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SMALL SAMPLE COMPARISONS FOR THE BLENDED WEIGHT CHI–SQUARE GOODNESS–OF–FIT TEST STATISTICS

Dong Wan Shin, Ayanendranath Basu and Sahadeb Sarkar

Department of Statistics, Ewha Womans University, Seoul, Republic of Korea
Department of Mathematics, University of Texas at Austin, Austin, TX, USA
Department of Statistics, Oklahoma State University, Stillwater, OK, USA

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ABSTRACT

The small sample properties of the family of blended weight chi–square (BWCS) goodness-of-fit tests are investigated. Like the power divergence family, this family is a very rich subclass of a more general class of goodness–of–fit tests called the disparity tests (Basu and Sarkar 1994a). Use of the standard asymptotic chi–square distribution in small samples can give quite inaccurate critical regions for most members of the BWCS family. We derive three other asymptotic approximations of the exact distributions in order to obtain more accurate significance levels for the BWCS tests. Two of these approximations are computationally simple to use in practice. Numerical comparisons are made for the equiprobable null hypothesis, for various multinomial sample sizes and numbers of cells. Exact power comparisons show that under specific alternatives to the equiprobable null hypothesis there may be other members in the BWCS family that have more power than the commonly used Pearson's chi–square.

1. INTRODUCTION

Let $\mathbf{X} = (X_1,\dots,X_k)$ denote the vector of observed frequencies for k categories for a sequence of n observations on a multinomial distribution with probability vector $\boldsymbol{\pi} = (\pi_1,\dots,\pi_k)$, $\Sigma_{i=1}^k \pi_i = 1$. Let $\mathbf{p} = (p_1, p_2, \dots, p_k) = (X_1/n, \dots, X_k/n)$ and let $\boldsymbol{\pi}_0 = (\pi_{01}, \dots, \pi_{0k})$ be a prespecified probability vector with $\pi_{0i} > 0$ for each i and $\Sigma_{i=1}^k \pi_{0i} = 1$. Several test statistics are available for testing the simple null hypothesis

$$H_0: \boldsymbol{\pi} = \boldsymbol{\pi}_0. \tag{1.1}$$

There are the well–known Pearson's chi–square and the log likelihood ratio test statistic as well as some other less used goodness–of–fit test statistics like the Freeman–Tukey statistic, the modified likelihood ratio statistic and the Neyman's chi–square.

Cressie and Read (1984) and Read and Cressie (1988) developed a class of goodness-of-fit test statistics called the family of power divergence statistics denoted by $\{I^\lambda: \lambda \in \mathbb{R}\}$ which contains as members the Pearson's chi–square, the log likelihood ratio statistic, the Freeman–Tukey statistic, the modified likelihood ratio statistic and the Neyman modified chi–square for $\lambda = 1, 0, -1/2, -1$ and -2 respectively. Read (1984a) studied small sample properties of the I^λ statistics, and compared the performance of the asymptotic χ^2 and three other alternative approximations of the exact distribution of the I^λ test statistics in small samples under the equiprobable null hypothesis (also known as the symmetric hypothesis):

$$H_0: \boldsymbol{\pi} = \boldsymbol{\pi}_0^* = (1/k, 1/k, \dots, 1/k). \tag{1.2}$$

An even more general class of goodness–of–fit test statistics, called the disparity tests, which contains the family of power divergence statistics as a subclass has been introduced by Basu and Sarkar (1994a), hereafter referred to as B&S. The disparity tests can be used to test simple as well as composite hypotheses. In the case of composite hypotheses the disparity test statistics are computed using the minimum disparity parameter estimators

(Lindsay 1994; Basu and Sarkar 1994b, 1994c; Sarkar and Basu 1995). B&S show that another subfamily of the disparity tests called the blended weight chi–square family of tests denoted by $\{BWCS_\alpha,\ 0 \le \alpha \le 1\}$, like the power weighted divergence statistics, contains a member ($BWCS_{1/3}$) that provides an excellent alternative to the usual Pearson's chi–square and the log likelihood ratio tests for testing whether the observed multinomial variables are sufficiently close to their null expected values. For testing (1.1) the blended weight chi–square $\{BWCS_\alpha,\ 0 \le \alpha \le 1\}$ test statistic is defined by

$$2nBWCS_\alpha(\mathbf{p}, \boldsymbol{\pi}_0) = n\sum_{i=1}^{k} \frac{(p_i - \pi_{i0})^2}{[\alpha p_i + (1-\alpha)\pi_{i0}]}. \tag{1.3}$$

The Pearson's chi–square (Pearson 1900) and the Neyman's chi–square (Neyman 1949) statistics are the $BWCS_0$ and $BWCS_1$ tests respectively, and their denominators are combined with different weights to obtain all the $BWCS_\alpha$ family members.

In this paper we examine the inaccuracy in using the upper percentage points of the usual χ^2 approximation for the null distribution $F_E(\cdot)$ of the $BWCS_\alpha$ statistics for testing (1.1). The significance levels produced by this approximation can be considerably different from the desired nominal levels for many $BWCS_\alpha$ tests. We derive and examine three other asymptotic approximations of F_E in order to achieve significance levels that are closer to the nominal levels, for small sample sizes. We also measure their maximum approximation error over the entire range. For various multinomial distributions Yarnold (1970), Odoroff (1970), Larntz (1978), Read (1984a) and Rudas (1986) gave simulation results on the error incurred in using the standard chi–square approximation for one or more of the power weighted divergence statistics. The specific simple null hypothesis used in our numerical experiment is the equiprobable hypothesis (1.2), significance of which is discussed by Read (1984a, p. 930). For different multinomial sample sizes and various numbers of cells recommendations are made on which approximations to use to obtain most accurate critical regions for different members of the $BWCS_\alpha$ family of tests. Exact powers of the $BWCS_\alpha$ tests are also compared for various α values under specific alternatives to the equiprobable null hypothesis. Exact power comparisons show that several other members of the $BWCS_\alpha$ family have more power than the most commonly used Pearson's chi–square ($BWCS_0$) under some alternatives.

The format for the remainder of this paper is as follows. In Section 2 we briefly review the disparity tests for the simple null hypothesis (1.1). We present three alternative approximations of the exact null distributions of the $BWCS_\alpha$ test statistics in Section 3. Section 4 contains a discussion of small sample comparisons of these three approximations as well as the chi–square approximation under the null hypothesis (1.2). In Section 5 we present exact power comparisons for various α values under some specific alternatives to the symmetric null hypothesis. Finally, some concluding remarks are given in Section 6.

2. DISPARITY GOODNESS–OF–FIT TESTS

First, we briefly describe the disparity tests for testing the simple null hypothesis. For more details for this case as well as for the composite null hypothesis case see B&S. Let G be a strictly convex function on $[-1, \infty)$ with $G(0) = 0$. Then, the disparity test statistic for the simple null hypothesis (1.1) generated by G is defined by

$$D_{\rho_G} = 2n\rho_G(\mathbf{p},\boldsymbol{\pi}_0)$$

where

$$\rho_G(\mathbf{p},\boldsymbol{\pi}_0) = \sum_{i=1}^{k} G\left(\frac{p_i}{\pi_{0i}} - 1\right)\pi_{0i}.$$

Letting $\delta_i = (\pi_{0i}^{-1}p_i - 1)$, we see that the Pearson chi–square statistic, the log likelihood ratio chi–square and the power divergence family are generated by

$$G(\delta) = \delta^2,\ \ G(\delta) = (\delta+1)\log_e(\delta+1),\ \ G(\delta) = [(\delta+1)^{\lambda+1} - 1]/(\lambda(\lambda+1))$$

respectively. The blended weight chi–square $\{BWCS_\alpha,\ 0 \le \alpha \le 1\}$ family is generated by $G(\delta) = 2^{-1}\delta^2/(\alpha\delta + 1)$. The statistic ρ_G is standardized to $2n\rho_G$ so that the latter converges to a chi–square statistic under the simple null hypothesis (B&S, Theorem 3.1) and under the assumptions that G is thrice differentiable, $G^{(3)}(0)$ is finite and $G^{(3)}$ is continuous at 0, where $G^{(3)}$ denotes the third derivative of G. In the following section we consider the three other approximations of the null distributions of the $BWCS_\alpha$ tests one of which ($F_N(t)$ in (3.6) below) is obtained under the specific equiprobable null hypothesis (1.2).

3. APPROXIMATIONS OF THE EXACT NULL DISTRIBUTIONS

Suppose the null hypothesis H_0: $\boldsymbol{\pi} = \boldsymbol{\pi}_0$ is true. Then, by Theorem 3.1 of B&S for each value of the family parameter α, we have

$$F_E(t) = F_{\chi^2(k-1)}(t) + o(1) \quad \text{as } n \to \infty \tag{3.1}$$

for all t, provided k is fixed. Let $F_{\chi^2(\nu)}(\cdot)$ denote the χ^2 distribution function with ν degrees of freedom. The chi–square distribution $F_{\chi^2(k-1)}$ is the usual approximation used to compute critical regions for the well–known Pearson's chi–square and the log likelihood ratio test statistics. Following Read (1984a) we present three closer approximations to F_E. The first is the moment corrected χ^2 distribution whose mean and variance agree to the second order with those of F_E, and is defined by

$$F_C(t) = F_{\chi^2(k-1)}(d_\alpha^{-1/2}(t-c_\alpha)) \tag{3.2}$$

where

$$c_\alpha = (k-1)[1 - d_\alpha^{1/2}] + n^{-1}a_\alpha, \quad d_\alpha = 1 + [n(2(k-1))]^{-1}b_\alpha$$

with

$$a_\alpha = \alpha(-S + 3k - 2) + \alpha^2(3S - 6k + 3),$$

$$b_\alpha = (2 - 2k - k^2 + S) + \alpha^2(39S - 9k^2 - 66k + 36) + \alpha(-18S + 6k^2 + 36k - 24)$$

and $S = \Sigma_{i=1}^{k}\pi_{0i}^{-1}$. The terms c_α and d_α are the asymptotic means and variances of the $BWCS_\alpha$ tests to the order $o(n^{-1})$. We derive the above expressions using equation (5.1) of B&S. The expectations of the terms involved in the equation (5.1) of B&S can be found in Read and Cressie (1988, Appendix A11).

Using a general asymptotic probability result for lattice random variables of Yarnold (1972), Read (1984b) derived the asymptotic expansion of the null limiting distribution of the I^λ statistics under the hypothesis (1.1). Let $W_j = n^{1/2}(p_j - \pi_{0j})$, j=1,2,...,k. The normalized vector $\mathbf{W} = (W_1,...,W_r)$, where r=k−1, takes values in the lattice

$$L = \left\{ \mathbf{w}=(w_1,...,w_r):\ \mathbf{w} = n^{1/2}(n^{-1}\mathbf{m} - \tilde{\boldsymbol{\pi}}_0) \text{ and } \mathbf{m} \in M \right\}$$

where $\tilde{\boldsymbol{\pi}}_0 = (\pi_{01},...,\pi_{0r})$ and $M = \{\mathbf{m} = (m_1,...,m_r):\ m_j,\ j=1,...,r$ are nonnegative integers satisfying $\Sigma_{j=1}^r m_j \le n\}$. Following the method of Read (1984b), we exploit Theorem 2 of Yarnold (1972), which gives a useful expression for the probability of lattice random variables belonging to an extended convex set B (for the definition see Definition 2.1 of Read 1984b), and derive an approximation of the exact distribution function $F_E(t)$ of $2nBWCS_\alpha(\mathbf{p}, \boldsymbol{\pi}_0)$ up to the order n^{-1} by considering the extended convex set

$$B_\alpha(t) = \left\{\mathbf{w} =(w_1,...,w_r):\ 2nBWCS_\alpha(n^{-1}(\mathbf{m}, m_k);\ \boldsymbol{\pi}_0) < t\right\} \tag{3.3}$$

where

$$w_k = -\Sigma_{j=1}^r w_j,\quad \mathbf{m} = n^{1/2}\mathbf{w} + n\tilde{\boldsymbol{\pi}}_0,\quad m_k = n^{1/2}w_k + n\tilde{\pi}_{0k}.$$

Using a fourth order Taylor series expansion of $2nBWCS_\alpha(\mathbf{p}, \boldsymbol{\pi}_0)$ (as a function of p_i around π_{0i}) we get the following.

THEOREM 1. *The asymptotic expansion for the distribution function* $F_E(t)$ *of the* $2nBWCS_\alpha(\mathbf{p},\boldsymbol{\pi}_0)$ *is given by*

$$F_E(t) = J_1^\alpha + J_2^\alpha + J_3^\alpha + O(n^{-3/2}), \tag{3.4}$$

where J_1^α, J_2^α *and* J_3^α *are defined by* J_1, J_2 *and* J_3 *respectively in Theorem 2.1 of Read (1984b) with* $B = B_\alpha(t)$ *defined in (3.3). Furthermore,*

$$J_1^\alpha = F_{\chi^2(k-1)}(t) + \frac{1}{24n}\Big\{ 2(1-S)F_{\chi^2(k-1)}(t) +$$

$$[\,3(3S-k^2-2k) - 18(S-k^2)\alpha + 9(S-3k^2+2k)\alpha^2\,]F_{\chi^2(k+1)}(t) +$$

$$[-6(2S-k^2-2k+1)+12(4S-3k^2-3k+2)\alpha +$$

$$18(-3S+3k^2+2k-2)\alpha^2]F_{\chi^2(k+3)}(t) +$$

$$[(9\alpha^2-6\alpha+1)(5S-3k^2-6k+4)]F_{\chi^2(k+5)}(t)\Big\}$$

and an approximation of J_2^α *to the first order is given by*

$$\hat{J}_2^\alpha = \Big\{ N_\alpha(t) - n^{(k-1)/2}V_\alpha(t) \Big\}\Big\{ e^{-t/2}(2\pi n)^{-(k-1)/2}Q^{-1/2} \Big\}$$

where $S = \Sigma_{i=1}^{k}\pi_{0i}^{-1}$, $Q = \Pi_{i=1}^{k}\pi_{0i}$,

$N_\alpha(t)$ = *number of multinomial X vectors such that* $2nBWCS(\mathbf{p};\ \boldsymbol{\pi}_0) < t$,

$V_\alpha(t)$ = *the volume of* $B_\alpha(t)$

$$= \left(\frac{(\pi t)^{(k-1)/2}}{\Gamma\{(k+1)/2\}}\right)Q^{1/2}\Big\{1 + \frac{t[9(S-3k^2+2k)\alpha^2]}{24n(k+1)}\Big\} + O(n^{-3/2}).$$

By Theorem 2.1 of Read (1984b) the term J_3^α is $O(n^{-1})$. Since the members of the family of $2nBWCS_\alpha(\mathbf{p},\boldsymbol{\pi}_0)$ tests are asymptotically equivalent (B&S, Theorem 3.1) we have $n(J_3^\alpha - J_3^0) = o(1)$ as $n \to \infty$. Therefore, all the α-dependent terms in J_3^α are $O(n^{-3/2})$. In view of the expansion in (3.4), J_3^α can be regarded as independent of α. Because the evaluation of J_3^α is complicated in nature (see e.g. Yarnold 1972, for J_3^0), as was done by Read (1984b) in the case of power divergence goodness–of–fit statistics, we ignore the term J_3^α in (3.4) and as a closer approximation of $F_E(t)$ than $F_{\chi^2(k-1)}(t)$ we propose to use:

$$F_S(t) = J_1^\alpha + \hat{J}_2^\alpha. \tag{3.5}$$

We derive the third approximation of the exact null distribution of the $BWCS_\alpha$ test for the situation when the number of cells k increases with the sample size n. We assume that $n/k \to a$ for $0 < a < \infty$ fixed. In such situations the asymptotic null distribution of the $BWCS_\alpha$ tests is normal. In the case of the Pearson's chi–square and the log likelihood ratio statistics Koehler and Larntz (1980) examined the applicability of the normal approximations for moderate sample sizes with moderately many cells. Under the symmetric null hypothesis (1.2), using Theorem 2.4 in Cressie and Read (1984), which follows from Holst (1972), with

$$f_i(x) = \frac{[(\frac{kx}{n}) - 1]^2}{\alpha(\frac{kx}{n}) + (1-\alpha)}$$

it can be shown that $F_E(t) = F_N(t) + o(1)$ as $n \to \infty$, where

$$F_N(t) = \Pr\{ N(0,1) < \sigma_n^{-1}(t-\mu_n) \}, \tag{3.6}$$

N(0,1) denotes a standard normal random variable,

$$\mu_n = nE\left\{ \frac{[(Y/a) - 1]^2}{\alpha(Y/a) + (1-\alpha)} \right\}$$

and

$$\sigma_n^2 = a^2k\left\{\mathrm{var}\left[\frac{[(Y/a) - 1]^2}{\alpha(Y/a) + (1-\alpha)}\right] - a\mathrm{Cov}^2\left[Y/a, \frac{[(Y/a) - 1]^2}{\alpha(Y/a) + (1-\alpha)}\right]\right\}$$

for $0 \leq \alpha < 1$, and Y is a Poisson(a) random variable.

4. SMALL SAMPLE COMPARISON OF THE APPROXIMATIONS

Among the three approximations F_C, F_N and F_S, F_C is computationally the simplest and F_N is the next best. In our comparative study it is shown that F_S provides the best approximation if compared across the entire distribution F_E, but F_C emerges as the best choice in approximating the right tail of F_E. This fact together with its computational ease, makes the use of F_C in place of the standard $F_{\chi^2(k-1)}$ desirable as well as practical for most $BWCS_\alpha$ tests.

In our numerical study all comparisons are made under the null hypothesis (1.2). For any fixed n, k and α, we compute the exact null distribution F_E following the procedure described in Read (1984a, Sec 2.1). We use two methods to measure the approximation errors associated with the four approximations $F_{\chi^2(k-1)}$, F_C, F_S and F_N for small sample sizes. The graphical results that we present here correspond to n=20 and k=5. However, we studied all the combinations (n,k) for n=10, 15, 20 and k=2, 3, 5, and the corresponding results are discussed in Section 4.3.

4.1. *Comparison of the Upper Percentiles of* F_E, $F_{\chi^2(k-1)}$, F_C, F_S and F_N

We compute the 90–th and 99–th percentiles of F_E, $F_{\chi^2(k-1)}$, F_C, F_S and F_N for various multinomial distributions with parameters n, k and $\pi = \pi_0^* = (1/k, 1/k, \ldots, 1/k)$. For $\gamma = 0.10$, 0.01 and for i = E, $\chi^2(k-1)$, C, S, we compute $t_{\gamma,i}$ such that

$$t_{\gamma,i} = \min\{t: \Pr[U \leq t] \geq 1-\gamma\}. \tag{4.1}$$

where U has the distribution F_i. Once the terms c_α, d_α, μ_n and σ_n are calculated, the $100(1-\gamma)$–th percentiles $t_{\gamma,C}$ and $t_{\gamma,N}$ of F_C and F_N respectively are computed as $t_{\gamma,C} = c_\alpha + d_\alpha^{1/2} t_{\gamma,\chi^2(k-1)}$ and $t_{\gamma,N} = \mu_n + \sigma_n z_\gamma$ where z_γ is the $100(1-\gamma)$–th percentile of the N(0,1) distribution. Calculation of the percentiles of F_E and F_S involves consideration of all $\binom{n+k-1}{k-1}$ possible multinomial vectors.

The 90–th and 99–th percentiles of the five distributions are illustrated in Figures 1 and 2 for $\alpha \in [0,1]$ and (n,k)= (20,5). In the figures CV–CHI, CV–E, CV–C, CV–S and CV–N respectively denote the percentiles of $F_{\chi^2(k-1)}$, F_E, F_C, F_S and F_N. The solid horizontal lines represent the percentiles of the $\chi^2(4)$ distribution. The percentage points of F_S and F_C approximate those of F_E much better than $F_{\chi^2(4)}$, especially at the 10% level. The approximation F_N performs well at the 1% level.

4.2. *Maximum Approximation Error*

In the previous section, we have compared two right tail percentiles of the five distributions. In this section, for a fixed (n,k,α) we compare the

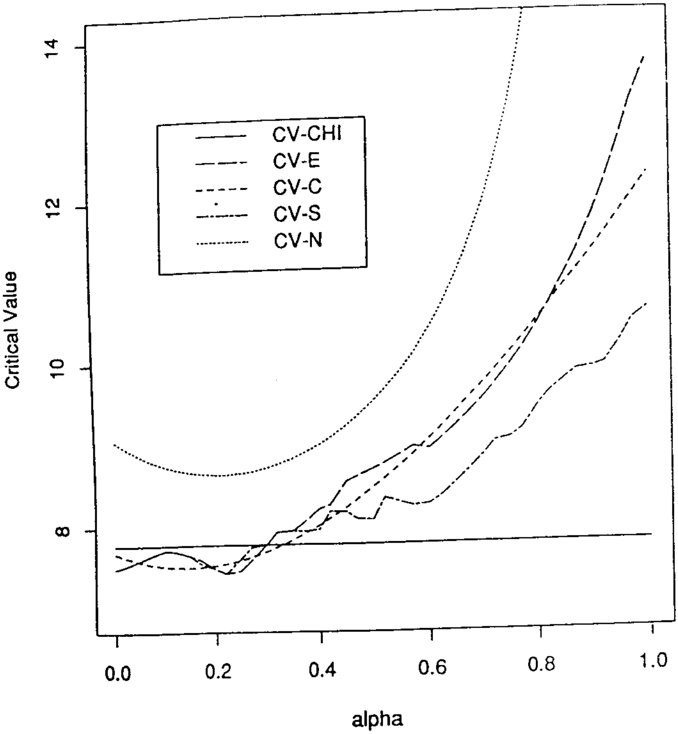

Figure 1. True and approximate critical values for the equiprobable null hypothesis at the 10% nominal level (n=20, k=5). In the graph CV–CHI, CV–E, CV–C, CV–S and CV–N denote the critical values corresponding to $F_{\chi^2(k-1)}$, F_E, F_C, F_S and F_N respectively.

worst error made across an entire approximating distribution in estimating F_E. It is called the maximum approximation error and following Read (1984a, Sec 2.3) is defined by

$$M_i = \max_{\mathbf{x}} \left| F_E(2nBWCS_\alpha(\tfrac{\mathbf{x}}{n}, \pi_0^*)) - F_i(2nBWCS_\alpha(\tfrac{\mathbf{x}}{n}, \pi_0^*)) \right| \tag{4.2}$$

for a fixed α and i= $\chi^2(k-1)$, C, S, N, where $BWCS_\alpha(\cdot,\cdot)$ is defined in (1.3)

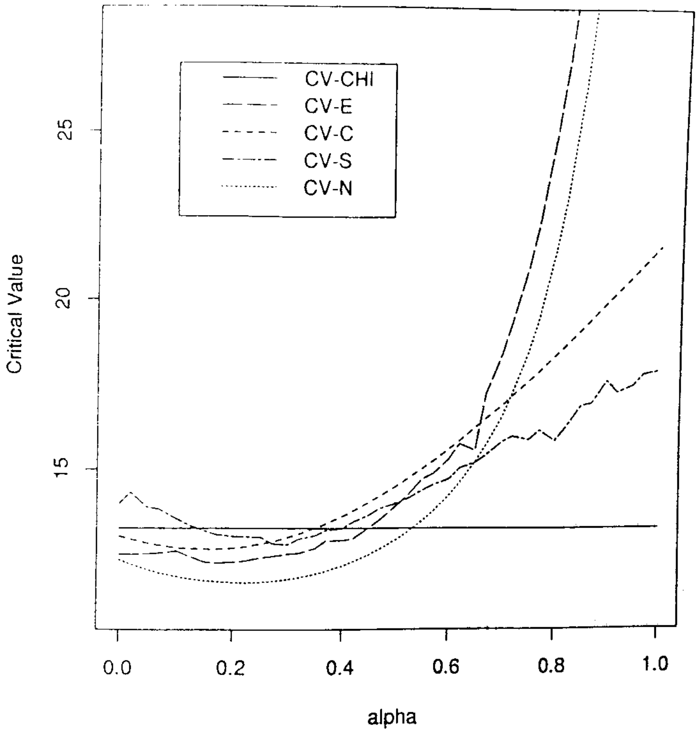

Figure 2. True and approximate critical values for the equiprobable null hypothesis at the 1% nominal level (n=20, k=5). In the graph CV–CHI, CV–E, CV–C, CV–S and CV–N denote the critical values corresponding to $F_{\chi^2(k-1)}$, F_E, F_C, F_S and F_N respectively.

and **x** represents the observed value of the multinomial random vector **X**. The sign associated with the maximum difference M_i is also recorded. The results for n=20 and k=5 are graphically represented in Figure 3 where M_i, i= $\chi^2(k-1)$, C, S, N, are denoted by MCHI, MC, MS and MN respectively. In general, the maximum approximation error is minimum for F_S.

4.3. *Conclusion*

On the basis of the findings in Figures 1–3 and the other (n,k)

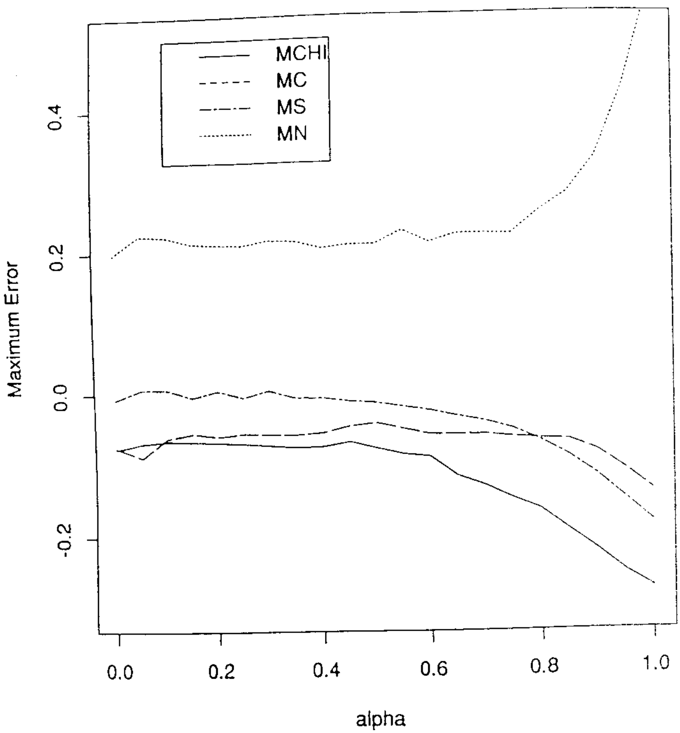

Figure 3. Maximum approximation errors for the equiprobable null hypothesis (n=20, k=5). In the graph MCHI, MC, MS and MN denote the maximum approximation errors for $F_{\chi^2(k-1)}$, F_C, F_S and F_N respectively.

combinations that we studied numerically, we can make the following observations. In general, it appears that the range of α values where the limiting chi–square distribution reasonably approximates the critical values of the exact distribution for moderate values of n is [0, 0.4]. For small to moderate sample sizes the use of the chi–square critical values to approximate the exact critical values is not recommended outside the above interval. In such situations one may use F_S or F_C; F_N may also be used if k is moderately large. On the whole, however, F_C is the best choice when one takes into account the computational aspect of the three approximations.

The $BWCS_{1/3}$ statistic, which belongs to the acceptable range of α values [0, 0.4], has been recommended by B&S to be a good alternative to

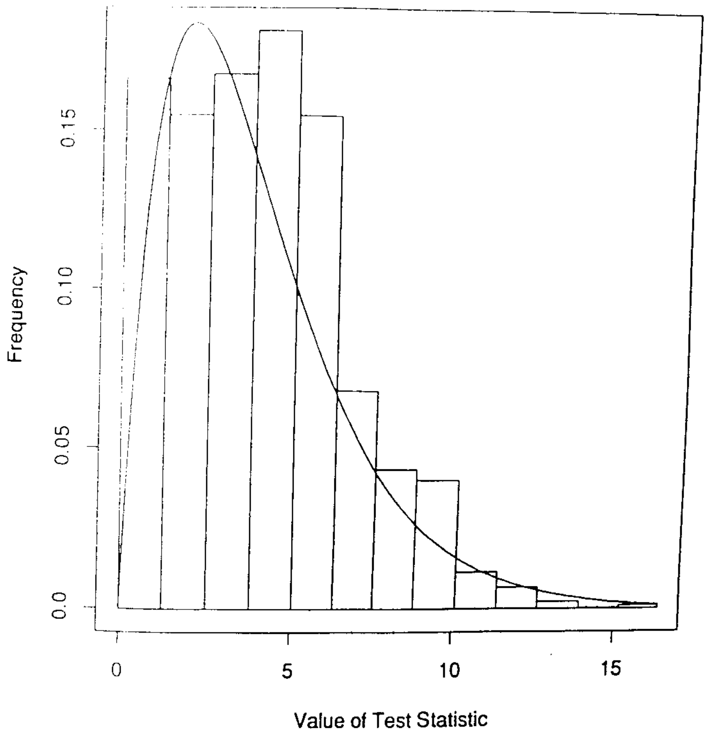

Figure 4. Histogram of exact distribution for the $2nBWCS_{1/3}$ test statistic (n=10, k=5) together with the $\chi^2(4)$ density.

the usual goodness–of–fit tests like the Pearson's chi–square and the likelihood ratio statistic. In Figure 4 we also illustrate how well the right tail of the exact distribution of the $BWCS_{1/3}$ test statistic is approximated by $F_{\chi^2(k-1)}$ for n=10 and k=5. The height of each bar in the histogram equals the exact probability that the $2nBWCS_{1/3}$ statistic belongs to the particular interval.

5. EXACT POWER COMPARISONS

In the last section we discussed how one can obtain excellent approximations of the exact critical regions for members of the $BWCS_\alpha$

TABLE I

Exact Power Function for the $BWCS_\alpha$ Randomized Size .05 Test (n=20, k=5).

α	δ 1.5	0.5	–0.9
1.00	0.2574	0.0785	0.5893
0.90	0.2574	0.0785	0.5893
0.70	0.3853	0.0839	0.5739
0.50	0.5627	0.1024	0.4485
0.30	0.6366	0.1120	0.3777
0.10	0.6907	0.1211	0.2851
0.00	0.6997	0.1228	0.2720

statistics. In this section we present small sample powers of the $BWCS_\alpha$ tests for testing (1.2) against

$$H_1:\ \pi_i = \begin{cases} \{1 - \delta/(k-1)\}/k & i=1,2,\ldots,(k-1), \\ (1+\delta)/k & i=k, \end{cases} \tag{5.1}$$

where $-1 \le \delta \le k-1$ is fixed. We have computed exact powers for three alternative hypotheses defined by $\delta = -0.9$, 0.5 and 1.5, as in Read (1984a). For a multinomial distribution with n=20 and k=5 we compute the randomized $BWCS_\alpha$ tests of size 0.05 for α = 0.00, 0.10, 0.20,..., 1.00. The results are presented in Table 1.

Table 1 shows that the exact power of the test statistics increases as α increases when δ is negative. In such situations, therefore, the Pearson's chi–square is the least powerful test within the $BWCS_\alpha$ family and tests with higher values of α will perform much better. However, when δ is positive the exact power of the tests decreases with α and in such cases the Pearson's chi–square will be the most powerful test.

6. DISCUSSION

In this paper we have studied the properties of the $BWCS_\alpha$ family of goodness–of–fit tests in small samples. By studying the equiprobable null hypothesis we have recommended range of α for which the exact distribution of the goodness–of–fit tests may be reasonably approximated by the chi–square distribution. Three other approximations are provided for the exact distributions of the statistics which often produce better results when α lies outside the interval [0, 0.4]. In particular, we recommend the use of the moment corrected chi–square distribution F_C which appears to be the optimal choice when the accuracy of the approximation and the computational ease are both taken into consideration.

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