$

for $2 \leqslant n \leqslant N_2$.

Hence, for all $t$,

$$1 = \nu_t\left(\sum_{k=2}^{N_2} G-x_n+tx_k\right), \qquad \ldots \quad (3.3)$$

for $2 \leqslant n \leqslant N_2$, since $\mu(G-x_n) = \mu(G-x_2) > 0$, for $2 \leqslant n \leqslant N_2$.

Observe that, by (vi), $\nu_t(G-x_2+tx_1) = 0$; otherwise, $G-x_2+tx_2 = G-x_1+tx_1$ which implies that $G-x_1 = G-x_2$. Consequently, from (3.3), $N_2 \geqslant 3$, and so, $G_0$ contains at least three disjoint cosets of $G$. Now consider the $2\times(N_2-1)$ array $M_2$ :

$$\begin{matrix} G-x_2+tx_2 & G-x_2+tx_3 & G-x_2+tx_4 \ldots G-x_2+tx_{N_2} \\ G-x_3+tx_2 & G-x_3+tx_3 & G-x_3+tx_4 \ldots G-x_3+tx_{N_2}. \end{matrix}$$

Observe that the $\nu_t$-measure of each row of $M_2$ is equal to 1. Now proceed, as in (v), to show that there exist integers $k_1$ and $k_2$, $2 \leqslant k_1 \leqslant N_2$, $2 \leqslant k_2 \leqslant N_2$, $k_1 \neq k_2$, so that

$$G - x_2 + x_3 = G - t(x_{k_2} - x_{k_1}), \qquad \dots \quad (3.4)$$

for infinitely many values of $t$. It follows, from (3.4), like in (v), that $k_1 = k_2$. Consequently, by (3.4), $G - x_2 = G - x_3$. This is a contradiction! Hence our initial assumption must be false and it follows that $\mu(G - z_1) = \mu(G_0)$.

To complete the proof of the theorem, observe that, by (iii) and (vii), we have $\mu(G - z_1) = \mu(G_0) = 1$.

In view of the last sentence of the previous section, we have the analogue of Theorem 3.1 for stable and Gaussian measures if the measures are $K$-regular and are defined on the Borel $\sigma$-algebra of a complete LCTVS. In the following corollary, we show, however, that the same result can be recovered from Theorem 3.1 even if the stable measures $\mu$ is defined on a measurable vector space $(E, \mathcal{F})$ provided $\mu$ has the index; i.e. there exists an $\alpha > 0$ such that for every $a > 0$, $b > 0$, $T_a\mu * T_b\mu = T_{(a^\alpha + b^\alpha)^{1/\alpha}} \mu * \delta_x$, for some $x \in E$. This corollary contains and extends various results of Dudley and Kanter (1974); we do not, however, deal with 0—1 laws when $G$ belongs to the completed $\sigma$-algebra.

Corollary 3.2 : *Let $(E, \mathcal{F})$ be a measurable vector space and let $G$ be a rational subspace of $E$, $G \in \mathcal{F}$. Then*

(i) *If $\mu$ is a strictly stable p.m. of index $\alpha$ on $(E, \mathcal{F})$, then for all $z \in E$, $\mu(G - z) = 0$ or 1.*

(ii) *If $\mu$ is a stable p.m. of index $\alpha$ on $(E, \mathcal{F})$, then $\mu(G) = 0$ or 1.*

*Proof* : (i) Assume $\mu$ is strictly stable of index $\alpha$ and set $\mu^s = T_s 1/\alpha\, \mu$. Then $\{\mu^s \mid s > 0\}$ is a semigroup with $\mu^1 = \mu$ and (2.1), (2.2) are satisfied for all $r > 0$, with $x(m) = \theta$, and $c = s^{1/\alpha}$. Then it is easy to see that $\mu$ is a $r$-semi-stable p.m. for all $0 < r < 1$. Choose $r_0$, $0 < r_0 < 1$, so that $r_0^{1/\alpha}$ is rational. Then $Q(r_0^{1/\alpha}) = Q$. Now apply Theorem 3.1 to obtain the desired result.

(ii) Let $\mu$ be a stable p.m. of index $\alpha$ and assume that $\mu(G) > 0$. Let $\nu = \mu * T_{-1}\mu$ be the symmetrization of $\mu$. Then $\nu$ is a strictly stable p.m. of index $\alpha$. Observe that

$$\nu(G) = \int_E \mu(G + y)\, \mu(dy) \geqslant \int_G \mu(G + y)\, \mu(dy)$$
$$= (\mu(G))^2 > 0.$$

Thus, by (i), $\nu(G) = 1$, and so $\mu(G+y) = 1$ a.s. $[\mu]$ which implies that $\mu(G) = 1$.

The following corollary shows that the method of proof of Theorem 3.1 also yields the $0-1$ dichotomy theorem for quasi-stable measures recently obtained by Fernique who uses a non-trivial inequality of Kantor for his proof. Our proof, as we noted earlier, uses only elementary facts about convolution. Now we recall the definition of quasi-stable p.m. as introduced by Fernique. Let $\mu$ be a p.m. on a measurable vector space $(E, \mathscr{F})$, then $\mu$ is said to be quasi-stable if $\mu^{*2} = T_c\mu$, for some $c > 0$, $c \neq 1$.

Corollary 3.3 : *Let $(E, \mathscr{F})$ be a measurable vector space and $\mu$ be quasi-stable on $E$. Let $G$ be $Q(c)$ vector space which belongs to $\mathscr{F}$. Then $\mu(G-z) = 0$ or $1$, for every $z \in E$.*

*Proof*: Let $\mu(G-z_1) > 0$ and let $\mathscr{H}' = \{G-x : \mu(G-x) > 0\}$ and define $G_0$ as in the beginning of the proof of Theorem 3.1 with $\mathscr{H}$ replaced by $\mathscr{H}'$. Since

$$0 < \mu(G_0) = T_c\mu(G_0) = \mu^{*2}(G_0) = \int_{G_0} \mu(G_0-x)\, \mu(dx)$$

(as $x \in G_0^c$ implies $\mu(G_0-x) = 0$), we have $\mu(G_0) = 1$. Now the definition of quasi-stability implies $\mu^{*2^m} = T_{c^m}\mu$; hence $\mu = T_{(1/c)^m}\mu^{*2^m} = T_{(1/c)^m}\mu^{*2^m-1} * T_{(1/c)^m}\mu$. Setting $(1/c)^m = t(m)$ and $T_{(1/c)^m}\mu^{*2^m-1} = \nu_t$, we see that $\mu = \nu_t * T_t\mu$. Now repeating the proof of (iv) to (vii) of Theorem 3.1 without any change at all, one shows $\mu(G-z_1) = 1$. Completing the proof.

The following corollary shows that nondegenerate $r$-semistable p.m's cannot have positive point mass.

Corollary 3.4 : *Let $\mu$ be a nondegenerate $r$-semistable measure of index $\alpha$ on a measurable vector space $(E, \mathscr{F})$. Assume that $\{x\} \in \mathscr{F}$, for all $x \in E$. Then $\mu\{x\} = 0$, for all $x \in E$.*

*Proof*: Let $G = \{0\}$ and $x \in E$. If $\mu\{G+x\} = \mu\{x\} > 0$, then, by Theorem 3.1, $\mu\{x\} = 1$. Hence $\mu$ is degenerate, a contradiction.

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