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AN INEQUALITY FOR MOMENTS OF SUMS OF TRUNCATED ϕ-MIXING RANDOM VARIABLES AND ITS APPLICATIONS

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SUMMARY. Let $\{X_n\}$ be a sequence of ϕ-mixing random variables. For $d > 1$, let $Y_{n,d} = X_n$ or 0 according as $|X_n| \leqslant d$ or not. In this paper, nontrivial bounds for moments of sums of $\{Y_{n,d}\}$ are obtained. These bounds are then applied to get the convergence rate in the Marcinkiewicz-Zygmund strong law for ϕ-mixing sequences.

1. Introduction

Let $\{X_n\}$ be a sequence of random variables. Let $\mathcal{B}_1^m$ be the σ-field generated by $X_1, \ldots, X_m$ and $\mathcal{B}_m^\infty$ be the σ-field generated by $X_m, X_{m+1}, \ldots$. The sequence $\{X_n\}$ is called ϕ-mixing if there exists a sequence

$$1 \geqslant \phi_1 \geqslant \phi_2 \quad \ldots$$

such that

$$\lim_{n \to \infty} \phi_n = 0$$

and

$$|P(A \cap B) - P(A)\, P(B)| \leqslant \phi_m P(A)$$

for all $A \in \mathcal{B}_1^h$, $B \in \mathcal{B}_{m+h}^\infty$, $h \geqslant 1$ and $m \geqslant 1$.

Let $d > 1$ be a real number and let

$$Y_i = \begin{cases} X_i & \text{if } |X_i| \leqslant d \\ 0 & \text{otherwise.} \end{cases}$$

For any real number $k \geqslant 2$ and integers $n \geqslant 1$, $h \geqslant 0$, let

$$D(n, k, h) = E\left|\sum_{i=1}^{n} Y_{i+h}\right|^k$$

and

$$D(n, k) = \sup_{h \geqslant 0} D(n, k, h).$$

In this paper we obtain bounds for $D(n, k)$, which are useful in studying convergence rates for partial sums of ϕ-mixing random variables. See Ghosh and Babu (1977) and Babu, Ghosh and Singh (1978). In the last section we give some applications of these bounds. Throughout this note K denotes a generic constant. The object of the paper is to prove the following proposition.

2. Main result

Proposition : *Let $\{X_n\}$ be a ϕ-mixing sequence with*

$$\sum_{n=1}^{\infty} \phi_n^{\delta} < \infty \qquad \dots \quad (2.1)$$

for some $0 < \delta \leqslant 1$ and satisfying for some $p > 0$ and $M > 1$,

$$\sup_{n \geqslant 1} E|X_n|^p \leqslant M. \qquad \dots \quad (2.2)$$

If $p > 1$, we further assume that $E(X_n) = 0$ for all $n \geqslant 1$, and $0 < \delta \leqslant \max\left(\frac{1}{p}, 1-\frac{1}{p}\right)$. Then for each $k \geqslant 2$, $q > 0$ and $p > q$, there exists a constant $a = a(k, p, q, M, \phi)$ such that

$$D(n, k) \leqslant a[n^{k/2}+nd^{k-p}], \qquad \dots \quad (2.3)$$

for all $1 \leqslant n \leqslant d^q$.

Proof: The main idea of the proof is borrowed from Doob (1953), (see Lemma 7.4, p. 225). We first note that for $r, s \geqslant 1$ such that $\frac{1}{r}+\frac{1}{s} = 1$,

$$|E(XY)-E(X)E(Y)| \leqslant 2\phi_h^{1/r}(E|X|^r)^{1/r}(E|Y|^s)^{1/s}, \qquad \dots \quad (2.4)$$

whenever X is measurable with respect to $\mathcal{B}_1^h$, $(E|X|^r)^{1/r} < \infty$, Y is measurable with respect to $\mathcal{B}_{h+n}^{\infty}$, $(E|Y|^s)^{1/s} < \infty$, and $h \geqslant 1$. If $r = 1$, $(E|Y|^s)^{1/s}$ is to be interpreted as ess.sup $|Y|$. For a proof of this inequality see Billingsley (1968, 170-171). Since $\phi_n^{\rho} \leqslant \phi_n^{\eta}$ for all $0 < \eta \leqslant \rho$ and $n \geqslant 1$, without loss of generality we can assume that $\delta = \max\left(\frac{1}{p}, 1-\frac{1}{p}\right)$ if $p > 1$, and $\delta = 1$ if $p \leqslant 1$. Observe that if $p > 1$, then

$$|E(Y_i)| = |E(X_i - Y_i)| \leqslant d^{1-p}E|X_i|^p \leqslant Md^{1-p}$$

and that if $p \leqslant 1$

$$|E(Y_i)| \leqslant E|Y_i| \leqslant d^{1-p}E|X_i|^p \leqslant Md^{1-p}.$$

So in any case

$$|E(Y_i)| \leq M d^{1-p}. \qquad \ldots \quad (2.5)$$

Next note that if $p \geq \theta > 1$, then

$$E|Y_i|^{\theta} \leq E|X_i|^{\theta} \leq 1+M \qquad \ldots \quad (2.6)$$

and that if $\theta > 1$ and $\theta \geq p$, then

$$E|Y_i|^{\theta} \leq d^{\theta-p} E|X_i|^{p} \leq M d^{\theta-p}. \qquad \ldots \quad (2.7)$$

Hence

$$(E|Y_i|^{1/\delta})^{\delta}(E|Y_j|^{1/(1-\delta)})^{1-\delta} \ll \begin{cases} d^{2-p} & \text{if } \delta p \leq 1 \text{ and } p(1-\delta) \leq 1 \\ 1 & \text{if } \ p > 1,\ 1 \leq \delta p \\ & \text{and } 1 \leq p(1-\delta). \end{cases} \qquad \ldots \quad (2.8)$$

Since, for $p > 1$, we assumed $\delta = \max\left(\frac{1}{p},\ 1-\frac{1}{p}\right)$, we have by (2.4), (2.5) and (2.8) that for $i, j \geq 1$.

$$|E(Y_i Y_{i+j})| \leq 2\phi_j^{\delta}(E|Y_i|^{1/\delta})^{\delta}(E|Y_{i+j}|^{1/(1-\delta)})^{1-\delta} + |E(Y_i)E(Y_{i+j})|$$

$$\ll \phi_j^{\delta}(1+d^{2-p}) + d^{2-2p}. \qquad \ldots \quad (2.9)$$

Since

$$D(n, 2, h) \leq \sum_{i=1}^{n} E(Y_{i+h}^2) + 2\sum_{i=1}^{n-1}\sum_{j=1}^{n-i} |E(Y_{i+h}Y_{i+j+h})|,$$

we have by (2.1), (2.6), (2.7) and (2.9) that if $1 \leq n \leq d^q$, then

$$D(n, 2) \ll n + nd^{2-p} + n^2 d^{2-2p}$$

$$\ll n(1+d^{2-p}).$$

To prove the proposition by induction, assume next that $1 \leq n \leq d^q$ and for some integer $m \geq 2$,

$$D(n, m) \ll n^{m/2} + nd^{m-p} \qquad \ldots \quad (2.10)$$

and prove a similar inequality for $D(n, k)$, where $k - m = \varepsilon$ and $0 < \varepsilon \leq 1$.

Fix an integer $h \geqslant 0$. Define $Z_n = \sum_{i=1}^{n} Y_{i+h}$, $Z_{n,t} = Z_{2n+t} - Z_{n+t}$ and $S_{n,t} = \sum_{i=1} Y_{n+h+i}$, $t > 1$ is an integer. Then

$$E|Z_n + Z_{n,t}|^k \leqslant E[(|Z_n| + |Z_{n,t}|)^m (|Z_n|^\epsilon + |Z_{n,t}|^\epsilon)]$$

$$\leqslant \sum_{j=0}^{m} \binom{m}{j} E[|Z_n|^{m-j} |Z_{n,t}|^j (|Z_n|^\epsilon + |Z_{n,t}|^\epsilon)]. \quad \ldots \quad (2.11)$$

By (2.4) we have for $j = 1, \ldots, m$,

$$\left| E|Z_n|^{k-j} |Z_{n,t}|^j - E|Z_n|^{k-j} E|Z_{n,t}|^j \right| \leqslant 2\phi_t^{\epsilon/(m+1)} D(n, k). \quad \ldots \quad (2.12)$$

It follows from (2.1), (2.11), (2.12) and Hölder's inequality that

$$E|Z_n + Z_{n,t}|^k \leqslant 2(1 + Kt^{-\epsilon/2\delta m}) D(n, k) + KD^{k/m}(n, m). \quad \ldots \quad (2.13)$$

Since $n \leqslant d^q$, we have

$$(nd^{m-p})^{k/m} = (nd^{k-p})(nd^{-p})^{\epsilon/m} \leqslant n^\alpha d^{k-p} \quad \ldots \quad (2.14)$$

for some $\alpha \in (0, 1)$, which depends only on k, p and q. Observe that for any $v > 0$,

$$\sup_{h \geqslant 0} E|Y_h|^v \leqslant H(v) = \begin{cases} 2M & \text{if } v \leqslant p. \\ Md^{v-p} & \text{if } v > p. \end{cases} \quad \ldots \quad (2.15)$$

Using Minkowski's inequality (2.13) and (2.14) we obtain that

$$D(2n, k, h) = E|Z_n + Z_{n,t} + S_{n,t} - S_{2n,t}|^k$$

$$\leqslant 2\{[(1 + Kt^{-\epsilon/2\delta m}) D(n, k) + K(n^{k/2} + n^\alpha H(k))]^{1/k} + 2tH^{1/k}(k)\}^k$$

$$\leqslant 2[(1 + Kt^{-\epsilon/2\delta m}) D(n, k) + K(n^{k/2} + n^\alpha H(k))](1 + Ktn^{-\alpha/k})^k$$

$$\ldots \quad (2.16)$$

Taking $2n = 2^j \leqslant d^q$ and $t = [2^{\epsilon j/2k}]$ in (2.16), we obtain that

$$D(2^j, k) \leqslant 2(1 + Kb^j) D(2^{j-1}, k) + K2^{jk/2} + K2^{j\alpha} H(k), \quad \ldots \quad (2.17)$$

where $[x]$ denotes the largest integer $\leqslant x$ and $b = 2^{-\min(\epsilon\alpha/48mk,\ \alpha/2k)}$.

Note that $0 < b < 1$. Repeating (2.17) j times and observing that $\prod_{s=1}^{\infty}(1+Kb^s) < \infty$, we obtain that

$$D(2^j, k) \leqslant K\left[2^j D(1, k)+2^{jk/2}\left(\sum_{s=0}^{j} 2^{-s(k-2)/2}\right)+H(k)2^{j/2}\left(\sum_{s=0}^{j} 2^{s(1-\varepsilon)}\right)\right]$$

$$\leqslant 2^j d^{k-p}+2^{jk/2}.$$

The last step above follows from (2.15).

To complete the proof we use a binary decomposition of n for any positive integer $n \leqslant d^q$ and obtain inequalities similar to (2.22) and (2.23) of Ghosh and Babu (1977). The details are omitted.

3. Applications

In this section we apply the moment inequality proved above to obtain the following theorems. Theorem 1 is similar to the results of Lai (1977). The method adopted here is different from his and is comparatively simple.

Theorem 1 : *Let $\{X_n\}$ be a ϕ-mixing sequence satisfying, for some $p > 0$ and $\varepsilon > 1$,*

$$\sup_{n \geqslant 1} E\{|X_n|^p(\log(1+|X_n|))^{\varepsilon}\} < \infty. \quad \ldots \quad (3.1)$$

Let $2\alpha > 1$ and $p\alpha > 1$.

(a) *If $p \leqslant 1$ and $\sum_{n=1}^{\infty} \phi_n < \infty$, then*

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\sup_{j \leqslant n}\left|\sum_{i=1}^{j} X_i\right| > n^{\alpha}\right) < \infty. \quad \ldots \quad (3.2)$$

(b) *If $p > 1$, $E(X_n) = 0$ for all $n \geqslant 1$ and $\sum_{n=1}^{\infty} \phi_n^{\delta} < \infty$, for some $0 < \delta \leqslant \max(1/p, (p-1)/p)$, then (3.2) holds.*

Theorem 2 : *Suppose $\{X_n\}$ is a stationary ϕ-mixing sequence satisfying, for some $p > 0$ and $\varepsilon > 0$,*

$$E\{|X_1|^p(\log(1+|X_1|))^{\varepsilon}\} < \infty. \quad \ldots \quad (3.3)$$

Then the assertions in Theorem 1 hold.

Remarks : Lai (1977) has proved a result similar to Theorem 2, assuming further for some $\eta > 1$ and $m \geqslant 1$, as $x \to \infty$,

$$\sup_{i > m} P(|X_1| > x,\ |X_i| > x) = O(P^{\eta}(|X_1| > x)). \qquad \dots \quad (3.4)$$

But the result is proved under $E|X_1|^p < \infty$ and without any condition on ϕ. Theorem 1 generalizes these results to non-stationary mixing sequences. Our proofs depend on the repeated use of the proposition given in Section 2.

We first prove the following lemmas :

Lemma 1 : *Let $A > 0$ and $\beta > 0$. Under the assumptions either of Theorem 1 or Theorem 2, there exists a $K(A, \beta) > 0$ such that for all $n \geqslant 3$, $h \geqslant 0$ and $1 \leqslant j \leqslant n-h$,*

$$P\left(\left|\sum_{i=1}^{j} X_{i+h,\,n}\right| > An^{\alpha}\right) \leqslant K(A, \beta) j n^{-\alpha p} (\log n)^{-2}, \qquad \dots \quad (3.5)$$

where $X_{i,n} = X_i$ or 0 according as $|X_i| \leqslant n^{\alpha} (\log n)^{-\beta}$ or not.

Proof : Put $S_0 = 0$ and $S_j = \sum_{i \leqslant j} X_{i,n}$. By the proposition, for each $\theta \geqslant 2$, there exists a $K(\theta)$ such that

$$E|S_{j+h} - S_h|^{\theta} \leqslant K(\theta)(j^{\theta/2} + j n^{\alpha(\theta-p)} (\log n)^{-\beta(\theta-p)}).$$

Since $2\alpha > 1$, by taking $\theta = \max(3, (2p\alpha+4)/(2\alpha-1),\ p+4\beta^{-1})$ and using Chebyshev's inequality, we obtain that

$$\begin{aligned} P(|S_{j+h} - S_h| > An^{\alpha}) &\leqslant A^{-\theta} n^{-\theta\alpha} E|S_{j+h} - S_h|^{\theta} \\ &\leqslant 2A^{-\theta} K(\theta)\, j n^{-p\alpha} (\log n)^{-2}. \end{aligned}$$

This completes the proof.

Lemma 2 : *Let $\beta > 0$ and $A > 0$. Let $X_{i,n}$ and S_i be as in Lemma 1. Then*

$$P(\sup_{j \leqslant n} |S_j| > 3An^{\alpha}) = O(n^{1-p\alpha} (\log n)^{-2}).$$

Proof : We follow the method given in Billingsley (1968). For $1 \leqslant i \leqslant n$, let

$$E_i = (\sup_{j < i} |S_j| \leqslant 3An^{\alpha},\ |S_i| > 3An^{\alpha}).$$

Notice that the events $\{E_i,\ 1 \leqslant i \leqslant n\}$ are disjoint and

$$E = \bigcup_{i=1}^{n} E_i = (\sup_{j \leqslant n} |S_j| > 3An^{\alpha}).$$

We have

$$P(E) \leqslant \sum_{i=1}^{n} P(E_i \cap (|S_n| < An^{\alpha})) + P(|S_n| > An^{\alpha})$$

$$\leqslant \sum_{i=1}^{n} P(E_i \cap (|S_n - S_i| > 2An^{\alpha})) + P(|S_n| > An^{\alpha}). \quad \ldots \quad (3.6)$$

Let q be a positive integer such that $4\phi(q) < 1$. Let $n_0 > q$ be such that $4K(A, \beta)n_0^{1-p\alpha} < 1$, where $K(A, \beta)$ is as in (3.5). Then for $1 \leqslant i \leqslant n-q$, $n > n_0$, we have, by the ϕ-mixing inequality, that

$$P(E_i \cap (|S_n - S_i| > 2An^{\alpha}))$$
$$\leqslant P(E_i \cap (|S_n - S_{i+q}| > An^{\alpha})) + P(|S_{i+q} - S_i| > An^{\alpha})$$
$$\leqslant P(E_i)[P(|S_n - S_{i+q}| > An^{\alpha}) + \phi(q)] + P(|S_{i+q} - S_i| > An^{\alpha}). \quad \ldots \quad (3.7)$$

Now by Lemma 1, (3.6) and (3.7), we have for all $n \geqslant n_0$,

$$P(E) \leqslant \frac{1}{2} P(E) + \sum_{i=n-q+1}^{n} P(|S_n - S_i| > 2An^{\alpha})$$

$$+ \sum_{i=1}^{n-q} P(|S_{i+q} - S_i| > An^{\alpha}) + P(|S_n| > An^{\alpha})$$

$$\leqslant \frac{1}{2} P(E) + O(n^{1-p\alpha}(\log n)^{-2}).$$

This completes the proof of the lemma.

Lemma 3 : *Let W be a non-negative valued random variable with $E(W) < \infty$. Then for any $\theta > 1$,*

$$\Sigma n^{\theta-1} P(W > n^{\theta}) < \infty.$$

Proof : $$\sum_{n=1}^{\infty} n^{\theta-1} P(W > n^{\theta})$$

$$= \sum_{n=1}^{\infty} n^{\theta-1} \sum_{k=n}^{\infty} P(k^{\theta} < W \leqslant (k+1)^{\theta})$$

$$= \sum_{k=1}^{\infty} P(k^{\theta} < W \leqslant (k+1)^{\theta}) \left(\sum_{n=1}^{k} n^{\theta-1} \right)$$

$$= O\left(\sum_{k=1}^{\infty} k^{\theta} P(k^{\theta} < W \leqslant (k+1)^{\theta}) \right)$$

$$= O(E(W)) < \infty.$$

Proofs of Theorems 1 *and* 2 : By Lemma 2, it follows for any $\beta > 0$, that

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P(\sup_{j \leq n} |S_j| > n^{\alpha}) < \infty.$$

Since

$$P\left(\sup_{1 \leq j \leq n} \left|\sum_{i=1}^{j} X_i\right| > n^{\alpha}\right) \leq P(\sup_{j \leq n} |S_j| > n^{\alpha})$$

$$+ \sum_{i=1}^{n} P(|X_i| > n^{\alpha} (\log n)^{-\beta}),$$

it is enough to show that

$$\sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P(|X_i| > n^{\alpha}(\log n)^{-\beta})) < \infty.$$

Put $W_i = |X_i|^p (\log(1+|X_i|))^{\varepsilon}$. In the case of Theorem 1, there exists a $t > 0$, $n_1 > 0$ such that for all $n \geq n_1$,

$$\sum_{i=1}^{n} P(|X_i| > n^{\alpha}(\log n)^{-\beta}) \leq \sum_{i=1}^{n} P(|W_i| > t n^{\alpha p}(\log n)^{\varepsilon-\beta p})$$

$$\leq t^{-1} (\log n)^{\beta p-\varepsilon} n^{1-p\alpha} (\sup_{i \geq 1} E|W_i|).$$

Since in this case (3.1) holds with some $\varepsilon > 1$, Theorem 1 follows by taking $\beta = (\varepsilon-1)/2p$.

To prove Theorem 2, take $\beta = \varepsilon/p$. There exists a $b > 0$ and $n_2 > 0$ such that for all $n \geq n_2$,

$$\sum_{i=1}^{n} P(|X_i| > n^{\alpha}(\log n)^{-\beta}) \leq n P(|W_1| > bn^{p\alpha}).$$

Now Lemma 3, with $\theta = \alpha p$ and $W = |W_1| b^{-1}$, yields Theorem 2.

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