Sankhyā : The Indian Journal of Statistics
1978, Volume 40, Series A, Pt. 4, pp. 347-368.

ON SOME NONUNIFORM RATES OF CONVERGENCE TO NORMALITY

By MALAY GHOSH

Indian Statistical Institute and *Iowa State University*
and
RATAN DASGUPTA

Indian Statistical Institute

SUMMARY. For row sums of independent random variables in a triangular array nonuniform rates of convergence to normality are studied. These results unify and extend earlier available results in this direction. Included as special cases are rates for standardized sums of i.i.d random variables. Next these results are used in proving moment convergences of such sums, in deriving L_p versions of the Berry-Esseen theorem, and also in finding probabilities of moderate deviations. Also, some general results regarding nonuniform rates of convergence to normality for nonlinear statistics are given. Applications are made in the case of L-statistics.

1. Introduction

Consider a sequence $\{X_i;\ i \geqslant 1\}$ of independent and identically distributed random variables with $E(X_1) = 0$, $E(X_1^2) = 1$. Let $S_n = \sum_1^n X_i$, $F_n(t) = P(n^{-1/2} S_n \leqslant t)$; t real, $n \geqslant 1$. The celebrated Berry-Esseen theorem states that if $\rho = E|X_1|^3 < \infty$, then $\sup|F_n(t) - \Phi(t)| \leqslant C\rho n^{-1/2}$, where C is a universal constant and $\Phi(t)$ is the distribution function of a $N(0, 1)$ variable. This result was later strengthened by Katz (1963). He showed that under the assumption $E[X_1^2\, g(X_1)] < \infty$, where $g(x)$ is a nonnegative, even, nondecreasing function on $[0, \infty)$ satisfying

$$\lim_{|x| \to \infty} g(x) = \infty, \qquad \dots \quad (1.1)$$

and

$$|x|/g(x) \text{ is defined for all } x \text{ and nondecreasing on } [0, \infty), \qquad \dots \quad (1.2)$$

one has

$$\sup |F_n(t) - \Phi(t)| = O((g(n^{1/2}))^{-1}). \qquad \dots \quad (1.3)$$

More recently Michel (1976) has given rates of convergence of $F_n(t)$ to $\Phi(t)$ depending on both n and t under the assumption $E|X_1|^{c+2} < \infty$ for some $c > 0$. These bounds reflect that rates of convergence to normality are much faster for large values of $|t|$ than for its small values. These nonuniform rates (as we shall see later) are quite useful in studying probabilities of moderate deviations, in getting L_p versions of the Berry-Esseen theorem, and also in studying certain moment convergences.

In this paper, we first generalize and extend Michel's (1976) results to row sums of independent random variables in a triangular array. Consider the double sequence $\{X_{ni};\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ of random variables, where the random variables within each row are independently distributed and satisfy $E(X_{ni}) = 0$ and

$$\sup_{n \geqslant 1} \sup_{1 \leqslant i \leqslant n} E[|X_{ni}|^{2+c} u(X_{ni})] < \infty, \qquad \ldots \quad (1.4)$$

where,

$u(x)$ is a nonnegative, even, nondecreasing function on $[0, \infty)$ with $u(x) < |x|^{\epsilon} + L$ for all $\epsilon > 0$, with some $L > 0$ and $\lim_{x \to 0} u(x) = 0$ when $c = 0$. ... (1.5)

Examples of functions $u(x)$ satisfying (1.5) are $u(x) = 1$, $u(x) = \log(1+|x|)$, $u(x) = \log \log(e+|x|)$ etc. Define $s_n^2 = \sum_{i=1}^{n} EX_{ni}^2 (n \geqslant 1)$. It is also assumed that

$$\inf_{n \geqslant 1} n^{-1} s_n^2 > 0. \qquad \ldots \quad (1.6)$$

In the special case when the X_i's are i.i.d r.v's, our assumptions are identical with those of Katz (1963) when $0 < c < 1$ or $c = 0$ and

$$\lim_{|x| \to \infty} u(x) = \infty, \quad \text{or} \quad c = 1 \quad \text{and} \quad u(x) = 1.$$

We define now $S_n = \sum_{1}^{n} X_{ni}$, $F_n(t) = P(S_n s_n^{-1} \leqslant t)$. In Section 2, we derive some nonuniform bounds for $|F_n(t) - \Phi(t)|$ for different values of t, and use there to study the speed of convergence of $1 - F_n(t)$ to zero as $t \to \infty$, the speed of convergence of the moments of $|s_n^{-1} S_n|$ to those of the $|N(0, 1)|$ variable, and certain L_p-versions of the Berry-Esseen theorem. In Section 3, the results of Section 2 are extended to certain nonlinear statistics. As an application, we include in Section 4 the L-statistics.

2. The results on row sums of random variables in a traingular array

First we prove two theorems giving rates of convergence of $F_n(t)$ to $\Phi(t)$ depending on both n and t. In the special case of sums of i.i.d random variables, these include more general versions of Theorems 1 and 2 of Michel (1976). For sums of i.i.d random variables, our theorems are quite in the spirit of Katz's (1963) extension of the classical Berry-Esseen theorem.

Theorem 1: *Suppose* (1.4)-(1.6) *hold. Then for*

$$t^2 \leqslant K\left(\frac{1}{2}c \log n + \log u(\sqrt{n})\right) K > 0,$$

there exists positive constants b and r (depending on u, c and K) such that

$$|F_n(t)-\Phi(t)| \leqslant bw \exp\left[-\frac{1}{2}t^2(1-3r)\right] + \sum_{i=1}^{n} P(|X_{ni}| > rs_n|t|) \quad \ldots \quad (2.1)$$

where

$$w = w(n, |t|, c) = (n^{1/2}|t|)^{-c} \quad (u(rs_n|t|))^{-1} \text{ or } n^{-1/2}$$

according as $0 \leqslant c < 1$ *or* $c \geqslant 1$.

Theorem 2: *Suppose* (1.4) - (1.6) *hold. Then for*

$$t^2 \geqslant K\left[\frac{c}{2} \log n + \log u(\sqrt{n})\right]$$

there exist $b(>0)$, $r(>0)$ *depending on u, c and K such that*

$$|F_n(t)-\Phi(t)| \leqslant b[n^{c/2}u(\sqrt{n})]^{-\frac{1}{2}(K-1)} \; |t|^{-2(K+1)} + \sum_{i=1}^{n} P(|X_{ni}| > rs_n|t|).$$

$$\ldots \quad (2.2)$$

Proof of Theorem 1 : Throughout the proof $b_1, b_2, \ldots$ denote positive constants which might depend on u and c, but not on n and t. The theorem is obvious for $t = 0$. We prove the theorem only for $t > 0$, as the proof when $t < 0$ is analogous. For $0 < t \leqslant 1$, the theorem follows immediately from Katz's (1963) theorem. For $t > 1$, let

$$Y_i = Y_{ni} = X_{ni} I_{[|X_{ni}| \leqslant rs_n t]}, \quad i = 1, \ldots, n, \qquad \ldots \quad (2.3)$$

I being the usual indicator function. Define $S'_n = \sum_{1}^{n} Y_i \; (n \geqslant 1)$. Then,

$$|P(s_n^{-1}\bar{S}'_n \leqslant t) - F_n(t)| \leqslant \sum_{i=1}^{n} P(|X_{ni}| > rs_n t). \qquad \ldots \quad (2.4)$$

Next define

$$f_i(t) = f_{n,i}(t) = E\exp(tY_i/s_n), \quad i = 1, \ldots, n; \qquad \ldots \quad (2.5)$$

$$m_i(t) = f_i^{-1}(t)E[Y_i \exp(tY_i/s_n)], \quad i = 1, \ldots, n;$$

$$\bar{m}_n(t) = n^{-1}\sum_{i=1}^{n} m_i(t); \qquad \ldots \quad (2.6)$$

$$m_i^2(t)+\sigma_i^2(t) = f_i^{-1}(t)E[Y_i^2 \exp(tY_i/s_n)], \quad i = 1, \ldots, n;$$

$$\bar{\sigma}_n^2(t) = n^{-1}\sum_{i=1}^{n} \sigma_i^2(t); \qquad \ldots \quad (2.7)$$

$$H_n(z) = P((S_n' - n\bar{m}_n(t))/(\sqrt{n}\bar{\sigma}_n(t)) \leqslant z). \qquad \ldots \quad (2.8)$$

Then standard methods (see e.g., Cramér, 1938 or Bahadur and Ranga Rao, 1960) yield

$$P(s_n^{-1}S_n' > t) = A_n(t)\int_{B_n(t)}^{\infty} \exp(-ts_n^{-1}\, n^{\frac{1}{2}}\bar{\sigma}_n\, z)dH_n(z), \qquad \ldots \quad (2.9)$$

where

$$A_n(t) = \prod_{i=1}^{n} f_i(t) \exp(-ts_n^{-1}\, n\bar{m}_n(t)); \qquad \ldots \quad (2.10)$$

$$B_n(t) = (ts_n - n\bar{m}_n(t))/(\sqrt{n}\bar{\sigma}_n(t)). \qquad \ldots \quad (2.11)$$

Using (1.4) - (1.6), one has the estimates

$$|EY_i| = O((rs_nt)^{-(c+1)}u^{-1}(rs_nt)), \qquad 1 \leqslant i \leqslant n; \qquad \ldots \quad (2.12)$$

$$0 \leqslant EX_{ni}^2 - EY_i^2 = O((rs_nt)^{-c}u^{-1}(rs_nt)), \quad 1 \leqslant i \leqslant n; \qquad \ldots \quad (2.13)$$

$$E|Y_i|^3 = \begin{cases} O(1) & \text{if } c \geqslant 1, \\ O((rs_nt)^{1-c}(u(rs_nt))^{-1}) & \text{if } 0 \leqslant c < 1. \end{cases} \qquad \ldots \quad (2.14)$$

Now, using (2.12) - (2.14)

$$\left| f_i(t) - 1 - \frac{t^2}{2s_n^2}EX_{ni}^2 \right| \leqslant bwn^{-1}\exp\left(\frac{5}{4}rt^2\right). \qquad \ldots \quad (2.15)$$

Next we show that $w\exp\left(\frac{5}{4}rt^2\right) = o(1)$, by proper choice of $r > 0$. For $0 \leqslant c < 1$

$$w\exp\left(\frac{5}{4}rt^2\right) \leqslant (n^{\frac{1}{2}}t)^{-c}\, u^{-1}(rs_nt)\, (n^{c/2}u(\sqrt{n}))^{\frac{5}{4}rk} = o(1) \qquad \ldots \quad (2.16)$$

if $r < \min\left(\frac{4}{5}K^{-1}, 1\right)$. Again, for $c \geqslant 1$, since $u(x) < |x|^{\epsilon}+L$ for all $\epsilon > 0$, one gets

$$w \exp\left(\frac{5}{4} rt^2\right) = n^{-\frac{1}{2}}\left(n^{c/2}u(\sqrt{n})\right)^{\frac{5}{4}rk} = o(1) \qquad \dots \quad (2.17)$$

if $r < 4/(5Kc)$. Choose $0 < r < \min\left(1, (5K)^{-1}(c \vee 1)^{-1}\right)$, so that both (2.16) and (2.17) hold. Now from (2.15) - (2.17),

$$\sum_{i=1}^{n} \log f_i(t) = \frac{1}{2} t^2+o\left(w \exp\left(\frac{5}{4}rt^2\right)\right). \qquad \dots \quad (2.18)$$

Next note that

$$E[Y_i \exp(ts_n^{-1} Y_i)] = ts_n^{-1} EX_{ni}^2+o\left(n^{-\frac{1}{2}}w \exp\left(\frac{5}{4} rt^2\right)\right). \qquad \dots \quad (2.19)$$

$$E\left[Y_i^2 \exp\left(ts_n^{-1} Y_i\right)\right] = EX_{ni}^2+o\left(n^{-\frac{1}{2}}w \exp\left(\frac{5}{4} rt^2\right)\right). \qquad \dots \quad (2.20)$$

Hence, from (2.6), (2.7), (2.15), (2.19) and (2.20) one gets

$$m_i(t) = ts_n^{-1} EX_{ni}^2+o\left(n^{-\frac{1}{2}} w \exp\left(\frac{5}{4} rt^2\right)\right); \qquad \dots \quad (2.21)$$

$$m_i^2(t)+\sigma_i^2(t) = EX_{ni}^2 +o\left(n^{-\frac{1}{2}} w \exp\left(\frac{5}{4} rt^2\right)\right). \qquad \dots \quad (2.22)$$

Thus,

$$\bar{m}_n(t) = tn^{-1} s_n+o\left(n^{-\frac{1}{2}} w \exp\left(\frac{5}{4} rt^2\right)\right); \qquad \dots \quad (2.23)$$

$$\bar{\sigma}_n^2(t) = n^{-1} s_n^2+o\left(n^{-\frac{1}{2}} w \exp\left(\frac{5}{4} rt^2\right)\right). \qquad \dots \quad (2.24)$$

Hence, from (2.10), (2.18) and (2.23),

$$A_n(t) = \exp\left(\frac{1}{2}t^2\right)\left[1+o\left(w \exp\left(\frac{5}{4} rt^2\right)\right)\right]\exp(-t^2)\left[1+o\left(w \exp\left(\frac{3}{2} rt^2\right)\right)\right]$$

$$= \exp\left[\left(-\frac{1}{2}t^2\right)+o\left(w \exp\left(\frac{3}{2} rt^2\right)\right)\right], \qquad \dots \quad (2.25)$$

where $w \exp\left(\frac{3}{2} rt^2\right) = o(1)$ by choosing $r (> 0)$ appropriately small.

Also, from (2.11), (2.23) and (2.24) one gets,

$$B_n(t) = o\left(w \exp\left(\frac{3}{2} rt^2\right)\right) \quad \ldots \quad (2.26)$$

Finally, from (2.9) one gets

$$|P(s_n^{-1} S_n' \leqslant t)-\Phi(t)| = |P(s_n^{-1} S_n' > t)-\Phi(-t)|$$

$$= \left| A_n(t) \int_{B_n(t)}^{\infty} \exp(-ts_n^{-1} n^{\frac{1}{2}} \bar{\sigma}_n z)\, dH_n(z)-\Phi(-t) \right|$$

$$\leqslant I_1+I_2+I_3, \quad \ldots \quad (2.27)$$

where

$$I_1 = \left| A_n(t) \int_{B_n(t)}^{\infty} \exp(-ts_n^{-1} n^{\frac{1}{2}} \bar{\sigma}_n z)\, d(H_n(z)-\Phi(z)) \right|; \quad \ldots \quad (2.28)$$

$$I_2 = \left| A_n(t)-\exp\left(-\frac{1}{2} t^2\right)\right| \int_{B_n(t)}^{\infty} \exp(-ts_n^{-1} n^{\frac{1}{2}} \bar{\sigma}_n z)\, d\Phi(z); \quad \ldots \quad (2.29)$$

$$I_3 = \left| \exp\left(-\frac{1}{2} t^2\right) \int_{B_n(t)}^{\infty} \exp(-ts_n^{-1} n^{\frac{1}{2}} \bar{\sigma}_n z)\, d\Phi(z)-\Phi(-t) \right|. \quad \ldots \quad (2.30)$$

Using (2.24)-(2.26), the Berry-Esseen theorem, the c_r—inequality with $r = 3$ and (2.14) one gets

$$I_1 \leqslant \exp\left[-\frac{1}{2} t^2+o\left(w \exp\left(\frac{3}{2} rt^2\right)\right)\right]$$

$$\exp\left[-t\left(1+o\left(w \exp\left(\frac{5}{4} rt^2\right)\right)\right)\left(o\left(w \exp\left(\frac{3}{2} rt^2\right)\right)\right)\right] \cdot \sup|H_n(z)-\Phi(z)|$$

$$\leqslant b_1 \exp\left[-\frac{1}{2} t^2+o\left(w \exp\left(\frac{3}{2} rt^2\right)\right)\right] \frac{\sum_{i=1}^{n} E|Y_i-m_i(t)|^3}{n^{3/2}\bar{\sigma}^3}$$

$$\leqslant b_2 \exp\left(-\frac{1}{2} t^2\right)\left[1+o\left(w \exp\left(\frac{3}{2} rt^2\right)\right)\right]$$

$$w \exp(rt^2)\left[1+o\left(w \exp\left(\frac{5}{4} rt^2\right)\right)\right]^{-3}$$

$$\leqslant bw \exp\left(-\frac{1}{2}t^2+rt^2\right); \quad \ldots \quad (2.31)$$

$$I_2 \leqslant \left| A_n(t)-\exp\left(-\frac{1}{2}t^2\right)\right| \exp\left(\frac{1}{2}t^2 s_n^{-2} n\sigma_n^2\right) \Phi(-B_n(t)-ts_n^{-1}n^{\frac{1}{2}}\bar{\sigma}_n)$$

$$\leqslant \exp\left(-\frac{1}{2}t^2\right) o\left(w\exp\left(\frac{3}{2}rt^2\right)\right) \exp\left\{\left(\frac{1}{2}t^2\right)\left(1+o\left(n^{-\frac{1}{2}}w\exp\left(\frac{5}{4}rt^2\right)\right)\right)\right\}$$

$$[B_n(t)+ts_n^{-1}n^{\frac{1}{2}}\sigma_n]^{-1}\exp\left(-\frac{1}{2}(B_n(t)+ts_n^{-1}n^{\frac{1}{2}}\bar{\sigma}_n)^2\right)$$

$$\leqslant bw\exp\left(-\frac{1}{2}t^2+\frac{3}{2}rt^2\right); \qquad \ldots (2.32)$$

$$I_3 = \left| \exp\left(-\frac{1}{2}t^2+\frac{1}{2}t^2 s_n^{-2} n\bar{\sigma}_n^2\right) \Phi(-B_n(t)-ts_n^{-1}n^{\frac{1}{2}}\bar{\sigma}_n)-\Phi(-t)\right|$$

$$\leqslant b_1\left[1+o\left(w\exp\left(\frac{3}{2}rt^2\right)\right)\right] o\left(w\exp\left(\frac{3}{2}rt^2\right)\right) \exp\left(-\frac{1}{2}t^2\right)$$

$$+b_2\exp\left(-\frac{1}{2}t^2\right) o\left(w\exp\left(\frac{3}{2}rt^2\right)\right)$$

$$\leqslant bw\exp\left(-\frac{1}{2}t^2+\frac{3}{2}rt^2\right). \qquad \ldots (2.33)$$

The theorems now follow from (2.9) and (2.27)—(2.33).

Proof of Theorem 2 : The result is trivially true for $K = 1$ by using the same truncation as of Theorem 1. For $K > 1$, first note that for $t > 0$

$$\Phi(-t) \leqslant t^{-1}(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}t^2\right)$$

$$= t^{-1}(2\pi)^{-\frac{1}{2}}\exp\left[-\frac{t^2(K-1)}{2K}-\frac{t^2}{2K}\right].$$

$$\leqslant t^{-1}(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{t^2}{2K}\right)(n^{c/2}u(\sqrt{n}))^{-\frac{1}{2}}(K-1)$$

$$\leqslant b(n^{c/2}u(\sqrt{n}))^{-\frac{1}{2}(K-1)}\,t^{-2(K+1)}.$$

The rest of the proof of the theorem for $t > 0$ follows the lines of Michel (1976) by taking $r = (2K(K+1))^{-1}$ and

$$h = t^{-1}n^{-\frac{1}{2}}\left[(K-1)\left(\frac{1}{2}c\log n+\log u(\sqrt{n})\right)+2K(K+1)\log t\right].$$

For $t < 0$, the proof is similar.

Remark 1 : In the special case of i.i.d.r.v's when $u(x) = 1$ for all x, $c > 0$, Michel's Theorem 1 follows as a special case of our Theorem 1 by taking $K = 2(1+c^{-1})$. However, with the same choice of K, Theorem 2 of Michel does not follow from (2.2). However, again in the case of i.i.d.r.v's, under the conditions of Theorem 2, one can for $c > 0$, obtain instead of (2.2) the bound

$$|F_n(t)-\Phi(t)| \leqslant bn^{(-Kc-2)/4}(u(\sqrt{n}))^{(-Kc-2)/2c}\,t^{-2Kc}+nP(|X_1| > rn^{\frac{1}{2}}|t|)$$

taking

$$0 < r < \min[c(Kc-2)^{-1},\ (2+c)(Kc)^{-2}]$$

and

$$h = t^{-1}n^{-\frac{1}{2}}\left[(Kc-2)\left(\frac{1}{2}\right)\log n +c^{-1}\log u(\sqrt{n})+(Kc)^2\log t\right] \text{ for } K > 2c^{-1}.$$

Then taking $K = 2(1+c^{-1})$ one obtains a stronger form of Michel's (1976) Theorem 2.

From Theorems 1 and 2 (by a proper choice of $K > 0$) it is easy to derive the following nonuniform Berry-Esseen theorem which generalizes Theorem 3 of Michel (1976) and Theorem 3 of Nagaev (1965). Included also is a corresponding uniform Berry-Esseen theorem of Katz (1963).

Theorem 3 : *There exists a constant $b(>0)$ depending only on c and u, such that for all t*

$$|F_n(t)-\Phi(t)| = b(1+|t|^{2+c})^{-1}[n^{\frac{1}{2}c}u(\sqrt{n})\wedge n^{\frac{1}{2}}]^{-1} \qquad \ldots \quad (2.34)$$

where $u\wedge v = \min(u, v)$.

Remark 1 : (a) The order of t in (2.34) can be improved in general.. From Theorems 1 and 2 it is easy to obtain

$$|F_n(t)-\Phi(t)| \leqslant b[n^{c/2}u(\sqrt{n})\wedge n^{1/2}]^{-1}\,|t|^{-k^*}+\sum_{i=1}^{n} P|(X_{ni}| > rs_n|t|)$$

$$\leqslant b[n^{c/2}u(\sqrt{n})\wedge n^{1/2}]^{-1}|t|^{-k^*}+b_1n^{-c/2}|t|^{-(2+c)}\,(u(r\sqrt{n}t))^{-1}$$

for any $k^* > 0$ and $b\ (>0)$ being a constant depending on k^*, and hence

$$|F_n(t)-\Phi(t)| \leqslant b_2 n^{-1/2}(1+|t|^{2+c})^{-1}(1+u(t))^{-1} \text{ if } c \geqslant 1$$

$$\leqslant b_2[n^{c/2}u(\sqrt{n}]^{-1}(1+|t|^{2+c})^{-1}\left(1+\frac{u(r\sqrt{n}t)}{u(\sqrt{n})}\right)^{-1}$$

for $0 \leqslant c < 1$ which is further inprovoment of (2.34).

Next Theorems 1 and 2 are used for proving convergence of moments of $Y_n = |s_n^{-1}S_n|$ to those of $T = |N(0, 1)|$. Related results of von Bahr (1965) and Michel (1976) are special cases of the following theorem, where $\lambda_1, \lambda_2, \lambda_3$ are positive constants.

Theorem 4 : *Suppose that the assumptions of Theorems* 1 *and* 2 *are satisfied with* $u'(x) < \lambda_1 + \lambda_2 x^{\lambda_3}$, $0 < x < \infty$; *then*

$$|E(Y_n^{2+c}u(Y_n)) - E(T^{2+c}u(T))| = \begin{cases} o(n^{-c}) & \text{if } \lim_{x\to\infty} u(x) = \infty, 0 \leqslant c < 1 \\ O(n^{-c^*}) & \text{otherwise,} \end{cases} \quad \dots \quad (2.35)$$

where $c^* = \frac{1}{2}\min(c, 1)$.

We omit the proof of this theorem as it follows the lines of Michel (1976). The bound (2.35) might not appear very useful when $c = 0$. But even in that case if $\lim_{|x|\to\infty} u(x) = \infty$, the l.h.s of (2.35) converges to zero.

Erickson (1973) has derived L_p-version of the Berry-Esseen theorem. Our next theorem also provides a L_p version of the Berry-Esseen theorem, although the assumptions and final results are different from Erickson's. We write $\|\cdot\|_p$ for the usual L_p-norm with respect to Lebesgue measure.

Theorem 5 : *Suppose the assumptions of Theorems* 1 *and* 2 *are satisfied. Then, for* $p \geqslant 1$,

$$\|(1+|t|)^{2+c-q/p}(F_n(t)-\Phi(t))\|_p = o(n^{c/2}u(\sqrt{n}) \wedge n^{\frac{1}{2}})^{-1} \quad \dots \quad (2.36)$$

for any $q>1$.

Proof : Note that

$$\|(1+|t|)^{2+c-q/p}(F_n(t)-\Phi(t))\|_p = \left[\int_{-\infty}^{\infty}(1+|t|)^{(2+c)p-q}|F_n(t)-\Phi(t)|^p dt\right]^{1/p}. \quad \dots \quad (2.37)$$

Now, from Theorem 3, since for $q > 1$

$$\int_{-\infty}^{\infty} (1+|t|)^{-q} dt < \infty, \qquad \dots \quad (2.38)$$

the desired conclusion follows.

The next two theorems investigate whether the tail probabilities

$$1-F_n(t_n) \sim \Phi(-t_n) \quad \text{as} \quad t_n \to \infty. \qquad \dots \quad (2.39)$$

(By $a(n) \sim b(n)$ we mean $a(n)/b(n) \to 1$ as $n \to \infty$).

We shall see that as a consequence of Theorem 6, one can easily establish probabilities of moderate deviations (see Rubin and Sethuraman, 1965; Michel, 1974, 1976) in the special case $t_n = (c \log n)^{\frac{1}{2}}$.

Theorem 6 : *Suppose that the conditions of Theorem 1 are satisfied and $\{(x_{ni})^{2+c}\, \ddot{u}(x_{ni})\}$ is uniformly integrable. Then for*

$$t_n^2 \leqslant [c \log n + 2(c+1) \log |t_n| + 2 \log u(r\sqrt{n}t_n)] + M, \qquad \dots \quad (2.40)$$

where M is a positive constant, (2.39) *holds.*

The next theorem states that such a strong conclusion may not be possible if

$$t_n^2 = [c \log n + 2(c+1) \log |t_n| + 2 \log u(r\sqrt{n}t_n)] + M_n, \qquad \dots \quad (2.41)$$

where $M_n \to \infty$ as $n \to \infty$.

Theorem 7 : *Suppose that the conditions of Theorems 1 and 2 are satisfied, along with u, i assumption. Then if* (2.41) *holds,*

$$1-F_n(t_n) = o(t_n^{-2-c} \cdot n^{-c/2} u^{-1}(r\sqrt{n}t_n)) \ \text{as} \ t_n \to \infty. \qquad \dots \quad (2.42)$$

Once again we omit the proofs of these two theorems as they follow the lines of Michel (1976).

Remark 2 : Suppose

$$(\bullet) \qquad 0 < \underline{\lim} \frac{u(K'x)}{u(x)} \leqslant \overline{\lim} \frac{u(K'x)}{u(x)} < \infty \ \forall \ K' > 0;$$

then (2.40) and (2.41) reduce to

$$t_n^2 \leqslant K\Big[\frac{c}{2} \log n + 2(c+1) \log |t_n| + 2 \log u(\sqrt{n}t_n)\Big] + M$$

and

$$t_n^2 = K\left[\frac{c}{2}\log n + 2(c+1)\log|t_n| + 2\log u(\sqrt{n}t_n)\right] + M_n$$

respectively. Also the order of approximation in (2.42) reduces to

$$o(t^{-2-c}\, n^{-c/2}\, u^{-1}(\sqrt{n}\, t_n)).$$

The condition (•) is satisfied for $u(x) = \log^m(1+|x|)$, $m \geqslant 0$, $\log\log(e+|x|)$ and in general for slowly varying functions.

As an example, consider the case when $u(x) = \log^m(1+|x|)$, $m \geqslant 0$, c and m not both zeros. Then, (2.40) reduces to

$$t_n^2 \leqslant c\log n + (c+2m+1)\log\log n + M \text{ if } c > 0.$$

If $c = 0$, then (2.40) reduces to $t_n^2 \leqslant 2m\log\log n + \log\log\log n + M$.

3. Rates of convergence for general nonlinear statistics

In this section we consider nonlinear statistics of the form

$$T_n = s_n^{-1}S_n + R_n$$

where

$$S_n = \sum_{i=1}^{n} X_{ni}, \quad s_n^2 = \sum_{i=1}^{n} EX_{ni}^2,$$

$\inf_{n \geqslant 1} n^{-1}s_n^2 > 0$, $X_{n1}, X_{n2}, \ldots, X_{nn}$ being independent random variables. Representation of the above type are fairly general and are obtainable, for example via Hájek's projection lemma (see Hájek, 1968). Suppose further that R_n satisfies

$$E(R_n^{2m}) = O(n^{-m}(\log n)^h) \text{ for some } h \geqslant 0. \ (h \text{ may be a function of } m) \quad \ldots \quad (3.1)$$

m being a positive integer. We shall verify (3.1) for L-statistics in Section 4, although other examples can be given. In a forthcoming paper rates of convergence of U-statistics along with other properties will be discussed. Assume (1.4) - (1.6) are satisfied.

Let $a_n(t) = |t|^d(n^{c/2}u(\sqrt{n}))^{-\eta}$, where $d > 1$ and u satisfies (1.5); $\eta > 0$ will be chosen later. Then for $t^2 \leqslant K\left[\frac{1}{2}c \log n+\log u(\sqrt{n})\right]$, Theorem 1 yields

$$|P(s_n^{-1}S_n \leqslant t\pm a_n(t))-\Phi(t\pm a_n(t))|$$

$$\leqslant b \exp\left[-\frac{1}{2}(t\pm a_n(t))^2(1-3r)\right][n^{c/2}u(\sqrt{n})\wedge\sqrt{n}]^{-1}$$

$$+\sum_{i=1}^{n} P(|X_{ni}| > rs_n|t-a_n(t)|). \qquad \ldots \quad (3.2)$$

Note $a_n(t) \to 0$ as $n \to \infty$ for $t^2 \leqslant K\left[\frac{1}{2}c \log n+\log u(\sqrt{n})\right]$; so $|t\pm a_n(t)|^2 = t^2(1+o(1))$.

Without any loss of generality, assume $t > 0$, as $t < 0$ can be handled similarly. Then,

$$|\Phi(t\pm a_n(t))-\Phi(t)| \leqslant a_n(t)(2\pi)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(t-a_n(t))^2\right]$$

$$\leqslant bt^d(n^{c/2}u(\sqrt{n}))^{-\eta} \exp\left[-\frac{1}{2}t^2\right]. \qquad \ldots \quad (3.3)$$

Again using Markov's inequality and (3.1) one has

$$P(|R_n| > a_n(t)) \leqslant a_n^{-2m}(t)\, ER_n^{2m}$$

$$= O(t^{-2md}(n^{c/2}u(\sqrt{n}))^{2m\eta}\, n^{-m}(\log n)^h). \qquad \ldots \quad (3.4)$$

If $c > 0$ choose η such that $n^{-c\eta/2} = n^{mc\eta}\ n^{-m}$, i.e., $\eta = \dfrac{2m}{(2m+1)c}$. For $c = 0$, $\eta(> 0)$ can be chosen arbitrarily. From (3.2) - (3.4) if $c > 0$ one gets for $t^2 \leqslant K\left[\frac{1}{2}c \log n+\log u(\sqrt{n})\right]$,

$$|P(T_n \leqslant t)-\Phi(t)| \leqslant b\left[\exp\left(-\frac{1}{2}t^2(1-3r)(n^{\frac{1}{2}c}u(\sqrt{n})\wedge n^{\frac{1}{2}}\right)^{-1}\right.$$

$$+|t|^d\, n^{-m(2m+1)}(u(\sqrt{n}))^{-(2m)/(2m+1)c} \exp\left(-\frac{1}{2}t^2\right)$$

$$\left.+|t|^{-2md}\, n^{-(m)/(2m+1)}(u(\sqrt{n}))^{4m^2/(2m+1)c}(\log n)^h\right]$$

$$+\sum_{i=1}^{n} P(|X_{ni}| > rs_n|t-a_n(t)|). \qquad \ldots \quad (3.5)$$

Now for $t > 0$, $t - a_n(t) = t(1 - t^{d-1}(n^{c/2}u(\sqrt{n}))^{-\eta})$. This equals zero if either $t = 0$ or $t = (n^{c/2}\, u(\sqrt{n}))^{\eta/(d-1)}$. Since the last value of t lies outside the region $t^2 \leqslant K\left[\frac{1}{2}\, c \log n + \log u(\sqrt{n})\right]$ and since

$$\inf_n \inf_{0 < t^2 < K\left[\frac{c}{2} \log n + \log u(\sqrt{n})\right]} |1 - t^{-1}a_n(t)| = \lambda = \lambda(K, c, u, d) > 0,$$

we have,

$$|P(T_n \leqslant t) - \Phi(t)| \leqslant \Big[\exp\left(-\frac{1}{2}\, t^2(1-3r)\right) (n^{c/2}\, u(\sqrt{n}) \wedge n^{\frac{1}{2}})^{-1}$$

$$+ |t|^d\, n^{-(m)/(2m+1)} u(\sqrt{n})^{-(2m)/(2m+1)c} \exp\left(-\frac{t^2}{2}\right)$$

$$+ |t|^{-2md}\, n^{-(m)/(2m+1)}\, (u(\sqrt{n}))^{4m^2/(m+1)c} (\log n)^h$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r\lambda\, s_n |t|\Big]. \qquad \ldots \quad (3.6)$$

For $c = 0$ similarly we have

$$|P(T_n \leqslant t) - \Phi(t)| \leqslant b\left[\exp\left(-\frac{t^2}{2}(1-3r)\right) (u(\sqrt{n}))^{-1}\right.$$

$$+ |t|^d\, (u(\sqrt{n}))^{-\eta} \exp\left(-\frac{1}{2}t^2\right) + |t|^{2md}\, (u(\sqrt{n}))^{2m\eta}\, .n^{-m}\, (\log n)^h$$

$$\left. + \sum_{i=1}^{n} P(\,|X_{ni}| > r\lambda s_n |t|)\right] \qquad \ldots \quad (3.7)$$

for some $\lambda > 0$, where $\eta > 0$ is arbitrary. We are now in a position to state

Theorem 8 : *Suppose* $\{X_{ni}\}$ *is a sequence of r.v.'s satisfying* (1.4) - (1.6). *Let* $T_n = s_n^{-1} S_n + R_n$ *where* $S_n = \sum_{i=1}^{n} X_{ni}$, $s_n^2 = \sum_{i=1}^{n} EX_{ni}^2$, $(n \geqslant 1)$ *and* R_n *satisfies* (3.1). *Then for* $t_n^2 \leqslant K\left[\frac{c}{2} \log n + \log u(\sqrt{n})\right]$, (3.6) *holds if* $c > 0$ *and* (3.7) *holds if* $c = 0$.

Yet another form of the inequality is obtainable by a different choice of $\eta > 0$. For $c > 0$ let $\eta > 0$ be such that $mc\eta - m < \frac{1}{2}c$, i.e., $\eta < \left(m - \frac{1}{2}c\right) \Big/ (mc)$; $c < 2m \leqslant c+2$. Then from (3.2) - (3.4) it follows that

$$|P(T_n \leqslant t) - \Phi(t)| \leqslant b.n^{-\gamma}\left[\exp\left(-\frac{1}{2}t^2\right) + n^{-c/2}\right]$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r\lambda s_n |t|) \qquad \ldots \quad (3.7a)$$

for some $\gamma > 0$. Note that under the additional assumption

$$\frac{1}{n}\sum_{i=1}^{n} E|X_{ni}|^{2+c}\, u(X_{ni})\, I_{[|X_{ni}| > \epsilon s_n]} \to 0, \quad \text{as } n \to \infty \qquad \ldots \quad (3.8)$$

for every $\epsilon > 0$, putting $|t| = (c \log n)^{\frac{1}{2}}$ one obtains probabilities of moderate deviations. (3.7a) will be used further to have a theorem analogous to Theorem 6.

Next we derive an error bound for $t^2 > K\left[\frac{c}{2}\log n + \log u(\sqrt{n})\right]$. First obtain from Theorem 2 with $t > 0$,

$$|F_n(t \pm a_n(t)) - \Phi(t \pm a_n(t))|$$

$$\leqslant b[n^{c/2}\, u(\sqrt{n})]^{-\frac{1}{2}(K-1)} |t - a_n(t)|^{-2(K+1)}$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > rs_n |t - a_n(t)|). \qquad \ldots \quad (3.9)$$

Note that

$$|\Phi(t \pm a_n(t)) - \Phi(t)| \leqslant a_n(t)\phi(t - a_n(t)). \qquad \ldots \quad (3.10)$$

Hence for $t \neq a_n(t)$,

$$|P(T_n \leqslant t) - \Phi(t)| \leqslant b[n^{c/2}\, u(\sqrt{n})]^{-\frac{1}{2}(K-1)} |t - a_n(t)|^{-2(K+1)}$$

$$+ [n^{c/2} u(\sqrt{n})]^{-\eta}\, t^d \exp\left[-\frac{1}{2}(t - a_n(t))^2\right]$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > rs_n |t - a_n(t)|).$$

$$+ P(|R_n| > a_n(t)). \qquad \ldots \quad (3.11)$$

Note that the solution of $t-a_n(t) = 0$ are $t = 0$ and $t = [n^{c/2}u(\sqrt{n})]^{\eta/(d-1)}$. So $|t-a_n(t)| \geq \lambda t$ if $|1-t^{-1}a_n(t)| > \lambda$ where $\lambda(0 < \lambda < 1)$ may depend on K, and is at our choice.

Hence, for $t \notin [(1\pm\lambda)\, n^{c/2}u(\sqrt{n})]^{\eta/(d-1)}$, one has

$$|P(T_n \leq t)-\Phi(t)| \leq b[n^{c/2}u(\sqrt{n})]^{-\frac{1}{2}(K-1)}|t|^{-2(K+1)}$$

$$+(n^{c/2}u(\sqrt{n}))^{-\eta}|t|^{d} \exp\left[-\frac{1}{2}\lambda^2t^2\right]$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r\lambda s_n|t|)$$

$$+b_2(n^{-c/2-\epsilon'}|t|^{-2(K+1)}). \qquad \ldots \quad (3.12)$$

To justify (3.12) note that

$$|P(|R_n| < a_n(t)) \leq \frac{ER_n^{2m}}{(a_n(t))^{2m}} = O\,(n^{-m}(\log n)^h(n^{c/2}u(\sqrt{n}))^{2m\eta}|t|^{-2md})$$

$$= O(n^{-c/2-\epsilon'}|t|^{-2(K+1)})$$

for some $\epsilon' > 0$ if

$$m(1-c\eta) > \frac{c}{2}+\epsilon'$$

i.e.,

$$1-c\eta > \frac{c+2\epsilon'}{2m}$$

i.e.,

$$\eta < \frac{2m-c-2\epsilon'}{2mc}.$$

Since $c < 2m \leq c+2$ this choice of $\eta\,(> 0)$ is possible for some $\epsilon' \in \left(0, \frac{1}{2}\,(2m-c)\right)$; d is chosen in such a way that

$$2md > 2(K+1), \text{ i.e., } d > \frac{1}{m}(K+1).$$

But for $t \in [(1\pm\lambda)n^{c/2}u(\sqrt{n})]^{\eta/(d-1)}$, choose $a_n^*(t) = |t|^{d^*}\,(n^{c/2}u(\sqrt{n}))^{-\eta}$ $(1 < d^* < d)$ and get a similar inequality as (3.12) with d^* replacing d for $t \notin [(1\pm\lambda)n^{c/2}u(\sqrt{n})]^{\eta/(d^*-1)}$. Since the two intervals

$$\{(1\pm\lambda)n^{c/2}u(\sqrt{n})]^{\eta/(d-1)} \text{ and } \{(1\pm\lambda)n^{c/2}u(\sqrt{n})\}^{\eta/(d^*-1)}$$

can be made disjoint, one gets (3.12) for all real t with

$$t^2 \geq K\left[\frac{c}{2}\log n+\log u(\sqrt{n})\right].$$

Now

$$\exp\left[-\frac{1}{2}\lambda^2 t^2\right] = \exp\left[-\frac{1}{2}\lambda^2 a t^2\right] \exp\left[-\frac{1}{2}\lambda^2(1-a)t^2\right]$$

$$\leqslant [n^{c/2}u(\sqrt{n})]^{-K\lambda^2 a/2} \exp\left[-\frac{1}{2}\lambda^2(1-a)t^2\right]. \quad \ldots \quad (3.13)$$

Choose a such that $(\eta+K\lambda^2 a/2) = \dfrac{K-1}{2}$, i.e., $a = \dfrac{K-1}{K\lambda^2}-2\eta$. For adequate choice of λ and η one has $0 < a < 1$. Hence the 2nd term of the r.h.s. of (3.12) is less than or equal to

$$[n^{c/2}u(\sqrt{n})]^{-(K-1)/2}|t|^d \cdot \exp\left[-\frac{1}{2}\lambda^2(1-a)t^2\right].$$

We now state the following

Theorem 9 : *Let the conditions of Theorem 8 hold; then for*

$$t^2 > K\left[\frac{c}{2}\log n+\log u(\sqrt{n})\right],$$

we have

$$|P(T_n \leqslant t)-\Phi(t)| \leqslant b[n^{c/2}u(\sqrt{n})]^{-(K-1)/2}.|t|^{-2(K+1)}$$
$$+\sum_{i=1}^{n} P(|X_{ni}| > r\lambda s_n|t|).$$
$$+b_2(n^{-c/2-\epsilon'}|t|^{-2(K+1)}). \quad \ldots \quad (3.14)$$

Remark 3 : It is interesting to note that whereas in Theorem 2 the order sharpens as K increases it is not the case with (3.12) because of the term $P(|R_n| > a_n(t))$.

In view of Theorems 8 and 9 with K sufficiently large we have for $c > 0, t \in (-\infty\, \infty)$

$$|P(T_n \leqslant t)-\Phi(t)|$$
$$\leqslant b[(n^{c/2}u(\sqrt{n}))^{-1}+n^{-(m)/(2m+1)}(u(\sqrt{n}))^{4m^2/(c(2m+1))}(\log n)^h]\,(1|+t|^{2+c}). \quad \ldots \quad (3.15)$$

where $2m$ is the largest even integer $\leqslant (c+2)$ and for $c = 0$

$$|P(T_n \leqslant t)-\Phi(t)| \leqslant b.[u(\sqrt{n})]^{-1}(1+t^2),\ t \text{ real}. \quad \ldots \quad (3.16)$$

Remark 4 : Note that if X_1 admits all moments, one has from (3.15),

$$|P(T_n \leqslant t)-\Phi(t)| \leqslant bn^{-\frac{1}{2}+\delta_1}(1+|t|^{\delta_2})^{-1}, \qquad \ldots \quad (3.16a)$$

for any positive δ_1 and δ_2. Also, the nonuniform bound (3.15) can be improved when from uniform Berry-Esseen bound one knows that

$$\sup |P(T_n \leqslant t)-\Phi(t)| = O((n^{c/2}u(\sqrt{n}) \wedge n^{\frac{1}{2}})^{-1}). \qquad \ldots \quad (3.17)$$

Then, without using (3.6) one may directly use (3.14) along with (3.17) to obtain

$$|P(T_n \leqslant t)-\Phi(t)| \leqslant b(n^{c/2}u(\sqrt{n}) \wedge n^{\frac{1}{2}})^{-1}(\log n)^{g(c)}\,(1+|t|^{2+c})^{-1}u(\sqrt{t})^{-1} \qquad \ldots \quad (3.18)$$

for some $g(c) > 0$.

In the light of (3.6), (3.7), (3.7a), (3.14) it is also possible to obtain theorems analogous to 4 - 7. Without going into the detailed proof we now state the results as follows.

Theorem 10 : *Suppose that the assumptions concerning $u(x)$ of Theorem. 4 hold. Let $Y_n = |T_n|$ and $T = |N(0, 1)|$. Then,*

$$|E(Y_n^{2+c}u(y_n))-E(T^{2+c}u(T))|$$

$$= \begin{cases} O(n^{-c/2} \vee n^{-m/(2m+1)}(u(\sqrt{n}))^{4m^2/(2m+1)c}(\log n)^h) & \text{if } c > 0 \\ o(\text{above quantity}) \text{ if } c > 0, \text{ and if} & \\ \dfrac{1}{n}\sum\limits_{t=1}^{n} E\left(|X_{nt}|^{2+c}u\left(\dfrac{X_{nt}}{r\lambda s_n}\right)\right) \to 0 \text{ as } n \to \infty & \\ o(1) \text{ if } c = 0 \text{ and if } \dfrac{1}{n}\sum\limits_{t=1}^{n} E\left(X_{nt}^2\, u\left(\dfrac{X_{nt}}{r\lambda s_n}\right)\right) \to 0 & \end{cases}$$

as $n \to \infty$. $\qquad \ldots \quad (3.19)$

Note that if a uniform Berry-Esseen bound (3.17) is known, by using (3.18) instead of (3.6) sometimes it is possible to obtain sharper orders in (3.19) for $c > 0$. This comment holds for the next theorem also.

Theorem 11 : *Let $G_n(t) = P(T_n \leqslant t)$. Then under the assumptions of Theorems 10 and 11, for any $q > 1$*

$$\|(1+|t|^{2+c-q/p}(G_n(t)-\Phi(t)\|_p = \begin{cases} O(n^{-c/2}\sqrt{n}^{m/(2m+1)}(\log n)^h, & \text{if } c > 0 \\ O(u(\sqrt{n}))^{-1} & \text{if } c = 0. \end{cases} \qquad \ldots \quad (3.20)$$

Remark 5 : If $c > 0$, we take $u(x) = 1$ in (1.4) to obtain (3.20). Using (3.8) and (3.7a) in the case $c > 0$ and (3.7) with $\eta = 1$ in the case $c = 0$, we have the following theorems.

Theorem 12 : *Let the conditions of Theorem* 10 *and assumption* (3.8) *hold. Then, for a sequence* $\{t_n\}$, $t_n \to \infty$ *with*

$$t_n^2 \leqslant [c \log n + 2(c+1) \log |t_n| + 2 \log u(r\lambda s_n t_n)] + M, \quad \dots \quad (3.21)$$

where M *is a positive constant*,

$$1 - G_n(t_n) \sim \Phi(-t_n) \text{ as } t_n \to \infty. \quad \dots \quad (3.22)$$

Theorem 13 : *Let the conditions of Theorems* 10 *and* 11 *and assumption* (3.8) *hold. Then, for a sequence* $\{t_n\}$, $t_n \to \infty$ *with*

$$t_n^2 = [c \log n + 2(c+1) \log |t_n| + 2 \log u(r\lambda s_n t_n)] + M_n \quad \dots \quad (3.23)$$

where $M_n \to \infty$ *as* $n \to \infty$,

$$1 - G_n(t_n) = o(t_n^{-2c}\ n^{-c/2}(u\,(r\lambda s_n t_n))^{-1}). \quad \dots \quad (3.24)$$

Remark 6 : If assumption (•) of Remark 3 holds then (3.21) and (3.23) reduce to

$$t_n^2 \leqslant [c \log n + 2(c+1) \log |t_n| + 2 \log u(\sqrt{n}\, t_n)] + M \quad \dots \quad (3.25)$$

and

$$t_n^2 = [c \log n + 2(c+1) \log |t_n| + 2 \log u(\sqrt{n} t_n)] + M_n. \quad \dots \quad (3.26)$$

Remark 7 : For Theorems 6 and 12 the restriction (1.6) may be relaxed. From the proof of above two theorems it follows that in the case $c > 0$ it is enough for (2.1) to have $\omega = n^{-\epsilon}$ with some $\epsilon > 0$. The ultimate condition replacing (1.6) turns out to be (as evident from (2.15))

$$s_n \geqslant n^{-1/(2+c^*)+\epsilon'} \quad \dots \quad (3.27)$$

for some $\epsilon' > 0$ where $c^* = \min (c, 1)$.

4. Rates of convergence for L-statistics

Let $X_{1n} \leqslant X_{2n} \leqslant \dots X_{nn}$ denote the order statistics corresponding to n i.i.d r.v.'s $X_1, \dots, X_n$ having a common distribution function F. Consider linear combinations of functions of order statistics of the form

$$T_n = \sum_{i=1}^{n} c_{in} h(X_{in}), \quad \dots \quad (4.1)$$

where the c_{in}'s are constant and h is some measurable function. Let $H = h \circ F^{-1} \circ G$ where $G(x) = 1-\exp(-x)$. Also, let $Z_{1n} \leqslant \ldots \leqslant Z_{nn}$ denote the order statistics corresponding to n i.i.d r.v's $Z_1, \ldots, Z_n$ each having the distribution function $G(x)$. Then T_n is identically distributed as $\sum_{i=1}^{n} c_{in} H(Z_{in})$. This representation is due to Chernoff, Gastwirth and Johns (1967). Note that Z_{in} has the same distribution as $\sum_{j=1}^{i} Z_j/(n-j+1)$ and, hence

$$\nu_{in} = E(Z_{in}) = \sum_{j=1}^{i} (n-j+1)^{-1}, \quad 1 \leqslant i \leqslant n. \qquad \ldots \quad (4.2)$$

Assuming that H is differentiable, by the first mean value theorem, one has

$$(T_n - \mu_n)/s_n = s_n^{-1} L_n + R_n, \qquad \ldots \quad (4.3)$$

where

$$\mu_n = \sum_{i=1}^{n} c_{in} \; H(\nu_{in}); \qquad \ldots \quad (4.4)$$

$$L_n = \sum_{i=1}^{n} c_{in} H'(\nu_{in})(Z_{in} - \nu_{in}); \qquad \ldots \quad (4.5)$$

$$R_n = s_n^{-1} \sum_{i=1}^{n} c_{in}(Z_{in} - \nu_{in})[H'(\theta_n Z_{in} + (1-\theta_n)\nu_{in}) - H'(\nu_{in})], \qquad \ldots \quad (4.6)$$

for some $0 < \theta_n < 1$; $s_n^2 = \sum_{i=1}^{n} \alpha_{in}^2$, $\alpha_{in} = (n-i+1)^{-1} \sum_{j=i}^{n} c_{jn} H'(\nu_{jn})$, $1 \leqslant i \leqslant n$.

Note that $\sum_{i=1}^{n} c_{in} H'(\nu_{in})(Z_{in} - \nu_{in})$ has the same distribution as $U_n = \sum_{i=1}^{n} \alpha_{in} (Z_i - 1)$ and $s_n^2 = \operatorname{var}(U_n)$. The above expansion was used by Bjerve (1977) in obtaining a uniform Berry-Esseen theorem of $O(n^{-\frac{1}{2}})$ for T_n. The asymptotic normality of T_n was proved by Chernoff *et al.* (1967).

Our aim in this section is to develop nonuniform Berry-Esseen bounds for T_n and obtain Theorems 10 - 13. For this we need a verification of (1.4), (1.6) and (3.1).

The following assumptions are made.

(I) $\sup_{n \geqslant 1} \max_{1 \leqslant i \leqslant n} |c_{in}| < \infty$;

(II) H is differentiable and $\sup_{0 < x < \infty} |H'(x)| < \infty$;

(III) H' is Lipschitz of order one over $(0, \infty)$;

(IV) $\varliminf_{n \to \infty} \; n^{-1} s_n^2 > 0$.

It is then easy to see that (1.4) and (1.6) are satisfied. To verify (3.1) first observe that in view of I, and III,

$$E|R_n|^{2m} \leqslant Cn^{-m}\, E\Big\{ \sum_1^n (Z_{in}-\nu_{in})^2 \Big\}^{2m}$$

$$= Cn^{-m}E\Big[\sum_{i=1}^n \Big\{ \sum_{j=1}^i (Z_j-1)/(n-j+1) \Big\}^2 \Big]^{2m}$$

$$= Cn^{-m}E\Big[\sum_{j=1}^n (Z_j-1)^2/(n-j+1)$$

$$+2 \sum_{j=1}^{n-1} \sum_{j'=j+1}^n (Z_j-1)(Z_{j'}-1)/(n-j+1)\Big]^{2m}$$

where in the above and in what follows C is a generic constant which might depend on m, but not on n. Also,

$$E\Big(\sum_{j=1}^n (Z_j-1)^2/(n-j+1)\Big)^{2m} \quad \ldots \quad (4.7)$$

$$= \sum_{j_1} \ldots \sum_{j_{2m}} \frac{E[(Z_{j_1}-1)^2 \ldots (Z_{j_{2m}}-1)^2]}{(n-j_1+1)\ldots(n-j_{2m}+1)}. \quad \ldots \quad (4.8)$$

Using Hölder's inequality $2m$ times,

$$E[(Z_{j_1}-1)^2 \ldots (Z_{j_{2m}}-1)^2] \leqslant \prod_{i=1}^{2m} E^{1/2m}(Z_{j_i}-1)^{4m}. \quad \ldots \quad (4.9)$$

But $E(Z_1-1)^{4m} \leqslant (4m)!$ Hence from (4.8) and (4.9)

$$E\Big(\sum_{j=1}^n (Z_j-1)^2/(n-j+1)\Big)^{2m} \leqslant (4m)!\, \Big(\sum_{j=1}^n \frac{1}{n-j+1} \Big)^{2m} \leqslant C(\log n)^{2m}.$$

$$\ldots \quad (4.10)$$

Further,

$$E\Big[\sum_{j=1}^n \sum_{j'=j+1}^n (Z_j-1)(Z_{j'}-1)/(n-j+1)\Big]^{2m}$$

$$= \sum_{j_1=1}^{n-1} \sum_{j_1'=j_1+1}^n \ldots \sum_{j_{2m}=1}^{n-1} \sum_{j_{2m}'=j_{2m}+1}^n \frac{E(Z_{j_1}-1)(Z_{j_1'}-1)\ldots(Z_{j_{2m}}-1)(Z_{j_{2m}'}-1)}{(n-j_1+1)\ldots(n-j_{2m}+1)}.$$

$$\ldots \quad (4.11)$$

Note that if any one of the pairs (j_i, j_i') of suffixes occurs only once, then the expectation vanishes, and hence every suffix should occur at least twice to make a nonzero contribution.

Subject to the condition that each pair of suffixes occurs at least twice, the maximum number of pairs that can occur is m. Also applying Holders inequality $4m$ times

$$|E(z_{j_1}-1)(z_{j_1'}-1)\dots(z_{j_{2m}}-1)(z_{j_{2m}'}-1)| \leqslant \prod_{j=1}^{4m} E^{1/4m}(z_j-1)^{4m}$$

$$= E(z_1-1)^{4m} \leqslant (4m)! \quad \dots \quad (4.12)$$

In view of the fact that $\sum_{i=a}^{b} \frac{1}{(n-i+1)p} \downarrow$ as $p\,(>0) \uparrow$ for any $a < b \leqslant n$ and that maximum number of pairs is m, each occuring at least twice, we have,

$$\text{l.h.s. of (4.11)} \leqslant (4m)!\left(\sum_{j=1}^{n-1} \frac{1}{(n-j+1)}\right)^m \leqslant (4m)!\,(\log n)^m \quad \dots \quad (4.13)$$

Thus, (3.1) is verified with $h = m$, and one gets Theorems 8 - 13 in this situation. The only thing to note is that (3.16a) should replace (3.15).

The conditions assumed on H are satisfied when for example $F(x) = G(x) = 1-\exp(-x)$ and h is the identity map. In the case of trimmed L-statistics $T_n = \sum_{[n\alpha]+1}^{[n\beta]} c_{in}H(Z_{in})$, $0 < \alpha < \beta < 1$, and weaker condition on H suffices. All we need assume there is $\sup_{a\leqslant x\leqslant b} |H'(x)| < \infty$ and H' is Lipschitz of order 1 on [a, b] where $a < -\log(1-\alpha)$, $b > -\log(1-\beta)$.

References

Bahadur, R. R., and Rao, R. R. (1960) : On deviations of sample mean. *Ann. Math. Statist.*, 23, 1015-1027.

von Bahr, B. (1965) : On the convergence of moments in the central limit theorem. *Ann. Math. Statist.*, 36, 808-818.

Bjerve, S. (1977) : Error bounds for linear combinations of order statistics. *Ann. Statist.*, 5, 357-369.

Chernoff, H., Gastwirth, J. L. and Johns, M. V., Jr. (1967) : Asymptotic distribution of linear combinations of functions of order statistics with applications to estimation. *Ann. Math. Statist.*, 38, 52-72.

Cramer, H. (1938) : Sur un nouveau théoréme limite de la Probabilities. *Actualites Sci. Indust.* No. 736.

Erickson, R. V. (1973): On an L_p version of Berry-Esseen theorem for independent and dependent variable. *Ann. Prob.*, **1**, 497-503.

Hajek, J. (1968): Asymptotic normality of simple linear rank statistics under alternatives. *Ann. Math. Statist.*, **39**, 325-346.

Katz, M. L. (1963): Note on the Berry-Esseen theorem. *Ann. Math. Statist.*, **34**, 1107-1108.

Michel, R. (1974): Results on probabilities of moderate deviations. *Ann. Prob.*, **2**, 349-353.

——— (1976): Nonuniform central limit bounds with applications to the probabilities of deviations. *Ann. Prob.*, **4**, 102-106.

Nagaev, S. V. (1965): Some limit theorems for large deviations. *Theo. Probability Appl.*, **10**, 214-235.

Rubin, H. and Sethuraman, J. (1965): Probabilities of moderate deviations. *Sankhyā*, **A**, **27**, 325-346.

Paper received: January, 1978.
Revised: July, 1978.