A gauge theoretical formulation of a frustrated Ising system

B. Basu
Indian Statistical Institute, Calcutta-700035, India

The gauge theoretical formulation of a frustrated Ising model is studied here. In this formulation the Chern–Simons term has played the key role. The inherent anisotropic feature of the system is associated with the order parameter which violates the commutativity of translation.

I. INTRODUCTION

The study of disordered systems has been a fast-growing field in condensed matter physics in the past 20 years. Spin glasses are a class of disordered magnetic materials that, in some sense, are related to ordinary magnets as glass in the crystalline solids. In spin glasses the interactions between the spins are "in conflict" with each other, due to some "quenched disorder" in the system. No conventional long-range order (ferromagnetism, antiferromagnetism, etc.) can occur, but nevertheless a fairly sharp cusp of the susceptibility χ indicates[1,2] a "transition" to a new state, where the spins are thought to be more or less "frozen in" in random directions. It is known that various classes of random interactions exist for spin glass behavior. It is essential that the randomness leads to "frustration,"[3] i.e., there is no ground state satisfactory for all bonds. The presence of both randomness and frustration is thought to be the requisite property of a spin glass. Since the statistical mechanics of systems containing both randomness and frustration is very hard, interest has arisen in studying "fully frustrated" systems: the nearest-neighbor Ising triangular[4] or face-centered cubic[5–8] antiferromagnetic or fully frustrated square (Villain's odd model[9]) and simple cubic lattices. Here we shall try to study the frustrated Ising system on the basis of certain geometrical and topological properties of spin system.

It is well known that an Ising model represents a system of fermions. In higher dimensions, a fermionic language of Ising model is still possible but much less transparent as nonlocal interactions are required. In the geometry developed below we are going to show that the above-mentioned equivalence can be realized transparently in three space dimensions. In a recent paper[10] it has been shown that the quantization procedure of a fermion involves the introduction of an anisotropic feature in complexified Minkowski space-time that gives rise to the internal helicities corresponding to fermion and antifermion. The anisotropic feature in the internal space may be visualized by the fixation of a particular axis that behaves like a vortex line attached to a space-time point. When such vortices are considered at the different sites of a lattice the system considered can be called an Ising system. This suggests that a statistical system can be described in the language of gauge theory which takes into account the fiber structure of "direction vector" attached to a space-time point. Indeed this leads to a SL(2,c) gauge theory and the Ising Hamiltonian can be considered to be represented by a current–current interaction in the continuum limit, when the current is constructed out of the non-Abelian SL(2,c) gauge fields B_μ. In a different paper,[11] with the help of this non-Abelian gauge field the Ising system has been studied. Here we shall study the frustrated Ising system from this gauge theoretical formulation.

In Sec. II, we recapitulate the gauge theoretic formulation of a spin system. In Sec. III, we study the frustrated Ising model from this view point and, finally, in Sec. IV, we consider the order parameter for such a system.

II. ISING MODEL AND FERMIONS

In a recent paper[10] it has been shown that the quantization procedure of a fermion involves the introduction of an anisotropic feature in the internal space so that the internal variable appears as a "direction vector." The two opposite orientations of the direction vector corresponds to particle and antiparticle. To have equivalence with the Feynman path integral we have to take into account complexified space-time when the coordinate is given by $z_\mu = x_\mu + i\xi_\mu$, where ξ_μ corresponds to a direction vector attached to the space-time point x_μ.[12] Since for quantization we have to introduce Brownian motion processes both in the external and internal space, after quantization, for an observational procedure we can think of the mean position of the particle q_μ in the external observable space with a stochastic extension determined by the internal stochastic variable ξ_μ. The nonrelativistic quantum limit is obtained in the sharp point limit.[13] It has been shown that when we consider the internal space anisotropic in nature so that ξ_μ appears as a direction vector we can generate two internal helicities in terms of two spinorial variables giving rise to fermion and antifermion. This helps us to have a gauge theoretic extension of a relativistic quantum particle when the gauge group is given by SL(2,c). This inherent gauge structure seems to be the major ingredient of quantization procedure.

It is now noted that when a direction vector ξ_μ is attached to a space-time point x_μ, the latticization corresponds to an Ising system. Evidently, the inherent gauge field theoretic extension of a fermion then finds its relevance for an Ising system and we can consider a gauge theoretical formulation for an Ising model. When we consider that the two opposite orientations of the direction vector ξ_μ attached to the space-time point x_μ in the complexified Minkowski space-time having the coordinate $z_\mu = x_\mu + i\xi_\mu$ give rise to two opposite internal helicities corresponding to fermion and antifermion. We can formulate the "internal helicity" in terms of the two component spinorial variable $\theta(\bar{\theta})$.[14] In fact, for a massive spinor we can choose the chiral coordinate in this space as

$$z^\mu = x^\mu + (i/2)\lambda^\mu_\alpha \theta^\alpha \quad (\alpha = 1,2), \tag{1}$$

where we identify the coordinate in the complex manifold $z^\mu = x^\mu + i\xi^\mu$ with $\xi^\mu = (1/2)\lambda^\mu_\alpha \theta^\alpha$. We can now replace the chiral coordinate by the matrices

$$z^{AA'} = x^{AA'} + (i/2)\lambda^{AA'}_\alpha \theta^\alpha, \tag{2}$$

where

$$x^{AA'} = \begin{bmatrix} x^0 - x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 + x^1 \end{bmatrix}$$

and

$$\lambda^{AA'}_\alpha \in \mathrm{SL}(2,c).$$

With these relations, the twistor equation is now modified as

$$\bar{z}_\alpha z^\alpha + \lambda^{AA'}_\alpha \theta^\alpha \bar{\Pi}_A \Pi_{A'} = 0, \tag{3}$$

where $\bar{\Pi}_A$ ($\Pi_{A'}$) is the spinorial variable corresponding to the four momentum variable p_μ, the conjugate of x_μ, and is given by the matrix representation

$$p^{AA'} = \bar{\Pi}^A \Pi^{A'} \tag{4}$$

and

$$z^{a}=(\omega^{A},\Pi_{A'}),\quad \bar{z}_{a}=(\bar{\Pi}_{A'},\bar{\omega}^{A'})$$

with

$$\omega^{A}=i[x^{AA'}+(i/2)\lambda_{\alpha}^{AA'}\theta^{\alpha}]\Pi_{A'}.$$

Equation (3) now involves the helicity operator

$$S=-\lambda_{\alpha}^{AA'}\theta^{\alpha}\bar{\Pi}_{A}\Pi_{A'}, \tag{5}$$

which we identify as the internal helicity of the particle and which corresponds to the fermion number. We note here that the matrix representation of p^{μ} (which is the conjugate of x^{μ}) in the complex coordinate $z^{\mu}=x^{\mu}+i\xi^{\mu}$ is written as $p^{AA'}=\bar{\Pi}^{A}\Pi^{A'}$, which implies $(p^{\mu})^{2}=0$. Therefore the particle will have mass due to the nonvanishing character of the quantity $(\xi_{\mu})^{2}$.

We observe that the complex conjugate of the chiral coordinate corresponds to a massive particle with opposite internal helicity which will represent an antifermion. In the null plane where $(\xi^{\mu})^{2}=0$ we can write the chiral coordinate as

$$z^{AA'}=x^{AA'}+(i/2)\bar{\theta}^{A}\theta^{A'}, \tag{6}$$

where the coordinate ξ^{μ} is replaced by

$$\xi^{AA'}=(1/2)\bar{\theta}^{A}\theta^{A'}.$$

Therefore the helicity operator can be written as

$$S=-\bar{\theta}^{A}\theta^{A'}\bar{\Pi}_{A}\Pi_{A'}=-\bar{\varepsilon}\varepsilon, \tag{7}$$

where

$$\varepsilon=i\theta^{A'}\Pi_{A'},\quad \bar{\varepsilon}=-i\bar{\theta}^{A}\bar{\Pi}_{A}.$$

The corresponding twistor equation describes a massless spinor field. The state with the internal helicity $+\frac{1}{2}$ is the vacuum state of the fermion operator

$$\varepsilon|S=+\tfrac{1}{2}\rangle=0. \tag{8}$$

Similarly, the state with the internal helicity $-\frac{1}{2}$ is the vacuum state of the fermion operator

$$\bar{\varepsilon}|S=-\tfrac{1}{2}\rangle=0. \tag{9}$$

In case of a massive spinor, we can define a negative definite plane D^{-} where for the coordinate $z^{\mu}=x^{\mu}+i\xi^{\mu}$, ξ^{μ} belongs to the interior of the forward light cone ($\xi>0$) and as such represents the upper half-plane with the condition det $\xi^{AA'}>0$ and $\frac{1}{2}$ Tr $\xi^{AA'}>0$. The lower half-plane D^{-} is given by the set of all coordinates z^{μ} with ξ^{μ} in the interior of the backward light cone ($\xi<0$). The map $z\rightarrow z^{*}$ sends the upper half-plane to the lower half-plane. The space M of null plane (det $\xi^{AA'}=0$) is the Shilov boundary so that a function holomorphic in D^{-} (D^{+}) is determined by its boundary values. Thus if we consider that any function $\phi(z)=\phi(x)+i\phi(\xi)$ is holomorphic in the whole domain, the helicity $+\frac{1}{2}$ ($-\frac{1}{2}$) in the null plane may be taken to be the limiting value of the internal helicity in the upper (lower) half-plane.

In the sense of Minkowski space-time, the domain having the characteristics $\xi>0$ and $\xi<0$ in the upper and lower half-planes indicates that the domain is disconnected and anisotropic in nature. This indicates that the behavior of the angular momentum operator in such a region will be similar to that of a charged particle moving in the field of a magnetic monopole. In such a case, the wave function $\phi(z_\mu)=\phi(x_\mu)+i\phi(\xi_\mu)$ describes a particle moving in the external space-time having the coordinate x_μ with an attached direction vector ξ_μ. Thus the wave function should take into account the polar coordinates r, θ, ϕ, along with the angle $\varkappa$ specifying the rotational orientation around the direction vector ξ_μ. The eigenvalue of the operator $i\partial/\partial\varkappa$ corresponds to the internal helicity. For an extended body represented by the De Sitter group SO(4,1), θ, ϕ, and $\varkappa$ represent the three Euler angles.

In three space dimension, these three Euler angles have their correspondence in an axisymmetric system. In that case the anisotropy is introduced along a particular direction and the components of the linear momentum satisfy the relation

$$[p_i, p_j]=i\mu_M\epsilon_{ijk}x^k/r^3, \tag{10}$$

where μ_M is the strength of the magnetic monopole.

In the case of a charged particle moving in a field of a magnetic monopole, the three-momentum Π is defined as[15]

$$p=\Pi-\mu_M D(r), \tag{11}$$

where

$$D(\mathbf{r})=\frac{\mathbf{r}\times\boldsymbol{\eta}(\mathbf{r}\cdot\boldsymbol{\eta})}{\mathbf{r}[r^2-(\mathbf{r}\cdot\boldsymbol{\eta})^2]}$$

and $\boldsymbol{\eta}$ is a unit vector. We see that when $\mu_M=0$, $p=\Pi$, i.e., μ_M represents the measure of anisotropy.

In this space, the angular momentum operator J is defined as

$$\mathbf{J}=\mathbf{r}\times\mathbf{p}-\mu_M\mathbf{r} \tag{12}$$

so that $J^2=L^2-\mu_M^2$ and the eigenvalue of $\mathbf{J}^2$ is a conserved quantity and not the eigenvalue of $\mathbf{L}^2$. Fierz[16] and Hurst[17] have extensively studied the spherical harmonics incorporating the term μ_M. Following them we write

$$\begin{aligned} Y_\iota^{m,\mu_M}&=(1+x)^{-(m-\mu_M)/2}(1-x)^{-(m+\mu_M)/2}\\ &\times\frac{d^{\iota-m}}{dx^{\iota-m}}[(1+x)^{\iota-\mu_M}(1-x)^{\iota+\mu_M}]e^{i(m\phi-\mu_M\varkappa)} \end{aligned} \tag{13}$$

with $x=\cos\theta$. The quantities m and μ just represent the eigenvalues of the operators $i(\partial/\partial\phi)$ and $i(\partial/\partial\varkappa)$, respectively, when the wave function is written in terms of the angles θ, ϕ and $\varkappa$. For $m=\pm\frac{1}{2}$, $\mu_M=\pm\frac{1}{2}$, and $\iota=\frac{1}{2}$,

$$\begin{aligned} Y_{1/2}^{1/2,1/2}&=\sin(\theta/2)e^{(i/2)(\phi-\varkappa)},\\ Y_{1/2}^{-1/2,1/2}&=\cos(\theta/2)e^{(-i/2)(\phi+\varkappa)},\\ Y_{1/2}^{1/2,-1/2}&=\cos(\theta/2)e^{(i/2)(\phi-\varkappa)},\\ Y_{1/2}^{-1/2,-1/2}&=\sin(\theta/2)e^{(-i/2)(\phi-\varkappa)}. \end{aligned} \tag{14}$$

These represent spherical harmonics for half-orbital angular momentum ($l=\frac{1}{2}$) with $\mu_M=\pm\frac{1}{2}$. The fact that in such an anisotropic space the angular momentum can take the value $\frac{1}{2}$ is then found to be analogous to the result that a monopole-charged particle composite representing a dyon satisfying the condition $gq=2\pi$ have their angular momentum shifted by $\frac{1}{2}$ unit and their statistics shift accordingly.[18] Thus we note that a fermion may be viewed as a scalar particle moving with $l=\frac{1}{2}$ in an anisotropic space. An Ising model then corresponds to such a particle moving with $l=\frac{1}{2}$ having a fixed l_z value.

To study the topological properties of such a system we note that in the complexified space-time exhibiting the internal helicity states, we can now write the metric as $g_{\mu\nu}(x,\theta,\bar{\theta})$. It has been shown elsewhere[19] that this metric structure gives rise to the SL(2,c) gauge theoretical extension of such a particle and generates the field strength tensor $F_{\mu\nu}$ in terms of gauge fields B_μ, where B_μ are matrix valued having the SL(2,c) group structure. Here $F_{\mu\nu}$ is given by

$$F_{\mu\nu}=\partial_\nu B_\mu-\partial_\mu B_\nu+[B_\mu,B_\nu]. \tag{15}$$

This effectively helps us to write the configuration variable

$$Q_\mu=-i(\partial/\partial p_\mu+B_\mu), \tag{16}$$

where B_μ takes care of the stochastic extension of the particle.

The asymptotic zero curvature condition $F_{\mu\nu}=0$ helps us to write the non-Abelian gauge field as

$$B_\mu=U^{-1}\partial_\mu U,\quad U\in \mathrm{SL}(2,c).$$

With this substitution, we note that the term $F_{\mu\nu}F^{\mu\nu}$ in the Lagrangian gives rise to the Skyrme term $\mathrm{Tr}\,[\partial_\mu UU^+,\partial_\nu UU^+]^2$. Now the Skyrme Lagrangian is

$$L=M^2\,\mathrm{Tr}(\partial_\mu U^+\,\partial_\mu U)+\mathrm{Tr}[\partial_\mu UU^+,\partial_\nu UU^+]^2, \tag{17}$$

where the first term can be derived from the term like $B_\mu B^\mu$. Here M is a suitable constant having the dimension of mass. Thus we find that the quantization of a Fermi field considering an anisotropy in the internal space leading to an internal helicity description corresponds to the realization of a nonlinear sigma model, where the Skyrme term ($L_{\mathrm{Skyrme}}=\mathrm{Tr}[\partial_\mu UU^+,\partial_\nu UU^+]^2$) introduced for stabilizing the soliton automatically arises here as an effect of quantization. Thus in this picture fermions have solitonic feature and the fermion number has some topological origin. Indeed, for the Hermitian representation, we can take the group manifold as SU(2) and this leads to a mapping from the three-sphere S^3 to the group space S^3 [SU(2)$=S^3$] and the corresponding winding number is

$$q=\frac{1}{24\pi^2}\int dS^\mu\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}[U^{-1}\partial_\nu UU^{-1}\partial_\alpha UU^{-1}\partial_\beta U]. \tag{18}$$

Evidently, q is topological index and represents the fermion number. The Skyrme term which arises here as an effect of quantization does not manifestly express the internal anisotropy as it is invariant under P and T. So to incorporate this anisotropic feature in the Lagrangian, we should add the Wess–Zumino term. The W–Z action is given by

$$S_{\mathrm{wz}}=\frac{iN}{240\pi^2}\int_D \epsilon^{\mu\nu\lambda\sigma\rho}\,\mathrm{Tr}[U^{-1}\partial_\mu UU^{-1}\partial_\nu UU^{-1}$$
$$\times\partial_\lambda UU^{-1}\partial_\sigma UU^{-1}\partial_\rho U]d^5x,\quad x=\mathbf{x},t,x^5. \tag{19}$$

Here, the physical space-time is the boundary of the five dimensional domain. Witten[20] has shown that, for the existence of a consistent quantum description, N must be an integer.

In the geometry developed we have introduced spinor structure to each space-time point and have got a superspace. This geometry effectively gives rise to the SL(2,c) gauge fields (as the spinor–affine connection) having the field strength

$$F_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu+[B_\mu,B_\nu],$$

where B_μ is the matrix-valued potential. In superspace a given covariant tensor $F_{\mu\nu}$ does not have contravariant components $F^{\mu\nu}$. So, if we now demand SL(2,c) invariance, we will have to follow Carmelli and Malin[21] and choose the simplest Lagrangian density which is invariant under SL(2,c) transformations:

$$L=-\tfrac{1}{4}\,\mathrm{Tr}(\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}), \tag{20}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the completely antisymmetric tensor density in four dimensions with $\varepsilon^{0123}=1$. Applying the usual procedure of variational calculus, we get the field equations

$$\partial_\delta(\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta})-[B_\delta,\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}]=0. \tag{21}$$

Let us consider the infinitesimal generators of the group SL(2,c) in the tangent space as

$$g^1=\begin{bmatrix}0&0\\1&0\end{bmatrix},\quad g^2=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad g^3=\begin{bmatrix}0&1\\0&0\end{bmatrix}. \tag{22}$$

Then we can write

$$B_\mu=b^a_\mu g^a=\mathbf{b}_\mu\cdot\mathbf{g},\quad F_{\mu\nu}=f^a_{\mu\nu}g^a=\mathbf{f}_{\mu\nu}\cdot\mathbf{g}. \tag{23}$$

Evidently, in this space these SL(2,c) gauge fields will appear as background fields.

Thus to describe a matter field in this geometry, the Lagrangian will be modified by the introduction of this SL(2,c) invariant Lagrangian density. Hence the Lagrangian for a massless Dirac field is given by

$$L=-\bar{\psi}\gamma_\mu D_\mu\psi-\tfrac{1}{4}\,\mathrm{Tr}\,\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}\,, \tag{24}$$

where D_μ is the SL(2,c) gauge covariant derivatives defined by

$$D_\mu=\partial_\mu-igB_\mu\,, \tag{25}$$

where g is some coupling strength.

We can now construct a conserved current corresponding to this Lagrangian (neglecting the coupling with the gauge field)

$$\mathbf{J}^\mu=\bar{\psi}\gamma^\mu\psi+\varepsilon^{\mu\nu\alpha\beta}\mathbf{b}_\nu\times\mathbf{f}_{\alpha\beta}=\mathbf{J}_x{}^\mu+\mathbf{J}_\theta{}^\mu. \tag{26}$$

We find from (21) that

$$\varepsilon^{\mu\nu\alpha\beta}(\partial_\nu\mathbf{f}_{\alpha\beta}-\mathbf{b}_\nu\times\mathbf{f}_{\alpha\beta})=0. \tag{27}$$

Then we can write,

$$\mathbf{J}_\theta{}^\mu=\varepsilon^{\mu\nu\alpha\beta}\mathbf{b}_\nu\times\mathbf{f}_{\alpha\beta}=\varepsilon^{\mu\nu\alpha\beta}\partial_\nu\mathbf{f}_{\alpha\beta}\,. \tag{28}$$

So,

$$\partial_\mu \mathbf{J}_\theta{}^\mu = \epsilon^{\mu\nu\alpha\beta}\partial_\mu\partial_\nu \mathbf{f}_{\alpha\beta} = 0. \tag{29}$$

It is to be noted here that this $\mathbf{J}^\mu_\theta$ is the current corresponding to the spinorial variable "θ" that gives rise to the internal helicity and is thus associated with the "vortex lines" attached to the space-time point x_μ. We also note that $\mathbf{J}_\theta{}^\mu$ changes sign under space reflection. This picture enables us to think that the Ising interaction

$$-J\sum_i \sigma_i\sigma_{i+1}$$

(σ_i: orientation of the spin axis at a lattice site with value $+1$ or -1) can be associated with the current–current coupling $\mathbf{J}_\theta{}^\mu \cdot \mathbf{J}_\theta{}^\mu$ in the continuum limit.

However, in the Lagrangian (24), if the Dirac massless spinor is split in chiral forms and the internal helicity is identified with left (right) chirality corresponding to $\theta(\bar{\theta})$, then the three SL(2,c) gauge field equations give rise to the following conservation laws:[22]

$$\begin{aligned} &\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_R\gamma_\mu\psi_R)+J^1_\mu]=0,\\ &\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_\mu\psi_L+ig\bar{\psi}_R\gamma_\mu\psi_R)+J^2_\mu]=0,\\ &\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_\mu\psi_L)+J^3_\mu]=0. \end{aligned} \tag{30}$$

These three equations represent a consistent set of equations if we choose

$$J^1_\mu = -J^2_\mu/2, \quad J^3_\mu = J^2_\mu/2, \tag{31}$$

which evidently guarantees the vector current conservation. Then we can write

$$\partial_\mu(\bar{\psi}_R\gamma_\mu\psi_R + J^2_\mu) = 0, \quad \partial_\mu(\bar{\psi}_L\gamma_\mu\psi_L - J^2_\mu) = 0. \tag{32}$$

From these we find

$$\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi) = \partial_\mu J^5_\mu = -2\partial_\mu J^2_\mu. \tag{33}$$

Thus the anomaly is expressed here in terms of the second component of the SL(2,c) gauge field current J^2_μ. However, since in this formalism the chiral currents are modified by the introduction of J^2_μ, we note from (32) that the anomaly vanishes.

The charge corresponding to the gauge field part is

$$q = \int J_0{}^2\, d^3x = \int_{\text{surface}} \epsilon^{ijk}\, d\sigma_i\, f^2_{jk} \quad (i,j,k = 1,2,3). \tag{34}$$

Visualizing f^2_{jk} to be the magnetic field like components for the vector potential b^2_i, we see that q is actually associated with the magnetic pole strength for the corresponding field distribution. Thus we find that the quantization of a Fermi field associates a background magnetic field and the charge corresponding to the gauge field effectively represents a magnetic charge.

Thus term $\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{Tr}\, F_{\alpha\beta}F_{\gamma\delta}$ in the Lagrangian can be actually expressed as a four divergence of the form $\partial_\mu\Omega_\mu$, where

$$\Omega^\mu = -\frac{1}{16\pi^2}\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{Tr}\left[\frac{1}{2} B_\alpha F_{\beta\gamma} - \frac{2}{3}(B_\alpha B_\beta B_\gamma)\right]. \tag{35}$$

We recognize that the gauge field Lagrangian is related to the Pontryagin density

$$P=-\frac{1}{16\pi^2}\,\mathrm{Tr}\,{}^*F^{\mu\nu}F_{\mu\nu}=\partial_\mu\Omega^\mu, \tag{36}$$

where Ω_μ is the Chern–Simons term. The Pontryagin index

$$q=\int P\,d^4x \tag{37}$$

is then a topological invariant. As we know, the introduction of the Chern–Simons term modifies the axial vector current as

$$\widetilde{J}^5_\mu=J^5_\mu+2\hbar\Omega_\mu\,, \tag{38}$$

where $\partial_\mu\widetilde{J}^5_\mu=0$ though $\partial_\mu J^5_\mu\neq 0$. We find from Eq. (33) that the Chern–Simons term is effectively represented by the current J^2_μ constructed out of the SL(2,c) gauge fields. Thus we see that the anisotropy of the space-time builds the Chern–Simons term automatically in the system and is associated with the topological aspects of a fermion arising out of the quantization procedure.

In 2+1 dimension, the Hopf invariant is defined as

$$H=-\frac{1}{4\pi}\int d^3x\,\epsilon^{\hat\mu\hat\nu\hat\lambda}A_{\hat\mu}F_{\hat\nu\hat\lambda}\,. \tag{39}$$

Now if μ denotes a four-dimensional index then

$$\partial_\rho\epsilon^{\rho\mu\nu\lambda}A_\mu F_{\nu\lambda}=\tfrac{1}{2}\epsilon^{\rho\mu\nu\lambda}F_{\rho\mu}F_{\nu\lambda} \tag{40}$$

connects the Hopf invariance to the chiral anomaly.[23] So from our analysis above, we note that $\partial_\mu J^2_\mu$, J^2_μ being the second component of the current arising out of the background SL(2,c) gauge field is associated with the Hopf term in a 2+1-dimensional system. As we know, the Hopf term in a 2+1-dimensional system is analogous to the Wess–Zumino term in the Skyrme Lagrangian in 3+1 dimension. Thus we find that origin of the Wess–Zumino term is associated with the inherent gauge field current J^2_μ.

This is analogous to the equivalence of a three-dimensional Ising system and a Polyakov string. Indeed that a Polyakov string may be associated with a nonlinear σ model with a Weiss–Zumino term and topological aspects of a Liouville field suggests that a Polyakov string may be described by associating two vortex lines at the end points of a string when for a fermion (boson) the orientation of the vortex lines are in the same (opposite) direction.[24,25] This represents a three-dimensional Ising system. In a string picture, as for a fermion, we have two vortex lines of same orientation at the end points, each vortex line carrying a fermion number $\frac{1}{2}$. Since this corresponds to a three-dimensional Ising system, we have the effect of fractional statistics in three dimensions.

This analysis suggests that an Ising model Hamiltonian can be represented in the continuum limit by the interaction $J^2_\mu\cdot J^2_\mu$.[7] Also the current J^2_μ can be considered to be the source of the gauge field B_μ such that

$$J^2_{\mu(\theta)}=\Box B_{\mu+}\,,\quad J^2_{\mu(\bar\theta)}=\Box B_{\mu-}\,. \tag{41}$$

This suggests that for a three-dimensional Ising model, we can formulate a Z_2 gauge field which corresponds to the dual lattice.[11]

III. FRUSTRATED ISING SYSTEM

From our analysis above, it appears that the Ising spin system can well be represented by chiral currents J_μ^2 in the continuum limit and as such can be considered to represent a chiral spin liquid. However, the inherent Z_2 symmetry of the Hamiltonian suggests that chiral symmetry must be preserved. That is, for a ferromagnetic system we can have interactions like $J^2_{\mu(\theta)}J^2_{\mu(\theta)}$ or $J^2_{\mu(\bar\theta)}J^2_{\mu(\bar\theta)}$. Evidently these interactions will preserve chiral symmetry and as such will be reflection invariant and hence the overall system will not bear any specific signature of the chirality. For an antiferromagnetic system the corresponding interaction will be $J^2_{\mu(\theta)}J^2_{\mu(\bar\theta)}$ and, in general, this is also chiral invariant. However, for a system with an odd number of antiferromagnetic links, this may lead to the breakdown of chiral symmetry. According to the definition of frustration, this system will lead to frustrated spin system. We can associate the $J^2_{\mu(\theta)}J^2_{\mu(\theta)}$ interaction with the interaction in terms of the gauge potential $B_{\mu+}$, i.e., with $J^2_{\mu(\theta)}B_{\mu+}$. Similarly, the current $J^2_{\mu(\bar\theta)}$ will be associated with the gauge field $B_{\mu-}$, where $B_{\mu+}(B_{\mu-})$ represents the link variable. It is noted that to describe an antiferromagnetic link where the end lattice points have orientations in the opposite directions, the interaction in the continuum limit is given by $J^2_{\mu(\theta)}G_{\mu\nu}J^2_{\nu(\bar\theta)}$, where $G_{\mu\nu}$ is given by

$$G_{\mu\nu}=B_{\mu+}B_{\nu-}\ .$$

Evidently, this involves a change in chirality in the link variable. For a system with an even number of antiferromagnetic links, the change in chirality in one link is compensated by the change in chirality in another link. Hence there is no change in chirality in the closed loop and we can say that the whole system is invariant under chirality transformation. However, for a closed loop with an odd number of antiferromagnetic links, which is called a frustrated loop, the case will not be as above and the frustrated system will not be invariant under chirality transformation.

The change in chirality in a link where the gauge potential $B_{\mu+}$ changes to $B_{\nu-}$ may be visualized through the introduction of fictitious chiral current at a certain point on the link where the interaction may be written in the form

$$B_{\mu+}(\bar\psi_L\gamma_\mu\psi_L)(\bar\psi_R\gamma_\nu\psi_R)B_{\nu-}\ ,$$

where at that point, the left-handed spinor ψ_L changes to a right-handed spinor ψ_R through space reflection. For a system with an odd number of antiferromagnetic links this change in chirality of the fictitious spinors will lead to the change in chirality of the total system, i.e., there will be a change in chirality in a frustrated loop. This can be described by a chiral spin liquid bearing the chiral signature of the fictitious spinor ψ_L or ψ_R.

IV. ORDER PARAMETER OF A FRUSTRATED SPIN SYSTEM

Our foregoing discussion suggests that the order parameter of frustrated spin system can be denoted by a chiral fermion ψ_L or ψ_R. Therefore it may be inferred that the order parameter of a frustrated system is fermionic in nature, which is also commented by Kadanoff and Ceva[26] and also by Fradkin *et al.*[27]

As discussed in Sec. II, chiral fermion may be depicted by a scalar particle moving with $l=\frac{1}{2}$ in an anisotropic space with a specific l_z value. Indeed, in an anisotropic space the angular momentum ia given by

$$\mathbf{J}=\mathbf{r}\times\mathbf{p}-\mu_M\mathbf{r},$$

where μ_M corresponds to the measure of anisotropy given by

$$[p_i, p_j] = \mu_M e^{ijk} \frac{x^k}{r^3}, \quad i,j = 1,2,3,$$

and this behaves as a magnetic monopole charge. The spherical harmonics Y_ι^{m,μ_M} with $\iota = \frac{1}{2}$, $m = \pm\frac{1}{2}$, $\mu_M = \pm\frac{1}{2}$ are denoted as

$$Y_{1/2}^{1/2,1/2} = \sin(\theta/2) e^{i(\phi-\chi)/2},$$

$$Y_{1/2}^{-1/2,1/2} = \cos(\theta/2) e^{-i(\phi+\chi)/2},$$

$$Y_{1/2}^{1/2,-1/2} = \cos(\theta/2) e^{i(\phi+\chi)/2},$$

$$Y_{1/2}^{-1/2,-1/2} = \sin(\theta/2) e^{-i(\phi-\chi)/2}.$$

It is to be noted that the doublet

$$\psi_L = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \text{with } \phi_1 = Y_{1/2}^{1/2,1/2}(\theta,\phi,\chi) \quad \text{and } \phi_2 = Y_{1/2}^{-1/2,1/2}(\theta,\phi,\chi) \tag{42}$$

corresponds to a two-component spinor. The charge conjugate state of (42) is given by

$$\psi_R = \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} \quad \text{with } \tilde{\phi}_1 = Y_{1/2}^{-1/2,-1/2}(\theta,\phi,\chi) \quad \text{and } \tilde{\phi}_2 = Y_{1/2}^{1/2,-1/2}(\theta,\phi,\chi). \tag{43}$$

Now, writing $\rho_\alpha = \phi_\alpha$ $(\alpha = 1,2)$ and $\rho_{\alpha+2} = i\Pi_\alpha^+$ where Π_α is the conjugate of ϕ_α and is given by $\Pi_\alpha = (1/i)\partial/\partial\phi_\alpha$. We can construct a 4-component spinor

$$\rho = \begin{pmatrix} \phi \\ i\Pi^+ \end{pmatrix}$$

with its charge conjugate state

$$\rho^+ = \begin{pmatrix} \phi^+ \\ -i\Pi \end{pmatrix}.$$

The doublet

$$\xi = \begin{pmatrix} \rho \\ \rho^+ \end{pmatrix}$$

represents an eight component conformal spinor and ρ and ρ^+ represent two cartan semispinors as P and T reflection changes $\rho \rightleftarrows \rho^+$. Thus we can represent the spinorial variable ρ or ψ_L (ψ_R) as the order parameter for the three-dimensional frustrated Ising system that is fermionic in nature. On the other hand, if we want to write the spinorial variables in terms of the oscillator variables[28]

$$a_\alpha, \quad b_\alpha \quad (\alpha = 1,2),$$

then

$$a_\alpha = (\phi_\alpha + i\Pi_\alpha^+)/\sqrt{2},$$

$$b_\alpha=[\sigma_2(i\phi_\alpha^+-\Pi_\alpha)]/\sqrt{2}, \tag{44}$$

$$\tilde{b}_\alpha=i(\phi_\alpha-i\Pi_\alpha^+)/\sqrt{2}.$$

We can define the operators

$$K=a_\alpha^+a_\alpha,\quad L=b_\alpha^-b_\alpha=\tilde{b}_\alpha\tilde{b}_\alpha^+$$

from which we can define a new operator

$$\mu_M=(a_\alpha^+a_\alpha-b_\alpha^+b_\alpha)/2=(K-L)/2, \tag{45}$$

which denotes the internal orientation of the system and corresponds to the specific l_z value. We can take the eigenvalue of μ_M as the order parameter implying P and T violation.

Evidently, the order parameter cannot be classified in terms of the irreducible representations of $SU(2)_J$, where J is the total angular momentum as in the case of ^{3}He–A Phase.[29]

V. DISCUSSION

We have described here a gauge theoretical version of a frustrated spin system where the Chern–Simons term corresponding to the current $J^2_{\mu(\theta)}$ $(J^2_{\mu(\bar{\theta})})$ takes a specific role. In $2+1$ dimensions, this term effectively corresponds to the Hopf term. An Ising system in three or two dimensions can be described by a constant time surface in these dimensions when the interactions involve these currents in the continuum limit. A frustrated system is described by a chiral spin liquid where it bears the signature of a chiral spinor ψ_L or ψ_R. This may be considered to represent the order parameter. This may also be related to a Cartan semispinor characterizing the properties of P and T violation. This may also be characterized by solitonic features associated with the Chern–Simons term and may be associated to the Pontryagin index. Indeed, the Pontryagin index is given by $q=2\int J_0^2\, d^3x$ and is related to the chiral anomaly through the relation $q=-\int \partial_\mu J_\mu^5\, d^4x$.[30] Since the frustrated spin system in three dimensions may be associated with a chiral spin liquid where the Hamiltonian does not remain invariant under chirality transformation, the Pontryagin index behaves as a good topological index to specify the order parameter.

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[1] For reviews of experimental work see J. A. Mydosh, J. Mag. Magn. Mat. 7, 238 (1978).
[2] C. N. Guy, J. Phys. F 5, L242 (1978).
[3] G. Toulouse, Commun. Phys. 2, 115 (1977).
[4] J. Stephenson, J. Math Phys. 11, 720 (1970).
[5] A. Danielian, Phys. Rev. Lett. 6, 670 (1961).
[6] S. Alexander and P. Pincus, J. Phys. A 13, 263 (1980).
[7] M. K. Phani, J. L. Lebowitz, M. H. Kalos, and C. C. Tsai, Phys. Rev. Lett. 42, 572 (1979).
[8] O. J. Heilman, J. Phys. A: Math. Gen. 13, 1803 (1980).
[9] J. Villain, J. Phys. C: Solid State Phys. 10, 1717 (1977).
[10] P. Bandyopadhyay and K. Hajra, J. Math. Phys. 28, 711 (1987).
[11] G. Goswami and P. Bandyopadhyay, preprint (1991).
[12] P. Bandyopadhyay and K. Hajra, Phys. Lett. A 155, 7 (1992).
[13] E. Prugovecki, *Stochastic Quantum Mechanics and Quantum Space-Time* (Reidel, Dordrecht, 1985).
[14] P. Bandyopadhyay, Int. J. Mod. Phys. A 4, 4449 (1989).
[15] A. O. Barut and G. L. Bornzin, J. Math. Phys. 12, 841 (1971).

[16] M. Fierz, Helv. Phys. Acta **17**, 27 (1944).
[17] C. A. Hurst, Ann. Phys. **50**, 51 (1968).
[18] F. Wilczek, Phys. Rev. Lett. **48**, 1146 (1982).
[19] A. Bandyopadhyay, P. Chatterjee, and P. Bandyopadhyay, Gen. Relativ. Gravit. **18**, 1193 (1986).
[20] E. Witten, Nucl. Phys. B **223**, 422, 433 (1983).
[21] M. Carmeli and S. Malin, Ann. Phys. **103**, 208 (1977).
[22] A. Roy and P. Bandyopadhyay, J. Math. Phys. **30**, 2366 (1989).
[23] F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983).
[24] B. Basu and P. Banddyopadhyay, Nonlinear σ-model, Polyakov String and Three-Dimensional Ising System (submitted for publication).
[25] B. Basu and P. Bandyopadhyay, Europhys. Lett. **5**, 491 (1988).
[26] L. P. Kadanoff and H. Ceva, Phys. Rev. B **11**, 3918 (1971).
[27] E. Fradkin, B. A. Huberman, and S. H. Shenker, Phys. Rev. B **18**, 4789 (1978).
[28] T. Takabayasi, Prog. Theor. Phys. Suppl. Extra No. 339 (1965).
[29] P. Bandyopadhyay, K. Hajra, and B. Basu, Fractional Statistics, 3+1 Dimension and Superfluidity (submitted for publication).
[30] D. Banerjee and P. Bandyopadhyay, J. Math. Phys. **33**, 990 (1992).