Spin system, gauge theory, and renormalization group equation

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The gauge theoretical formulation of a spin system is studied here. It is pointed out that a quantum spin can be associated with a gauge bundle where the gauge group is SU(2). The Ising spin is obtained by breaking the group to U(1) coupled with a Z_2 symmetry giving rise to a Z_2 gauge theory in three dimension. The gauge structure of a spin system helps us to reformulate the block-variable transformation in terms of a gauge transformation.

I. INTRODUCTION

It has been argued by Jona-Lasinio[1] that Gell-Mann and Low type renormalization group equations which are essentially the outcome of the quantum field theory are equivalent in formalism with the Kadanoff–Wilson type renormalization group equations which essentially deal with block variables in Ising system. The main point of argument in showing this equivalence is that renormalization group equations in both these formalisms may be linked up with the limit theorems in probability theory. Indeed Bleher and Sinai[2] first emphasized in a rigorous mathematical way the link between the probabilistic ideas and renormalization group equations. Later on, Gallavotti, Knops, Cassandro, and Martin-Lof[3] have obtained some results of a more generalized character.

In this context, it may be added that Fisher[4] has put some objections on the equivalence of Gell-Mann–Low type field theoretical renormalization group equations and Kadanoff–Wilson approach involving block variables suggesting that these two approaches are "not as close as the use of the same word might suggest." Indeed, the main point of discrepancy lies in the fact that while the Kadanoff–Wilson renormalization group equation is connected with the distribution function for block variables and renormalization transformation expresses the distribution for larger blocks in terms of the distribution of smaller blocks, the Gell-Mann–Low type renormalization group equations in quantum field theory involve transformation which does not change the form of the distribution. In view of this, Jona-Lasinio has pointed out that one may characterize Kadanoff–Wilson formalism as a global approach and Gell-Mann–Low formalism as a local approach as it deals with the original variables and no block variable is introduced.

In this paper, we shall deal with the problem of equivalence of these two formalisms of renormalization group equations from the point of view of a gauge theoretical formulation of a spin system. The close analogy between two-dimensional spin systems and four-dimensional gauge theory has been extensively studied by many authors.[5] The analysis of renormalization group equations is shown to have same structures in both the systems. The striking similarity has also been observed during the phase transition phenomena which are associated with the condensation of topological objects. In the case of a two-dimensional spin system these are kinks whereas in four-dimensional gauge theory these objects are magnetic monopoles.[6] Here we shall point out that a quantum spin is associated with a Hermitian line bundle α over the configuration space when the classical spin is associated with an arbitrary vector. The quantum Heisenberg spin is obtained by lifting the Hamiltonian on sections of a gauge bundle where the gauge group is SU(2). The Ising spin is obtained when this group is broken to U(1) corresponding to the Z component of the spin with the property of Z_2 invariance. This in three dimension corresponds to Z_2 gauge theory. This gauge structure of a spin system helps us to reformulate the block variable transformation in terms of gauge transformations and in this way becomes identified with the renormalization group equations of quantum field theory in a direct way.

II. QUANTUM SPIN AND GAUGE BUNDLE

Many authors have studied the classical limit theorems for the partition function of quantum spin systems.[7] It has been pointed out that for a quantum system corresponding to the angular momentum **J** where the spin operator S satisfies the eigenvalue equation

$$S^2\psi=[S_x^2+S_y^2+S_z^2]\psi=J(J+1)\psi, \tag{1}$$

the $J\to\infty$ limit corresponds to the classical spin. Indeed, the classical spin can be associated with an arbitrary vector defined in the configuration space. Here we shall show that the quantization of a classical spin is associated with a Hermitian line bundle over the configuration space.

A spin system can be described by a "direction vector" ξ_μ attached to a space–time point x_μ when the two opposite orientations of the "direction vector" ξ_μ give rise to opposite helicities corresponding to spin up and down states. Now to have a direct correspondence of the "direction vector" ξ_μ with the helicity states, we describe the spin system over a configuration space by the coordinate $Z^\mu=x^\mu+i\xi^\mu=x^\mu+(i/2)\lambda_\alpha^\mu\theta^\alpha(\alpha=1,2)$, where we associate the "direction vector" ξ^μ with a two-component spinorial variable θ. Indeed, replacing the chiral coordinate by the matrices, we get

$$Z^{AA'}=x^{AA'}+\frac{i}{2}\lambda_\alpha^{AA'}\theta^\alpha, \tag{2}$$

where

$$x^{AA'}=\begin{bmatrix} x^0-x^1 & x^2+ix^3 \\ x^2-ix^3 & x^0+x^1\end{bmatrix}$$

and

$$\lambda_\alpha^{AA'}\in SL(2,C).$$

With these relations we can define a helicity with help of the twistor equation[8]

$$\bar Z_\alpha Z^\alpha+\lambda_\alpha^{AA'}\theta^\alpha\bar\pi_A\pi_{A'}=0, \tag{3}$$

where $\bar\pi_A$ $(\pi_{A'})$ is the spinorial variable corresponding to the four-momentum variable p_μ, the conjugate of x_μ and is given by the matrix representation

$$p^{AA'}=\bar\pi^A\pi^{A'} \tag{4}$$

and

$$Z^\alpha=(\omega^A,\pi_{A'}),\ \bar Z_\alpha=(\bar\pi_A,\bar\omega^{A'})$$

with

$$\omega^A = i\left[x^{AA'} + \frac{i}{2}\lambda_\alpha^{AA'}\theta^\alpha\right]\pi_{A'} .$$

Equation (3) now involves the helicity operator

$$S = -\lambda_\alpha^{AA'}\theta^\alpha\bar{\pi}_A\pi_{A'} \tag{5}$$

representing a spin up or down state. It is observed that the complex conjugate of the chiral coordinate (2) will give rise to the opposite helicity state.

Now we can define the upper half plane D^- where the coordinate $Z_\mu = x_\mu + i\xi_\mu$ is such that ξ_μ belongs to the interior of the forward light cone $\xi > 0$ with the condition det $\xi^{AA'} > 0$ and $\frac{1}{2}$ Tr $\xi^{AA'} > 0$. The lower half plane D^+ is given by the set of all coordinates Z_μ with ξ_μ in the interior of the backward light cone $\xi < 0$. The map $Z \to Z^*$ sends the upper half plane to the lower half plane. The space of M of null plane [$\det\xi^{AA'} = 0$] is the Shilov boundary so that a function holomorphic in D^+ (D^-) is determined by its boundary values. Thus if we consider that any function $\phi(z) = \phi(x) + i\phi(\xi)$ is holomorphic in the whole domain, the helicity $+\frac{1}{2}$ ($-\frac{1}{2}$) in the null plane may be taken to be the limiting value of the helicity in the upper (lower) half plane.

In the complexified space–time exhibiting the helicity states, we can now write the metric as $g_{\mu\nu}(x,\theta,\bar{\theta})$. It has been shown elsewhere[9] that this metric structure gives rise to the SL(2,C) gauge theory where the field strength tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + [B_\mu, B_\nu] \tag{6}$$

with $B_\mu \in$ SL(2,C). In view of this, we can have a gauge theoretical formulation of the spin system when we write the spin variable attached to a site X_μ by a coordinate formulation

$$Q_\mu = x_\mu + iB_\mu,\quad B_\mu \in \mathrm{SL}(2,C). \tag{7}$$

Demanding Hermiticity of the line bundle, we may restrict B_μ to the unitary group SU(2). This indicates that a Heisenberg quantum spin may be associated with the gauge bundle where the group structure is SU(2).

It is noted that the introduction of the "direction vector" ξ_μ at a space–time point x_μ suggests that the behavior of the angular momentum in such a coordinate system will be similar to that of a charged particle moving in the field of a magnetic monopole. Indeed, the wave function $\phi(z)$ should take into account the polar coordinates r,θ,ϕ along with the angle χ specifying the rotational orientation around the "direction vector" ξ_μ, the eigenvalue of the operator $i\partial/\partial\chi$ just corresponds to the helicity state. In an anisotropic space, the components of the linear momentum satisfies a commutation relation of the form

$$[p_i, p_j] = i\mu\epsilon_{ijk}\frac{x^k}{r^3}. \tag{8}$$

In such a space, the motion of a particle is equivalent to the motion of a charged particle in the field of a magnetic monopole. In this space, the angular momentum is given by

$$\mathbf{J} = \mathbf{r}\times\mathbf{p} - \mu\mathbf{r}. \tag{9}$$

The fact that in such an anisotropic space the angular momentum can take the value $\frac{1}{2}$ is found to be analogous to the result that a monopole charged particle composite representing a dyon satisfying the condition $gq = \frac{1}{2}$ have their angular momentum shifted by $\frac{1}{2}$ unit and their statistics shift accordingly.[10] Evidently a fermion can be described by a scalar particle moving with $l = \frac{1}{2}$

in an anisotropic space. Hence the l_z value denoting the spin orientation corresponds to the gauge bundle having the group structure U(1). In view of this, an Ising system can be represented by a collection of fermions and the gauge bundle having the group structure U(1) together with the reflection invariance property (Z_2 invariance) suggests that this can be represented by a Z_2 gauge theory in three dimension.

Now to study the spin–spin interaction in terms of the associated gauge fields, we note that the simplest Lagrangian density which is invariant under SL(2,C) transformation in spinor affine space is given by

$$L=-\tfrac{1}{4}\,\mathrm{Tr}\,\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}. \tag{10}$$

Following Carmeli and Malin,[11] if we apply the usual procedure of variational calculus, we get the field equations

$$\partial_\delta[\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}]-[B_\delta,\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}]=0. \tag{11}$$

Taking the infinitesimal generators of the group SL (2,C) in the tangent space as

$$g^1=\begin{bmatrix}0&0\\1&0\end{bmatrix},\quad g^2=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad g^3=\begin{bmatrix}0&1\\0&0\end{bmatrix} \tag{12}$$

we can write

$$\left.\begin{aligned}B_\mu&=b^a_\mu g^a=\mathbf{b}_\mu\cdot\mathbf{g}\\ F_{\mu\nu}&=f^a_{\mu\nu}g^a=\mathbf{f}_{\mu\nu}\cdot\mathbf{g}\end{aligned}\right\}\quad [a=1,2,3]. \tag{13}$$

From this we can now construct a conserved current corresponding to the Lagrangian[11]

$$\mathbf{J}^\mu_{(\theta)}=\epsilon^{\mu\nu\alpha\beta}\mathbf{b}_\nu\times\mathbf{f}_{\alpha\beta}. \tag{14}$$

Indeed, we find from Eq. (11) that

$$\epsilon^{\mu\nu\alpha\beta}[\partial_\nu\mathbf{f}_{\alpha\beta}-\mathbf{b}_\nu\times\mathbf{f}_{\alpha\beta}]=0. \tag{15}$$

This suggests that

$$\mathbf{J}^\mu_{(\theta)}=\epsilon^{\mu\nu\alpha\beta}\mathbf{b}_\nu\times\mathbf{f}_{\alpha\beta}=\epsilon^{\mu\nu\alpha\beta}\partial_\nu\mathbf{f}_{\alpha\beta}. \tag{16}$$

This gives

$$\partial_\mu\mathbf{J}^\mu_{(\theta)}=\epsilon^{\mu\nu\alpha\beta}\partial_\mu\partial_\nu\mathbf{f}_{\alpha\beta}=0. \tag{17}$$

From this we note that when the three-dimensional configuration space is represented as a constant time surface of a 3+1-dimensional configuration space, the Heisenberg spin–spin interaction can be represented in the zero lattice spacing limit as $\mathbf{J}^\mu_{(\theta)}\cdot\mathbf{J}^\mu_{(\theta)}$ interaction where the corresponding group structure of the gauge field B_μ is SU(2). It is noted that in this case, the three components of the current

$$J_{\mu(\theta)}=\begin{bmatrix} J^{(1)}_{\mu(\theta)} \\ J^{(2)}_{\mu(\theta)} \\ J^{(3)}_{\mu(\theta)} \end{bmatrix} \text{ represent an SU(2) triplet.}$$

However, for the Ising interaction, we can take the interaction $J^{\mu(i)}_{(\theta)}J^{\mu(i)}_{(\theta)}$ (no summation) in the continuum limit where the index i represents any one component of the $\mathbf{J}^{\mu}_{(\theta)}$. To be specific, we can take the second component $J^{\mu(2)}_{(\theta)}$ for this purpose. Now from the relation $J^{\mu(2)}_{(\theta)} = \epsilon^{\mu\nu\alpha\beta}\partial_\nu f^{(2)}_{\alpha\beta}$ we can write

$$J^{\mu(2)}_{(\theta)}=\epsilon^{\mu\nu\alpha\beta}\epsilon_{\nu\alpha\beta}C(x)=L^{\mu}C(x), \tag{18}$$

where L^{μ} behaves like a constant axial vector. Now from the relation $\partial_\mu J^{\mu(2)}=0$, we find that $C(X)$ is a constant. Thus from the field current relation

$$\begin{bmatrix} J^{(2)}_{\mu(\theta)} \\ J^{2}_{\mu(\bar\theta)} \end{bmatrix}=\Box\begin{bmatrix} A_{\mu+} \\ A_{\mu-} \end{bmatrix}, \tag{19}$$

we find

$$\Box\begin{bmatrix} A_{\mu+} \\ A_{\mu-} \end{bmatrix}=c\begin{bmatrix} +1 \\ -1 \end{bmatrix}L_{\mu}, \tag{20}$$

where we have utilized the relation $-J^{(2)}_{\mu(\theta)}=J^{(2)}_{\mu(\bar\theta)}$. Taking c as a normalization factor, the above relation is reduced to

$$\Box\begin{bmatrix} A_{\mu+} \\ A_{\mu-} \end{bmatrix}=\begin{bmatrix} +1 \\ -1 \end{bmatrix}L_{\mu}=\begin{bmatrix} L_{\mu+} \\ L_{\mu-} \end{bmatrix}. \tag{21}$$

Thus we observe that in the Z_2 symmetric case, the gauge field $A_\mu=[A_{\mu+},A_{\mu-}]$ appears as a disconnected one. The reflection invariance property suggests that $\begin{bmatrix} A_{\mu+} \\ A_{\mu-} \end{bmatrix}$ gives us the Z_2 gauge theory in three dimension when the analysis is carried out in 3+1 dimension at constant time. The constant nature of L_μ with the specific property $L_{\mu-}=-L_{\mu+}$ defines a gauge field

$$L_{\mu\pm}=\exp[i\tau_\pm] \quad \text{with} \quad \begin{bmatrix} \tau_+ \\ \tau_- \end{bmatrix}=\begin{bmatrix} 0 \\ \pi \end{bmatrix}.$$

Thus a vortex ($\tau_+=0$) or an antivortex ($\tau_-=\pi$) can be associated with a gauge field.

It is to be noted that the topological term in the Lagrangian (10) can be written as $\partial_\mu\Omega^\mu$, where Ω^μ is the Chern–Simons characteristic class. This is defined as

$$\Omega^\mu=-\frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}[\tfrac{1}{2}B_\nu F_{\alpha\beta}-\tfrac{2}{3}(B_\nu B_\alpha B_\beta)], \tag{22}$$

where the Pontryagin density is given by

$$P=-\frac{1}{16\pi^2}\,\mathrm{Tr}\,\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \tag{23}$$

The charge corresponding to the gauge field part is the Pontryagin index q where[12]

$$q=\int P\,d^4x=\int \partial_\mu\Omega^\mu\,d^4x. \tag{24}$$

From this we see that q is actually associated with the magnetic pole strength for the corresponding field distribution.

In 2+1 dimension, the corresponding Chern–Simons (CS) Lagrangian is given by

$$L_{cs}=\frac{\mu}{2}\int d^2x\,\epsilon^{\rho\sigma\tau}A_\rho\,\partial_\sigma A_\tau \tag{25}$$

where A_ρ is a gauge field. The corresponding topological invariant, known as the Hopf invariant which exhibits the Hopf map $S^3\to S^2$ $[\pi_3(s^2)=Z]$ is given by

$$H=-\frac{1}{4\pi}\int d^3x\,\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}. \tag{26}$$

If ρ denotes a four-dimensional index, then we find that

$$\partial_\rho\epsilon^{\rho\mu\nu\lambda}A_\mu F_{\nu\lambda}=\tfrac{1}{2}\epsilon^{\rho\mu\nu\lambda}F_{\rho\mu}F_{\nu\lambda} \tag{27}$$

which connects the Hopf invariant with the chiral anomaly. A three (two)-dimensional spin system may be visualized by projecting the Euclidean Pontryagin (Chern–Simons) theory on a three (two)-dimensional manifold. In view of this we note that the kinks in spin system may be transcribed in the language of magnetic monopoles when the gauge theoretic version of a spin system is taken into account.

III. QUANTUM SPIN, GAUGE BUNDLE, AND KADANOFF–WILSON FORMALISM

We have shown above that a quantum spin system can be represented by a gauge field where for the Heisenberg spin the gauge group is SU(2) and for the Ising system this splits into U(1) coupled with a Z_2 symmetry leading to a Z_2 gauge theory in three dimension. We shall here point out that this helps us to realize the Block variable transformation as a gauge transformation.

From our above analysis, we depict a spin system by a configuration variable where each coordinate point is given by

$$Z_\mu=x_\mu+iB_\mu, \tag{28}$$

B_μ being the corresponding gauge field. In view of this, the block spin variable may be constructed by considering

$$\Phi_n(k)=\sum_{i\in k_n}\sigma_i=\sum_{i\in k_n}B_{\mu_i}. \tag{29}$$

However, since B_μ is a gauge field, its value is arbitrary and thus by allowing a suitable gauge transformation, we can again replace $\Sigma_i B_{\mu_i}$ by B_μ. This implies that the magnitude of the direction vector attached to the site i is not of any consequence rather its orientation has its real significance. In the spin variable term, this means that the net orientation of the total block of spin variables is of main concern and not the magnitude of the block variable $\Sigma_i \sigma_i$. This is the main philosopy of Kadanoff–Wilson formalism.

Let us consider an infinite system of spin variables and construct a sequence of hosted lattices having spacing "a." Now we define a block variable

$$\Phi_n(k) = \sum_{i \in k_n} \sigma_i, \tag{30}$$

where the sum runs over all the spin σ_i in the box k_n which implies the recursion relation

$$\Phi_0(k) = \sigma_k,$$

$$\Phi_{n+1}(k) = \sum \Phi_n(i_n). \tag{31}$$

The probability distribution $\exp[-\beta H(\sigma)]$ of the σ variables induces naturally a probability distribution on the Φ variables when L_n, the size of the box, is larger than the coherence length ξ, the correlations of block variables belonging to different blocks will be essentially zero which implies that for large n, the probability distribution of $\Phi_n(k)$ is factorized and each Φ is the sum of practically uncorrelated variables. At the critical point $T = T_c$, ξ is infinite and hence the probability distribution of $\Phi_n(k)$ for large n should be well defined. Generally, the rescaled variables $\Psi = L_n^{-D/2} \Phi_n$, are introduced so that the new variables have finite variance when $n \to \infty$ at $T \neq T_c$. The crucial hypothesis in the renormalization group approach is connected with calculating the distribution function for block variables and the corresponding renormalization transformation is the transformation which expresses the distribution for larger block in terms of the distribution of smaller blocks.

In the gauge theoretical formulation of a spin system, let us define the site i having spin σ_i by the coordinate

$$Z_{\mu_i} = x_{\mu_i} + i B_{\mu_i}. \tag{32}$$

Now for a block of n spins, we may define a single spin having a new site j having the coordinate

$$Z_{\mu_j} = \sum_i x_{\mu_i} \Big/ n + i B_{\mu_j} = x_{\mu_j} + i B_{\mu_j} \tag{33}$$

which indicates that $\Sigma_i x_{\mu_i}/n$ is the c.m. coordinate and due to gauge freedom, the spin effect is again realized by the original gauge field B_μ. Now as discussed in Sec. II, in the Ising model the spin–spin interaction can be mapped into the current–current coupling in the continuum limit where the current $J^{\mu^{(2)}}$ is defined by

$$J^{\mu^{(2)}}_{(\theta)} = \epsilon^{\mu\nu\alpha\beta} \partial_\nu f^{(2)}_{\alpha\beta}.$$

Due to the gauge freedom, it is independent of the magnitude of the gauge field B_μ and hence for a block spin variable we note that the effective current

$$\sum_i J_i^{\mu^{(2)}} = \sum_i [\epsilon^{\mu\nu\alpha\beta} \partial_\nu f^{(2)}_{\alpha\beta}]_i \tag{34}$$

can be reduced to the original current $J_i^{\mu^{(2)}}$ through some suitable gauge transformation. Thus the interaction of two such block spins can be reduced to the interaction of the original spin variables. This indicates that the probability distribution of the block variables should be the same as the probability distribution of the original spin variables and hence the transformation $L' = SL$ preserves the partition function of the system apart from a multiplicative factor

$$Z_{S,L} = C_{S,L} Z_L . \tag{35}$$

This suggests a renormalization group transformation of the Hamiltonian

$$H_{SL} = R_S(H_L) \tag{36}$$

which is realized through the change in the coupling constant when the original spin variables are retained.

IV. DISCUSSION

We have shown above that a quantum spin can be associated with a gauge bundle and the block variable transformation can be associated with a gauge transformation in the gauge space. This helps us to relate the Kadanoff–Wilson formulation with a gauge transformation.

This gauge theoretical formulation of a spin system is associated with the Chern–Simons topology which has an important bearing. Indeed, from our analysis we may note that when the gauge field current J_μ is taken to represent a spin at a certain site in a lattice, the associated gauge field A_μ may be taken to lie on the bond and the corresponding vertex may now be related to the Jones polynomial of knot theory in Euclidean three-dimension and its generalization to quantum groups. To be more specific, since the Jones polynomial and its generalization can be identified with the expectation value of Wilson lines in three dimensional Chern–Simons gauge theories, the evaluation of these expectation values may be related to the two-dimensional integrable statistical mechanics when we project the three-dimensional system to the two-dimensional plane. This helps us to link up the spin systems with topological field theory in a direct way.

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