

ON RATES OF CONVERGENCE TO NORMALITY FOR ϕ -MIXING PROCESSES

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SUMMARY. For standardized sums of non-stationary ϕ -mixing random variables, uniform and non-uniform Berry-Esseen bounds are obtained. These bounds are applied to prove convergence of absolute moments of such sums to the corresponding moments of the normal (0, 1) distribution, and also in proving L_r -versions of the Berry-Esseen theorem. Further application of these bounds consists in proving probabilities of moderate deviations for non-stationary ϕ -mixing processes.

1. INTRODUCTION

Consider a sequence $\{X_n\}$ of random variables. Let \mathcal{M}_n^+ and \mathcal{M}_n^- denote the σ -fields generated by $\{X_1, \dots, X_n\}$ and $\{X_n, X_{n+1}, \dots\}$ respectively. Suppose there exists a sequence $\{\phi_n\}$ of real numbers such that $1 \geq \phi_1 \geq \phi_2 \geq \dots$, and

$$|P(A \cap B) - P(A)P(B)| \leq \phi_n P(A)$$

for all $A \in \mathcal{M}_1^+$, $B \in \mathcal{M}_{k+n}^-$, $k \geq 1$, $n \geq 1$. The sequence $\{X_n\}$ is called ϕ -mixing if $\lim_{n \rightarrow \infty} \phi_n = 0$.

Define $S_n = \sum_{i=1}^n X_i$, $\sigma_n^2 = V(S_n)$ and $F_n(t) = P(S_n \leq t\sigma_n)$. Assume that

$$E(X_n) = 0 \quad \text{for all } n \geq 1, \quad \dots \quad (1.1)$$

$$\sum_{n=1}^{\infty} \phi_n^4 < \infty, \quad \dots \quad (1.2)$$

$$\inf_{n \geq 1} n^{-1} \sigma_n > 0, \quad \dots \quad (1.3)$$

and for some $c > 0$ and $M > 1$

$$E|X_n|^{2+c} \leq M, \quad \text{for all } n \geq 1. \quad \dots \quad (1.4)$$

Under the above conditions, certain uniform and non-uniform rates of convergence to normality are proved in this paper, for $\sigma_n^{-1} S_n$. A uniform Berry-Esseen theorem is derived in Section 2. We have proved the

convergence of the absolute moments of $\sigma_n^{-1} S_n$ to the corresponding absolute moments of $|N(0, 1)|$ in Section 3. Also, proved in this section is a non-uniform Berry-Esseen theorem. These extend Michel's (1976) results for sums of i.i.d. r.v.'s. In the same section an L_r version of the Berry-Esseen theorem is proved. This result generalizes and strengthens a corresponding result of Erickson (1973). Finally in Section 4, another error bound for approximating $F_n(t)$ by $\Phi(t) = P(N(0, 1) \leq t)$ is derived. This is then utilized in finding a zone where $1 - F_n(t_n) \sim \Phi(-t_n)$ (i.e., the ratio goes to 1) as $n \rightarrow \infty$. A similar result for sums of i.i.d. r.v.'s, is available in Michel (1976).

We shall be using the following generalization of Ibragimov's (1962) result several times.

Lemma 1: Let $\{X_n\}$ be a ϕ -mixing process satisfying (1.1) and (1.2). Suppose for some $\delta \geq 2$, there exists a real number $N > 1$ such that for all $n \geq 1$,

$$E|X_n|^\delta \leq N. \quad \dots (1.5)$$

Let $d > 1$ and $Y_t = Y_{d,t} = X_t I(|X_t| < d)$, where $I(A)$ denotes the indicator function of the set A . Then for any real number $v \geq 2$, there exists $D(v) = D(v, \delta, N) > 0$, not depending on d , such that for all positive integers $u < d^2$ and $h \geq 0$,

$$E \left| \sum_{i=1}^u Y_{t+h} \right|^v \leq D(v) (u^{v/2} + uR(v)),$$

where

$$R(v) = R(v, \delta, d) = d^{v-\delta}.$$

Proof of this lemma is given in the appendix. Ghosh and Babu (1977) have proved a weaker version of this lemma under a stronger moment condition.

We conclude this section by stating another lemma which is used several times in this paper.

Lemma 2: Let $\{J_i\}$ be a sequence of random variables satisfying $0 < J_i \leq A$ for all i and for some $A > 0$. Suppose $J_i \in \mathcal{A}_{(j+m)}^{(j+m), m}$ for some integers $j \geq 0$ and $m > 0$. Then, for any positive integer n , we have

$$E \left(\prod_{i=1}^n J_i \right) \leq \prod_{i=1}^n M_i \quad \dots (1.6)$$

and

$$\left| E \left(\prod_{i=1}^n J_i \right) - \prod_{i=1}^n E(J_i) \right| \leq 2A\phi_j \sum_{r=1}^n \left(\prod_{\substack{1 \leq i \leq n \\ i \neq r}} M_i \right), \quad \dots (1.7)$$

where

$$M_i = E(J_i) + 2A\phi_j.$$

Proof: Using a result of Ibragimov (1962) (see, o.g., Lemma 1, page 170 of Billingsley, 1968) from now on referred to as Lemma A), we obtain that $E \left(\prod_{i=1}^n J_i \right) \leq M_n E \left(\prod_{i=1}^{n-1} J_i \right)$. Repeated application of this inequality gives us (1.6); (1.7) follows from (1.6) and Lemma A.

2. UNIFORM BERRY-ESSEEN BOUNDS

For any random variable X , let $F(X)$ denote the distribution function of X . Let Φ denote the distribution $N(0, 1)$. Let, for any bounded function f , $\|f\| = \sup_i |f(i)|$. We use Vinogradov's symbol \ll instead of the usual O -symbol whenever it is found convenient.

Theorem 1: *Let $\{X_n\}$ be a ϕ -mixing process satisfying (1.1), (1.2), (1.3) and (1.4). Then*

$$\|F(\sigma_n^{-1} S_n) - \Phi\| \ll n^{-\gamma(c)} \log n,$$

where $\gamma(c) = 2c^*/(6+5c^*)$ and $c^* = \min(c, 1)$.

Proof: First we present the blocking procedure which is to be used throughout the paper.

Let $p = p(\alpha, n) = [n^\alpha]$, $q = q(\beta, n) = [n^\beta]$, $k = k(\alpha, \beta, n) = [n/(p+q)]$ and $l = n - k(p+q)$, where $0 < \beta \leq \alpha < 1$ will be chosen according to the need.

Put

$$\begin{aligned} \xi_i &= \xi_{n,i} = \sum_{j=1}^p X_{(i-1)(p+q)+j}, & 1 \leq i \leq k \\ \eta_i &= \eta_{n,i} = \sum_{j=1}^q X_{ip+(i-1)q+j}, & 1 \leq i \leq k \end{aligned} \quad \dots (2.1)$$

and

$$\xi_{k+1} = \xi_{n,k+1} = \sum_{j=1}^l X_{kp+(p+q)+j} \quad \text{or } 0$$

according as $l \geq 1$ or not.

For the present use, we fix $\alpha = (2+5c^*)/(6+5c^*)$ and $\beta = (2+c^*)/(6+5c^*)$. Since for any two random variables X and Y , and $\epsilon > 0$,

$$\|F(X+Y) - \Phi\| \leq \|F(X) - \Phi\| + (2\pi)^{-1} \epsilon + P(|Y| > \epsilon), \quad \dots (2.2)$$

it is enough to prove that

$$P \left(\left| \sum_{i=1}^k \eta_i \right| > K \sigma_n n^{-\gamma(c)} \log n \right) \ll n^{-\gamma(c)} \log n \quad \dots (2.3)$$

for some $K > 0$ and

$$\left\| F \left(\sigma_n^{-1} \sum_{i=1}^{k+1} \xi_i \right) - \Phi \right\| \ll n^{-\gamma(c)} \log n. \quad \dots (2.4)$$

To establish (2.3), let $\eta_i^* = \eta_i I(|\eta_i| < (kq)^{1/2})$. By Lemma 1, the difference between the left-hand side of (2.3) and a similar quantity with η_i replaced by η_i^* is not more, in modulus, than

$$\sum_{i=1}^k P(|\eta_i| \geq (kq)^{1/2}) \ll k(kq)^{-1-\alpha/2} q^{1+\alpha/2} = k^{-\alpha/2} \ll n^{-\gamma(c)}.$$

The last inequality follows by the earlier choice of α and β . Hence, it suffices to show that

$$P \left(\left| \sum_{i=1}^k \eta_i^* \right| > Kn^{1-\gamma(c)} \log n \right) \ll n^{-\gamma(c)}. \quad \dots (2.5)$$

By Markov's inequality and by Lemma 2, we have

$$P \left(\sum_{i=1}^k \eta_i^* > Kn^{1-\gamma(c)} \log n \right) \leq n^{-K} E \left(\exp \left(n^{\gamma(c)-1} \sum_{i=1}^k \eta_i^* \right) \right) \leq n^{-K} \prod_{i=1}^k r_i, \quad \dots (2.6)$$

where

$$r_i = 2C\phi_p + E(\exp(n^{\gamma(c)-1}\eta_i^*))$$

and

$$C = \exp(n^{\gamma(c)-1}(kq)^{1/2}) \ll 1.$$

Now using Lemma 1 and

$$r_i \leq 2C\phi_p + 1 + n^{\gamma(c)-1} |E(\eta_i^*)| + n^{2\gamma(c)-1} E(\eta_i^*)^2, \quad \dots (2.7)$$

it can be shown that $\prod_{i=1}^k r_i = o(1)$. On choosing K large in (2.5), (2.6) and (2.7) plus similar inequalities with η_i^* replaced by $-\eta_i^*$ lead to (2.5).

We now turn to the proof of (2.4). Let $\{\xi'_i, 1 \leq i \leq k+1\}$ be a sequence of independent random variables with ξ'_i having the same distribution as that of ξ_i . Let $\theta_i^* = \sum_{i=1}^{k+1} V(\xi'_i)$. Then, following Katz's (1963) proof of Berry-Esseen

bounds for independent random variables, and using Lemma 4 of Iosifescu (1968), we conclude that the left-hand side of (2.4) is not more than

$$\begin{aligned} & \left\| F\left(\sigma_n^{-1} \sum_{i=1}^{k+1} \xi_i'\right) - \Phi \right\| + (k+1)\phi_q \\ & \leq \left\| F\left(\theta_k^{-1} \sum_{i=1}^{k+1} \xi_i'\right) - \Phi \right\| + k\phi_q \\ & \quad + \sup_t |\Phi(t) - \Phi(\sigma_n \theta_k^{-1} t)| \\ & \leq n^{-\gamma(c)} \log n + |\sigma_n - \theta_k| \theta_k^{-1}. \quad \dots (2.8) \end{aligned}$$

A few applications of Lemma A lead to

$$|\sigma_n^2 - \theta_k^2| \leq n^{1-\gamma(c)}, \quad \sigma_n^2 \leq n \quad \text{and} \quad n \leq \theta_k^2 \leq n. \quad \dots (2.9)$$

For similar calculations see Ghosh and Babu (1977).

The theorem now follows from (2.8) and (2.9).

Remark 2.1: If the sequence $\{X_n\}$ in Theorem 1 is stationary, and $0 < \sigma^2 = V(X_1) + 2 \sum_{t=1}^{\infty} \text{cov}(X_1, X_{1+t})$, then

$$\dots \quad \|F(n^{-1}\sigma^{-1}S_n) - \Phi\| \leq n^{-\gamma_1(c)} \log n, \quad \dots (2.10)$$

where $\gamma_1(c) = c/(c+4)$ if $c \leq 2/3$ and $= \gamma(c)$ if $c \geq 2/3$. Notice that $\gamma_1(c) < \gamma(c)$ if $c < 2/3$. The bound gets cruder due to the approximation of σ_n^2 by $n\sigma^2$.

3. NON-UNIFORM BERRY-ESSEEN BOUNDS

Let $K > 0$, $K_1 > 0$ and $\gamma > 0$ denote generic constants.

Theorem 2: Let $\{X_n\}$ be as in Theorem 1. Then for all $t^2 > (c+1) \log n$,

$$|P(S_n \leq t\sigma_n) - \Phi(t)| \leq Kn^{-c/2} |t|^{-2-c} (\log |t|)^{2+c}. \quad \dots (3.1)$$

Proof: Without loss of generality we assume that $t > 0$. Since $t^2 > (c+1) \log n$, we have $\Phi(-t) \leq n^{-c/2} t^{-2-c}$. So it is enough to show that $P(S_n > t\sigma_n)$ is not more than the right side of (3.1). We shall omit the details

as the proof follows the lines of Lemma 3 below. We only mention that in the proof of Lemma 3, we work with $X'_i = X_i I(|X_i| \leq n^{\lambda} t)$ and $\xi'_j = \xi_j I(|\xi_j| \leq s t n^{\lambda} / \log t)$ for some $s > 0$.

From Theorems 1 and 2 we immediately have the following.

Theorem 3: Under the hypothesis of Theorem 1, we have for all real t ,

$$|P(S_n \leq t \sigma_n) - \Phi(t)| \leq n^{-\gamma(t)} (1 + |t|^{2+\epsilon})^{-1} (\log(n(1 + |t|)))^{2+2\epsilon}.$$

Recently Erickson (1973) stressed the need of L_r version of Berry-Esseen bounds and presented some results for independent and m -dependent cases. The corollary given below follows from Theorems 1 and 3.

Corollary 1: If $\{X_n\}$ is as in Theorem 1, then for all $r \geq 1$

$$\|F(\sigma_n^{-1} S_n) - \Phi\|_r \leq n^{-\gamma(t)} (\log n)^{2/r},$$

where $\gamma(t)$ is defined in Section 2 and $\|\cdot\|_r$ denotes the r -th norm with respect to Lebesgue measure.

If $\{X_n\}$ is a ϕ -mixing process satisfying (1.1), (1.3), (1.4) and $\phi_n \leq e^{-\lambda n}$ for some $\lambda > 0$, then by similar calculations, we have

$$\|F(\sigma_n^{-1} S_n) - \Phi\|_r \leq n^{-\epsilon^*/(2+2\epsilon^*)} (\log n)^{2/r},$$

which sharpens Erickson's Theorem 2.

A few lemmas are needed to prove the next theorem. From now on in this section we assume that $\{X_n\}$ satisfies the hypothesis of Theorem 1.

Lemma 3: Let $b > 1/2$, $\epsilon > 0$ and let $X_{t,b} = X_t I(|X_t| \leq nb)$. Then we have for $t > 0$ and $\epsilon \log n \leq t^2 \leq n^{2+2b}$,

$$P\left(\left|\sum_{i=1}^n X_{t,b}\right| > 3t n^{\lambda}\right) \leq a(n, t), \quad \dots \quad (3.2)$$

where

$$a(n, t) = n^{-\epsilon/2} t^{-\epsilon-1} (\log n)^{2+2\epsilon}.$$

Proof: Clearly the left-hand side of (3.2) is zero if $t > n^{1+b}$.

Let $X'_i = X_i I(|X_i| \leq \min(nb, t n^{\lambda}))$. In view of Lemma 1, it is enough to prove (3.2) with $X_{t,b}$ replaced by X'_i . Now we use the blocking procedure as described in (2.1) with X_i replaced by X'_i and with $\alpha = \beta = 1/3$. Let

$$U_n = \sum_{j=1}^k \xi_j, \quad U'_n = \sum_{j=1}^k \eta_j \quad \text{and} \quad T_n = \xi_{k+1}.$$

Clearly

$$\sum_{i=1}^n X_i = U_n + U'_n + T_n. \quad \dots (3.3)$$

By Lemma 1, we have with $v = 2c+2$

$$\begin{aligned} P(|T_n| > tn^t) &\leq t^{-v} n^{-v/2} E|T_n|^v \\ &\ll n^{-v/2} t^{-v} (p^{v/2} + p(n^t)^{v-c-2}) \\ &\ll a(n, t). \end{aligned}$$

To complete the proof it is enough to show that

$$P(V_n > tn^t) \ll a(n, t), \quad \dots (3.4)$$

with $V_n = U_n, -U_n, U'_n$ and $-U'_n$. We shall only show that (3.4) holds with $V_n = U_n$. Proofs of the other three inequalities are similar. For $\varepsilon \log n \leq t^2 \leq n^{2+2b}$, put

$$\begin{aligned} y &= 2(c+1)(b+1)t^{-1}n^{-b} \log n, \\ \xi_j^* &= \xi_j I(|\xi_j| < 1/2y) \text{ and } U_n^* = \sum_{i=1}^k \xi_i^*. \end{aligned}$$

Another application of Lemma 1 with $v = 2c+2$ gives

$$|P(U_n > tn^t) - P(U_n^* > tn^t)| \leq \sum_{i=1}^k P(|\xi_i| \geq 1/2y) \ll a(n, t).$$

By Markov's inequality and Lemma 2, it follows that

$$P(U_n^* > tn^t) \leq e^{-ytn^t} E(e^{yU_n^*}) \ll a(n, t) \prod_{j=1}^k s_j, \quad \dots (3.5)$$

where

$$s_j = 2e\phi_p + E(e^{y\xi_j^*}) \leq 2e\phi_p + 1 + y |E(\xi_j^*)| + 2ey^2 E(\xi_j^*)^2.$$

We now estimate s_j . We have

$$\begin{aligned} |E(\xi_j^*)| &= |E(\xi_j) - E(\xi_j I(|\xi_j| \geq 1/2y))| \\ &\ll E(\xi_j) + yE(\xi_j^2). \end{aligned} \quad \dots (3.6)$$

Since

$$\begin{aligned} \xi_j &= \sum_{i=h+1}^{p+h} X'_i \text{ for some } h \geq 0, \text{ we have} \\ |E(\xi_j)| &\leq \sum_{i=h+1}^{p+h} E(|X_i| I(|X_i| > \min(n^b, t n^4))) \\ &\leq p \max(n^{-b(c+1)}, t^{-c-1} n^{-(c+1)/2}) \\ &\leq y^{-1} k^{-1} (n^{1-(c+1)b} (\log n)^{\dagger+1}) \leq y^{-1} k^{-1}. \quad \dots (3.7) \end{aligned}$$

The last step above follows because $b > 1/2$. Combining (3.6) and (3.7) and observing $k\phi_q \ll 1$, we obtain that

$$\log \prod_{j=1}^k \theta_j \leq 1 + y^2 \sum_{j=1}^k E(\xi_j^2) \leq 1 + k p y^2 \ll 1.$$

This completes the proof of the lemma.

Put $\eta = 1$ or $c - [c]$ according as c is an integer or not. Let $a = -1 + (c+2)/\eta$. Clearly $0 < \eta \leq 1$ and $a \geq c+1$. Let $\bar{X}_t = X_t I(|X_t| > n^a)$. By Minkowski's inequality we get that, for any $h \geq 0$ and $u \leq n$,

$$E \left| \sum_{t=1}^u \bar{X}_{t+h} \right|^{2+c-\eta} \leq M u^{c+2-\eta} n^{-a\eta} \leq M. \quad \dots (3.8)$$

Following the lines of proof of Lemma 1 with $c+2-\eta$ and η now playing the roles of m and ε respectively, and observing that in view of (3.8), the last term of (9) is bounded, we have

$$\text{Lemma 4: For all } u \leq n \text{ and } h \geq 0, E \left| \sum_{t=1}^u \bar{X}_{t+h} \right|^{2+c} \leq Ku.$$

Let $X_{t,a} = X_t I(|X_t| \leq n^a)$ and let $W_n = \sum_{t=1}^n X_{t,a}$. By Theorem 1, we get that

$$\begin{aligned} \sup_t |P(W_n < t\sigma_n) - \Phi(t)| &\leq \sum_{t=1}^n P(|X_t| > n^a) + \sup_t |P(S_n < t\sigma_n) - \Phi(t)| \\ &\leq n^{1-a(2+c)} + n^{-\gamma(c)} \log n \leq n^{-\gamma(c)} \log n \quad \dots (3.9) \end{aligned}$$

so $1 - a(c+2) < -1$.

We now state

Lemma 5 :

$$|E|\sigma_n^{-1} W_n|^{2+c} - \pi^{-1} 2^{(2+c)/2} \Gamma((c+3)/2)| \ll n^{-\gamma(c)} (\log n)^{2+c} \dots (3.10)$$

Proof: Since for $t \geq 1$, $\Phi(-t) \leq e^{-t^2}$, defining $v_n = (\log n)^{1/(c+2)}$, we have by (3.9) and by Lemma 3, the left-hand side of (3.10) is not more than

$$\begin{aligned} & \int_0^\infty |P(|W_n| > t^{1/(c+2)} \sigma_n) - 2\Phi(-t^{1/(c+2)})| dt \\ & \ll n^{-\gamma(c)} (\log n)^{2+c} + \int_{v_n}^\infty \Phi(-t^{-1/(c+2)}) dt + \int_{v_n}^\infty P(|W_n| > \sigma_n t^{1/(c+2)}) dt \\ & \ll n^{-\gamma(c)} (\log n)^{2+c} + \int_{v_n}^{n^{(a+1)/(c+2)}} P(|W_n| > \sigma_n t^{1/(c+2)}) dt \\ & \ll n^{-\gamma(c)} (\log n)^{2+c} + n^{-c/2} (\log n)^{2+c} \ll n^{-\gamma(c)} (\log n)^{2+c}. \dots (3.11) \end{aligned}$$

This completes the proof of Lemma 5.

From Lemmas 4 and 5 and from Minkowski's inequality, we immediately have

Theorem 4 :

$$|E|\sigma_n^{-1} S_n|^{2+c} - \pi^{-1} 2^{(2+c)/2} \Gamma((3+c)/2)| \ll n^{-c/2(c+2)}$$

Remark 3.1: Stein (1972) has shown that if $\{X_n\}$ is a sequence of stationary m -dependent process with $E\{X_n^2\} < \infty$, then

$$\|F(\sigma_n^{-1} S_n) - \Phi\| \ll n^{-1}.$$

Combining this result with our Theorem 2 we obtain that for $r \geq 1$ for stationary m -dependent case, that

$$\|F(\sigma_n^{-1} S_n) - \Phi\|_r \ll n^{-1} (\log n)^2,$$

which is slightly worse than the best possible order.

4. MODERATE DEVIATIONS

Theorem 5: Under the hypothesis of Theorem 1, we have for all $t^2 \leq (c+1) \log n$,

$$|P(S_n \leq t\sigma_n) - \Phi(t)| \leq Kn^{-\lambda} e^{-t^2/2} + O(b(n, t)), \quad \dots (4.1)$$

where $\lambda > 0$, $K > 0$ are some constants and

$$b(n, t) = n^{-\alpha/2} (1 + |t|)^{-2-\alpha}.$$

Remark 4.1: If $\{X_n\}$ is a stationary sequence then in (4.1), σ_n^2 can be replaced by $n \left(E(X_1^2) + 2 \sum_{i=1}^n E(X_1 X_{1+i}) \right)$ and $O(b(n, t))$ can be replaced by $o(b(n, t))$. If $\{|X_n|^{2+\epsilon}\}$ are uniformly integrable then also $O(b(n, t))$ can be replaced by $o(b(n, t))$ in (4.1).

Proof of Theorem 5: For $|t| \leq 1$, the result follows from Theorem 1. From now on we assume that $t > 1$. Proof for $t < -1$ is similar. Let $X'_i = X_i I(|X_i| \leq t n^{\lambda})$ and $S' = \sum_{i=1}^n X'_i$. Then

$$|P(S_n \leq t\sigma_n) - P(S'_n \leq t\sigma_n)| \leq \sum_{i=1}^n P(|X_i| > t n^{\lambda}) \ll b(n, t).$$

Next we use the blocking procedure described in (2.1) with X_i replaced by X'_i ; $0 < \beta < \alpha < 1$ will be chosen later.

Let

$$U_n = \sum_{i=1}^k \xi_i, \quad U'_n = \sum_{i=1}^k \eta_i,$$

$$T_n = \xi_{k+1} \text{ and } t_n = t(1 \pm 2n^{-\lambda}),$$

where $\lambda > 0$ will be chosen later. Clearly $S'_n = U_n + U'_n + T_n$. We complete the proof by showing

$$|\Phi(t_n) - \Phi(t)| \ll n^{-2} e^{-t^2} \quad \dots (4.2)$$

$$P(|T_n| > t n^{-\lambda} \sigma_n) \ll b(n, t) \quad \dots (4.3)$$

$$\text{and } P(|U'_n| > t n^{-\lambda} \sigma_n) \ll b(n, t) \quad \dots (4.4)$$

$$|P(U_n > t_n \sigma_n) - \Phi(-t_n)| \ll b(n, t) + n^{-2} e^{-t^2}. \quad \dots (4.5)$$

Proof of (4.2) is trivial. By Markov's inequality and by Lemma 1, the left-hand side of (4.3), for any $v \geq 2+c$, is not more than

$$\begin{aligned} (t n^{-\lambda} \sigma_n)^{-v} E |T_n|^v &\leq (p^{v/2} + p (t n^\lambda)^{v-2-c}) t^{-v} n^{-v(\lambda-1)} \\ &\leq (k^{-v/2} t^{-v} + k^{-1} b(n, t)) n^{\lambda v} \end{aligned}$$

(4.3) now follows if we choose $v = (c+2)/(1-\alpha)$ and $0 < \lambda < (1-\alpha)^2/(c+2)$.

To prove (4.4) we use Markov's inequality and Lemma 1 to get,

$$P(|U_n^*| > t n^{-\lambda} \sigma_n) \leq (t n^{-\lambda} \sigma_n)^{-v} [(kq)^{v/2} + (kq)(n^\lambda t)^{v-2-c}] \leq b(n, t),$$

where we chose $v = (c+2)/(\alpha-\beta)$ and $0 < \lambda < (\alpha-\beta)^2/(c+2)$.

Finally, to prove (4.5), let $\xi'_i = p^{-1}\xi_i$ and $\xi_{i0} = \xi'_i I(|\xi'_i| \leq s t_n k^\lambda)$, where $s(>0)$ will be chosen later. Another application of Lemma 1 gives

$$\begin{aligned} &\left| P\left(\sum_{i=1}^k \xi'_i > t_n p^{-1} \sigma_n\right) - P\left(\sum_{i=1}^k \xi_{i0} > t_n p^{-1} \sigma_n\right) \right| \\ &\leq \sum_{i=1}^k P(|\xi'_i| > s k^\lambda t_n) \leq b(n, t). \quad \dots (4.6) \end{aligned}$$

Next write

$$\begin{aligned} c_n &= (\sigma_n^2/kp)^\lambda, \quad b_k = n\sigma_n^{-1} t_n k^{-1} p^{-\lambda}, \\ f_i &= E(\exp(b_k \xi_{i0})), \quad g_k = E\left(\exp\left(b_k \sum_{i=1}^k \xi_{i0}\right)\right), \\ m_i &= f_i^{-1} E(\xi_{i0} \exp(b_k \xi_{i0})), \quad \bar{m} = k^{-1} \sum_1^k m_i, \end{aligned}$$

and

$$\bar{\sigma}^2 = k^{-1} \sum_{i=1}^k [f_i^{-1} E(\xi_{i0}^2 \exp(b_k \xi_{i0})) - m_i^2].$$

Then after some routine steps one gets

$$P\left(\sum_{i=1}^k \xi_{i0} > k^\lambda t_n c_n\right) = A_k \int_{B_k} \exp(-D_k x) dH_k(x), \quad \dots (4.7)$$

where

$$\begin{aligned} A_k &= g_k \exp(-b_k k \bar{m}), \quad B_k = (b_k n^{-1} \sigma_n^2 - m) k^\lambda \sqrt{\bar{\sigma}}, \\ D_k &= b_k k^\lambda \bar{\sigma}, \quad dG_k(x) = g_k^{-1} \exp(b_k x) dP\left(\sum_{i=1}^k \xi_{i0} \leq x\right) \end{aligned}$$

and

$$H_k(x) = G_k(\sigma k^t x + k \bar{m}).$$

We have by Lemma 1,

$$E(\xi_{t_0}^2) \leq K, E|\xi_{t_0}|^3 \leq K \text{ for } c \geq 1$$

and

$$E|\xi_{t_0}|^2 \leq K + n^{-\alpha + (1-c)/2} t^{1-c} \leq K$$

by choosing $\alpha > (1-c)/2$ for $0 < c < 1$. Also

$$|E(\xi_{t_0})| \leq K k^{-1-\gamma} \text{ for some } \gamma > 0.$$

Further, following the lines of proof of Lemma 6 of Ghosh and Babu (1977), one gets

$$\left| k^{-1} \sum_{i=1}^k E \xi_{i_0}^2 - n^{-1} \sigma_n^2 \right| \leq K n^{-\gamma} \text{ for some } \gamma > 0. \quad \dots (4.8)$$

Using these, one gets,

$$\sum_{i=1}^k \log f_i = \frac{1}{2} t_n^2 + O(n^{-\gamma}), \quad \dots (4.9)$$

$$\bar{m} = \sigma_n t_n k^{-1} p^{-1} + O(k^{-1-\gamma}), \quad \dots (4.10)$$

and

$$\bar{\sigma} = n^{-1} \sigma_n + O(n^{-\gamma}) \text{ for some } \gamma > 0. \quad \dots (4.11)$$

Then, $B_k = O(n^{-\gamma})$ and $D_k = t_n(1 + O(n^{-\gamma}))$ for some $\gamma > 0$. Also from Lemma 2, we obtain

$$\left| g_k - \prod_{i=1}^k f_i \right| \leq K k \phi_0 \prod_{i=1}^k (t_i + 2\phi_0 \exp(\theta n \sigma_n^{-1} t_n^2 (pk)^{-1})). \quad \dots (4.12)$$

As in (4.9), it follows, that the product on the right side above $\ll e^{\frac{1}{2} t_n^2}$ by choosing $1 - \alpha - 2\beta < 0$. Consequently the left-hand side of (4.12) $\ll n^{-\gamma} e^{\frac{1}{2} t_n^2}$ for some $\gamma > 0$. Hence $A_k = (1 + O(n^{-\gamma})) \exp -\frac{1}{2} t_n^2$. We shall now show that

$$\|H_k - \Phi\| \ll n^{-\gamma} \text{ for some } \gamma > 0.$$

Note that for any real number w

$$\begin{aligned} & \left| g_n^{-1} E \left(\exp \left((b_n + iw) \sum_{j=1}^k \xi_{j0} \right) \right) - \prod_{j=1}^k f_j^{-1} E \left(\exp \left((b_n + iw) \xi_{j0} \right) \right) \right| \\ & \leq \prod_{j=1}^k f_j^{-1} \left(\left| E \left(\exp \left((b_n + iw) \sum_{j=1}^k \xi_{j0} \right) \right) - \prod_{j=1}^k E \left(\exp \left((b_n + iw) \xi_{j0} \right) \right) \right| \right. \\ & \left. + \left| g_k - \prod_{j=1}^k f_j \right| \right) \ll n^{-\gamma} \end{aligned} \quad \dots (4.13)$$

for some $\gamma > 0$. The last step above follows from Lemmas 2 and A, as

$$\left| E \left(\exp \left((b_n + iw) \sum_{j=1}^k \xi_{j0} \right) \right) \right| \leq g_k.$$

Now arguing similarly as in Ghosh and Babu (1977), we obtain that

$$\|H_k - \Phi\| \ll n^{-\gamma} \text{ for some } \gamma > 0.$$

Finally, we write

$$\left| A_k \int_{B_k} \exp(-D_k x) dH_k(x) - \Phi(-t_n) \right| \leq I_1 + I_2 + I_3,$$

where

$$I_1 = A_k \left| \int_{B_k} \exp(-D_k x) d(H_k(x) - \Phi(x)) \right| \ll n^{-\gamma} \exp -\frac{1}{2} t_n^2,$$

$$I_2 = |A_k - \exp -\frac{1}{2} t_n^2| \int_{B_k} \exp(-D_k x) d\Phi(x) \ll n^{-\gamma} \exp -\frac{1}{2} t_n^2,$$

and

$$\begin{aligned} I_3 &= \left| \exp -\frac{1}{2} t_n^2 \int_{B_k} \exp(-D_k x) d\Phi(x) - \Phi(-t_n) \right| \\ &= \left| \left(\exp \left(-\frac{1}{2} t_n^2 + \frac{1}{2} D_k^2 \right) \right) \Phi(-B_k - D_k) - \Phi(-t_n) \right| \\ &\ll n^{-\gamma} \exp(-\frac{1}{2} t_n^2) \text{ for some } \gamma > 0. \end{aligned}$$

The proof of the theorem is complete.

Proof of Remark 4.1: Let $\{X_n\}$ be a stationary sequence. Let $t \geq 1$ and $\bar{X}_t = \bar{X}_t I(|X_t| \leq t a_n n^b)$, where $a_n \rightarrow 0$, $a_n = \max(n^{-1/(2+2c)}, h_n^{1/(4+2c)})$ and $h_n = E(|X_1|^{2+c} I(|X_1| > n^{c/(2+2c)}))$. It is easily seen that

$$nP(|X_1| > t a_n n^b) = o(b(n, t))$$

and also

$$|E(\xi_{t0})| = o(k^{-1}t^{-1}).$$

The rest of the proof is similar to that of Theorem 5.

Theorem 6: Let $\{X_n\}$ be as in Theorem 1. If $x_n \rightarrow \infty$ such that $x_n^2 - c \log n - (c+1) \log \log n \rightarrow -\infty$ as $n \rightarrow \infty$, then

$$P(S_n > \sigma_n x_n) \sim \Phi(-x_n) \quad \dots (4.14)$$

as $n \rightarrow \infty$.

Theorem 7: Let $\{X_n\}$ satisfy the hypothesis of Remark 4.1. If $x_n \rightarrow \infty$ such that $x_n^2 - c \log n - (c+1) \log \log n$ is bounded from above. Then (4.14) holds as $n \rightarrow \infty$.

Theorem 6 follows from Theorem 5 and Theorem 7 follows from Remark 4.1. In particular Theorem 7 proves Theorem 1 of Ghosh and Babu (1977) under weaker moment condition.

Appendix

Proof of Lemma 1: First observe that

$$|E(Y_t)| = |E(X_t - Y_t)| \leq d^{1-\delta} E|X_t|^\delta \leq Nd^{1-\delta}. \quad \dots (1)$$

and

$$E(Y_t^2) \leq E(X_t^2) \leq 1 + N \leq 2N. \quad \dots (2)$$

The lemma is proved by induction. Define

$$c(u, v, h) = E \left| \sum_{i=1}^u Y_{t+h} \right|^v \text{ and } c(u, v) = \sup_{h \geq 0} c(u, v, h). \quad \dots (3)$$

Then

$$c(u, 2, h) = \sum_{i=1}^u E(Y_{t+h}^2) + 2 \sum_{j=1}^{u-1} \sum_{i=1}^{u-j} E(Y_{t+h} Y_{t+h+j}). \quad \dots (4)$$

Using a lemma of Ibragimov (see Lemma 1, p. 170 of Billingsley, 1968) with $r = s = 2$, from (1) and (2) we obtain, for any i and j , that

$$\begin{aligned} |E(Y_t Y_{t+j})| &\leq 2 [\phi_j E(Y_t^2) E(Y_{t+j}^2)]^{1/2} + 4N^2 d^{2-2\delta} \\ &\leq 4(N\phi_j)^{1/2} + N^2 d^{2-2\delta}. \end{aligned} \quad \dots (5)$$

So from (1.2), (2), (4) and (5), it follows that $c(u, 2) \ll u$, for all $u \leq d^2$. To prove the lemma by induction, assume next that for $1 \leq u \leq d^2$ and for some integer $m \geq 2$,

$$c(u, m) \leq D(m)(u^{m/2} + u R(m)), \quad \dots (6)$$

and prove a similar inequality for $c(u, m + \epsilon)$, $0 < \epsilon \leq 1$. Fix an integer $h \geq 0$. Define $Z_u = \sum_{i=1}^u Y_{i+h}$, $Z_{u,t} = Z_{2u+t} - Z_{u,t}$ and $S_{u,t} = \sum_{i=1}^t Y_{u+h+i}$. Then,

$$\begin{aligned} E|Z_u + Z_{u,t}|^{m+\epsilon} &\leq E\{(|Z_u| + |Z_{u,t}|)^m (|Z_u|^\epsilon + |Z_{u,t}|^\epsilon)\} \\ &\leq \sum_{j=0}^m \binom{m}{j} E[|Z_u|^{m-j} |Z_{u,t}|^j |Z_u|^\epsilon + |Z_{u,t}|^\epsilon] \end{aligned} \quad \dots (7)$$

Using Lemma 1, p. 170 of Billingsley, 1968, again we obtain that

$$\begin{aligned} |E(|Z_u|^{m-j+\epsilon} |Z_{u,t}|^j - E|Z_u|^{m-j+\epsilon} E|Z_{u,t}|^j)| \\ \leq 2\phi_t^{j/(m+1)} c(u, m+\epsilon). \end{aligned} \quad \dots (8)$$

It follows now from (1.2), (7), (8) and Hölder's inequality, that

$$E|Z_u + Z_{u,t}|^{m+\epsilon} \leq 2(1 + K t^{-\epsilon/2m}) c(u, m+\epsilon) + K c^{(m+\epsilon)/m}(u, m). \quad \dots (9)$$

Since $u \leq d^2$, we have

$$(u R(m))^{(m+\epsilon)/m} = (u d^{m-\delta+\epsilon}) (u d^{-\delta})^{\epsilon/m} \leq R(m+\epsilon) u^\gamma \quad \dots (10)$$

for some $\gamma = \gamma(m+\epsilon, \delta) \in (0, 1)$. Using Minkowski's inequality, (9) and (10) we obtain that

$$\begin{aligned} c(2u, m+\epsilon, h) &= E|Z_u + Z_{u,t} + S_{u,t} - S_{2u,t}|^{m+\epsilon} \\ &\leq 2 \{[(1 + K t^{-\epsilon/2m}) c(u, m+\epsilon) + K(u^{(m+\epsilon)/2} + u^\gamma H(m+\epsilon))] \frac{1}{m+\epsilon} \\ &\quad + 2t H^{1/(m+\epsilon)}(m+\epsilon)\}^{m+\epsilon}, \end{aligned} \quad \dots (11)$$

where

$$\sup_{h \geq 0} E|Y_h|^\nu \leq H(\nu) = \begin{cases} 2N & \text{if } \nu \leq \delta \\ NR(\nu) & \text{if } \nu > \delta. \end{cases} \quad \dots (12)$$

Taking $2u = 2^\nu \leq d^2$ and $t = [2^{\nu/2m}]$ in (11) we obtain that

$$c(2^\nu, m+\epsilon) \leq 2(1 + Kb^\epsilon) c(2^{\nu-1}, m+\epsilon) + K(2^{\nu(m+\epsilon)/2} + 2^{\nu\gamma} H(m+\epsilon)), \quad \dots (13)$$

where $[x]$ denotes the largest integer $\leq x$ and $b = 2^{-s}/\epsilon m^2$. Note that $0 < b < 1$. Repeating (13) s times and observing that $\prod_{j=1}^{\infty} (1 + Kb^j) < \infty$, we obtain

$$c(2^s, m + \epsilon) \leq K \left(2^s c(1, m + \epsilon) + 2^{s(m+s)/2} \left(\sum_{j=0}^s 2^{-(m+s)j/2} \right) \right. \\ \left. + H(m + \epsilon) 2^{s^2} \left(\sum_{j=0}^s 2^{j(1-\gamma)} \right) \right) \leq K(2^s R(m + \epsilon) + 2^{s(m+s)/2}).$$

The last step follows from (12).

To complete the proof of Lemma 1, one uses a binary decomposition of u for any positive integer u ($1 \leq u \leq d^2$), and obtains inequalities similar to (2.22) and (2.23) of Ghosh and Babu (1977). The details are omitted.

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