Math. Ann. 296, 637–648 (1993)

Mathematische Annalen

On some convex sets and their extreme points

Rajarama Bhat[1], **V. Pati**[2], **and V.S. Sunder**[2]

[1] Stat.-Mathematics Unit, Indian Statistical Institute, New Delhi, India
[2] Stat.-Mathematics Unit, Indian Statistical Institute, RV College Post, Bangalore 560059, India

Mathematics Subject Classifications (1991): 46A55, 46L37

1 Introduction

In this note, we consider the problem of identifying the extreme points of two particular convex sets, one associated with a finite von Neumann algebra equipped with a faithful normal tracial state, and the other with such an algebra and a von Neumann subalgebra. (For work on similar problems – pertaining to the set of extreme points of certain sets of maps on operator algebras – see [1, 2, 5].)

To be precise, let M be a finite von Neumann algebra equipped with a faithful normal tracial state (henceforth denoted by 'tr') and let $D(M)$ denote the set of normal, unital, completely positive self maps of M which preserve tr. Then $D(M)$ is a convex set which is compact in the topology of pointwise σ-weak convergence, and so the set $\partial_e(D(M))$, of extreme points of $D(M)$, is non-empty. This is one of the two sets we are interested in. Taking a cue from the Birkhoff-von Neumann theorem, one might conjecture that, at least for "good" M, $\partial_e(D(M))$ consists of precisely the automorphisms of M. We show that this conjecture is valid when M is the algebra $M(2, \mathrm{K})$ of 2×2 matrices over K, where $\mathrm{K} = \mathbb{R}$ or $\mathbb{C}$, and invalid when $M = M(n, \mathbb{R})$, $n \geqq 3$, and $M = M(n, \mathbb{C})$, $n \geqq 4$.

Next, let N be a von Neumann subalgebra of M, with M as above, and consider the convex set

$$K(M, N) = \{x \in M : x \geqq 0, E_N(x) = 1\},$$

where E_N denotes the unique trace-preserving conditional expectation of M onto N. This set is clearly convex, and will be σ-weakly compact provided N is sufficiently 'ample' in M. (For instance, if M and N are (necessarily finite) factors and if the Jones index $[M:N]$ is finite, the above compactness holds because of a basic inequality due to Pimsner and Popa which states that, in this case, $x \leqq [M:N]E_N(x)$ whenever $x \in M$ and $x \geqq 0$.) In the just discussed case – i.e., when M and N are finite factors and the Jones index $[M:N] = \tau^{-1}$ is finite – we shall find it more convenient to change normalisations and consider the set

$$C(M, N) = \tau K(M, N) = \{x \in M : x \geqq 0, E_N(x) = \tau\} .$$

It is not hard to see that any projection in $C(M, N)$ is necessarily an extreme point of $C(M, N)$. As is well-known (see [6, Corollary 1.8]), projections in M with conditional expectation equal to τ are precisely the Jones projections which implement the conditional expectation of N onto a subfactor P such that $N \subset M$ is the result of applying the basic construction to the pair $P \subset N$. Thus the set $C(M, N) \cap \mathscr{P}(M)$ is a well-studied object. Here and elsewhere, we use the symbol $\mathscr{P}(M)$ to denote the lattice of projections in M. We obtain an upper bound for the trace of the support projection of any extreme point of $C(M, N)$ which suffices to prove that $C(M, N) \cap \mathscr{P}(M) = \partial_e C(M, N)$ in the special case when $\tau = \frac{1}{2}$. We also show, by example, that the equality $C(M, N) \cap \mathscr{P}(M) = \partial_e C(M, N)$ is not valid in general, for $[M : N] = n^2, n > 1$. We however leave open the possibility of equality when $N' \cap M = \mathbb{C}.1$.

2 Markov maps of $M(2, \mathrm{K})$

Throughout this section, M will denote a finite von Neumann algebra, with a distinguished faithful normal trace denoted by tr. By a *Markov map* of M, we shall mean a normal completely positive linear self-map $L : M \to M$ which preserves the identity of M as well as the trace tr – i.e., $L(1) = 1$ and $\mathrm{tr}(Lx) = \mathrm{tr}\, x$ for all x in M. We denote the set of Markov maps of M by $D(M)$. The terminology 'Markov map' is inspired by probabilistic considerations.

When $M = \mathbb{C}^n$, with $\mathrm{tr}\, \alpha = \alpha_1 + \alpha_2 + \ldots \alpha_n$, it is seen that $L \in D(M)$ if and only if there exists a doubly stochastic matrix $D = [d_{ij}]$ such that $(L\alpha)_i = \sum_j d_{ij}\alpha_j$ for all $\alpha \in \mathbb{C}^n$. (This example is the reason for our use of the notation $D(M)$ for the set of Markov maps of M.) It is the content of the Birkhoff-von Neumann theorem that the extreme points of this set are the permutation matrices; in other words, $\partial_e(D(M))$ is precisely the set of automorphisms of M in this case.

When $M = M(n, \mathbb{C})$, it is known that a linear self-map $L : M \to M$ is completely positive if and only if the matrix $[L(e_{ij})]$ is positive, where $\{e_{ij}\}$ is the usual set of matrix units in $M(n, \mathbb{C})$. For L as above, let us write $l_{ij} = L(e_{ij})$; note that if $x = [\xi_{ij}]$, then $Lx = \sum_{i,j} \xi_{ij} l_{ij}$ so that L preserves the trace if and only if, for all $[\xi_{ij}]$, we have $\sum_i \xi_{ii} = \sum_{i,j} \xi_{ij} \mathrm{tr}\, l_{ij}$, which clearly happens if and only if $\mathrm{tr}\, l_{ij} = \delta_{ij} \forall i, j$. Hence an equivalent description of $D(M(n, \mathbb{C}))$ is as the set

$$\left\{ L = [l_{ij}]_{1 \leqq i, j \leqq n} : l_{ij} \in M(n, \mathbb{C}),\ L \geqq 0,\ \mathrm{tr}\, l_{ij} = \delta_{ij},\ \sum_i l_{ii} = 1 \right\} .$$

Definition 2.1. Let K be $\mathbb{R}$ or $\mathbb{C}$. Let $L = [l_{ij}]$ be an $n \times n$ matrix of matrices $l_{ij} \in M(n, \mathrm{K})$. Motivated by the above discussion, we say L is a *Markov map* if it satisfies the three conditions:

(i) (complete positivity) The $n^2 \times n^2$ matrix $[l_{ij}]$ is positive semi-definite.
(ii) (identity preserving) $\sum_{i=1}^n l_{ii} = 1$
(iii) (trace preserving). $\mathrm{tr}\, l_{ij} = \delta_{ij}$.

In this section, we shall investigate the extreme points of $D(M(n, \mathbb{K}))$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. First we need some definitions:

Definition 2.2. A positive semi-definite (square) matrix is called a *correlation matrix* if all its entries on the main diagonal are equal to one. Let $\mathbf{K}_n$ denote the set of $n \times n$ correlation matrices with entries in $\mathbb{K} (= \mathbb{R}$ or $\mathbb{C})$.

Definition 2.3. A Markov map $L = [l_{ij}]$ is said to be an *inner automorphism* if there exists a unitary matrix $U \in U(n, \mathbb{K})$ such that $l_{ij} = U e_{ij} U^*$ for all $1 \leq i, j \leq n$, where e_{ij} are usual elementary matrices (standard matrix units).

It is clear that if $M = M(n, \mathbb{K})$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and if $A \in M$ is a correlation matrix, then the map

$$L_A : M \to M$$

$$X \mapsto A \cdot X$$

defines an element of $D(M)$ (where $\cdot$ denotes Schur or Hadamard multiplication: $[a_{ij}] \cdot [X_{ij}] = [a_{ij} X_{ij}]$). It is also clear (by taking $X = e_{ij}$, $1 \leq i, j \leq n$, the elementary matrix units) that the only way we can have $L_A(X) = UXU^*$ for all $X \in M$, for some unitary U in M is for U to be a diagonal unitary and A to have rank one. Namely, $U = \operatorname{diag}(\omega_i)$, where ω_i are numbers in $\mathbb{K}$ of modulus one, and $A = [\bar{\omega}_i \omega_j]$.

Lemma 2.4. *Let $M = M(n, \mathbb{K})$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $A \in M$ be a correlation matrix. Then*

$$L_A \in \partial_e(D(M)) \Leftrightarrow A \in \partial_e(\mathbf{K}_n)\,,$$

where $\mathbf{K}_n$ is defined above in Definition 2.2.

Proof. We shall show that there exist $\{L_k\}_{k=1}^m \subset D(M)$ such that $L_A = \sum_{k=1}^m \theta_k L_k$, where $\theta_k \in [0, 1]$ for all k and $\sum_{k=1}^m \theta_k = 1$, if and only if there exist $A_1, \ldots, A_m \in \mathbf{K}_n$ such that $L_k = L_{A_k}$ and $A = \sum_{k=1}^k \theta_k A_k$.

We only need to prove the 'only if' part of the above assertion. So, suppose $L_A = \sum_{k=1}^m \theta_k L_k$, $\theta_k \in [0, 1]$ for all k, and $\sum_{k=1}^m \theta_k = 1$. Let $\{e_{ij}\}_{i,j=1}^n$ denote the standard set of matrix units in M. For each fixed $i = 1, 2, \ldots, n$, we have

$$e_{ii} = L_A(e_{ii}) = \sum_{k=1}^m \theta_k L_k(e_{ii})\,.$$

Since e_{ii} is a rank projection and L_k is a Markov map, it follows easily that $L_k(e_{ii}) = e_{ii}$ for $1 \leq k \leq m$, and $1 \leq i \leq n$.

Next, fix $1 \leqq i, j \leqq n$ and note that for each $k = 1, 2, \ldots, m$, we have:

$$\begin{pmatrix} e_{ii} & e_{ij} \\ e_{ji} & e_{jj} \end{pmatrix} \geqq 0 \Rightarrow \begin{pmatrix} L_k(e_{ii}) & L_k(e_{ij}) \\ L_k(e_{ji}) & L_k(e_{jj}) \end{pmatrix} \geqq 0$$

$$\Rightarrow \exists a_{ij}^{(k)} \in \mathbb{K} \text{ such that } L_k(e_{ij}) = a_{ij}^{(k)} e_{ij}\,. \tag{1}$$

Hence $L_k(e_{ij}) = A_k \cdot e_{ij}$ for all i, j, where $A_k = [a_{ij}^{(k)}]$, $a_{ii}^{(k)} = 1$, and thus $L_k = L_{A_k}$. Since J (the matrix with all entries equal to one) is positive semidefinite, it follows that

$$A_k = L_{A_k}(J) = L_k(J) \geq 0$$

i.e., $A_k \in \mathbf{K}_n$, and the proof is complete.

Proposition 2.5. *Let $M = M(n, \mathbb{R})$, $n \geqq 3$ or $M = M(n, \mathbb{C})$, $n \geqq 4$. Then, there exist extreme points of $D(M)$ which are not given by inner automorphisms of M.*

Proof. It is known, cf. [2], and [3], that (in the notation of the last lemma) $\mathbf{K}_n$ contains extreme points of rank greater than one when $\mathbb{K} = \mathbb{R}$, $n \geqq 3$ or $\mathbb{K} = \mathbb{C}$, $n \geqq 4$. The Lemma 2.4 and the remarks preceding it complete the proof. □

We next look at $\partial_e(D(M(2, \mathbb{K})))$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and the main proposition is

Proposition 2.6. *The set of extreme points $\partial_e(D(M(2, \mathbb{K})))$ is precisely the set of inner automorphisms, viz. the set*

$$\{L = [l_{ij}] : l_{ij} = U e_{ij} U^*, \quad \text{where } U \in U(2, \mathbb{K})\}.$$

To prove this proposition, we need some notation and a couple of lemmas. By (i), (ii), (iii) of Definition 2.1, if $L \in D(M)$,

(i) the matrix

$$L = [l_{ij}] \stackrel{\text{def}}{=} \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} \tag{2}$$

is positive semi-definite as a 4×4 matrix in $M(2, \mathbb{K})$. In particular A and B are positive semi-definite.

(ii) $A + B = 1$.

(iii) $\operatorname{tr} A = \operatorname{tr} B = 1$, $\operatorname{tr} C = 0$.

It is well known that if B above is nonsingular, then the matrix

$$\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$$

in (2) is positive semi-definite if and only if

$$A \geqq C B^{-1} C^* . \tag{3}$$

Now since A is positive semi-definite of trace 1, we may write it as

$$A = A_{\rho, \varepsilon} = \begin{pmatrix} \frac{1}{2} - \rho & \varepsilon \\ \bar{\varepsilon} & \frac{1}{2} + \rho \end{pmatrix} \tag{4}$$

where $\rho \in \mathbb{R}$ and $\varepsilon \in \mathbb{K}$ are numbers satisfying $\rho^2 + |\varepsilon|^2 \leqq \frac{1}{4}$ since $\det A \geqq 0$. Thus, by (ii) above,

$$B = 1 - A = \begin{pmatrix} \frac{1}{2} + \rho & -\varepsilon \\ -\bar{\varepsilon} & \frac{1}{2} - \rho \end{pmatrix}$$

which clearly implies that

$$AB = \det A = \det B \tag{5}$$

so that $B^{-1} = (\det A)^{-1} A$, if A (equivalently B) is nonsingular. Combining this with (3), we see that if A is non-singular, then the matrix (2) is positive semidefinite if and only if

$$CAC^* \leqq (\det A) A .$$

Definition 2.7. Let us say that a trace zero (cf. (iii) above) 2×2 matrix C is a det A-contraction if $CAC^* \leqq (\det A) A$.

Lemma 2.8. *If $L \in \partial_e(D(M(2, \mathbb{K})))$, then A is singular.*

Proof. Let us assume, to the contrary, that $\det A \neq 0$, which implies by the positivity of A that $\det A > 0$. We will eventually show that L can be expressed as a non-trivial convex combination of two Markov maps. We first make the following:

Claim 1. If $U \in M(2, \mathrm{K})$ with $\operatorname{tr} U = 0$, then

$$U = U_1 \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} U_1^* \tag{6}$$

for some $U_1 \in U(2, \mathrm{K})$, where $\alpha, \beta \in \mathrm{K}$; further $|\alpha|, |\beta| \leqq 1 \Leftrightarrow \| U \| \leqq 1$.

The proof is quite elementary, and we omit it. □

Claim 2. If A is nonsingular then C is a $\det A$-contraction if and only if

$$C = A^{\frac{1}{2}} U B^{\frac{1}{2}} \tag{7}$$

where $U \in M(2, \mathrm{K})$ is a ($\det I$-) contraction, $\operatorname{tr} U = 0$ and $B = 1 - A$ as above.

We have remarked above that $\det A = \det B$ implies that B is also nonsingular, positive semi-definite, so the square roots of A and B make sense. Now by the hypothesis on C, and the fact that $B^{-1} = (\det A)^{-1} A$ by (5), we have

$$\begin{aligned} & CAC^* \leqq (\det A) A \\ \Leftrightarrow\ & C(\det A)^{-\frac{1}{2}} A^{\frac{1}{2}} A^{\frac{1}{2}} (\det A)^{-\frac{1}{2}} C^* \leqq A \\ \Leftrightarrow\ & (A^{-\frac{1}{2}} C B^{-\frac{1}{2}})(B^{-\frac{1}{2}} C^* A^{-\frac{1}{2}}) \leqq 1 \end{aligned}$$

which is equivalent to $U = (A^{-\frac{1}{2}} C B^{-\frac{1}{2}})$ being a contraction. This implies

$$\begin{aligned} \operatorname{tr} U &= \operatorname{tr}(A^{-\frac{1}{2}} C B^{-\frac{1}{2}}) \\ &= (\det A)^{-\frac{1}{2}} (\operatorname{tr}(A^{-\frac{1}{2}} C A^{\frac{1}{2}})) \\ &= (\det A)^{-\frac{1}{2}} (\operatorname{tr} C) = 0 \end{aligned}$$

since C has trace zero. This establishes the Claim 2. □

Claim 3. If L is extreme, A is nonsingular and U is as in Claim 2, then $U \in U(2, \mathrm{K})$.

This follows from Claims 1 and 2, and the fact that the extreme points in the unit disc have modulus one. □

Thus we assume henceforth that $C = A^{\frac{1}{2}} U B^{\frac{1}{2}}$, where $U \in U(2, \mathrm{K})$, $\operatorname{tr} U = 0$ and $B = 1 - A$. We can further assume, by Claim 1, that

$$U = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$$

and from Claim 3, $|\alpha| = |\beta| = 1$.

Now define:

$$V = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \quad W = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}. \tag{8}$$

Clearly, V and W are in $U(2, \mathrm{K})$.

Further define

$$\begin{pmatrix} \tilde{A}_1 & \tilde{C}_1 \\ \tilde{C}_1^* & \tilde{B}_1 \end{pmatrix} = \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} Ve_{11}V^* & Ve_{12}V^* \\ Ve_{21}V^* & Ve_{22}V^* \end{pmatrix} \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix}$$

$$\begin{pmatrix} \tilde{A}_2 & \tilde{C}_2 \\ \tilde{C}_2^* & \tilde{B}_2 \end{pmatrix} = \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} We_{22}W^* & We_{21}W^* \\ We_{12}W^* & We_{11}W^* \end{pmatrix} \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix}. \tag{9}$$

We now make the following assertions:

Claim 4. (a) $\operatorname{tr}\tilde{C}_1 = \operatorname{tr}\tilde{C}_2 = 0$.
(b) $\tilde{A}_i \geqq 0$, $\tilde{B}_i \geqq 0$ for $i = 1, 2$.
(c) $1 > t \stackrel{\text{def}}{=} \operatorname{tr}\tilde{A}_1 = \operatorname{tr}\tilde{B}_1 > 0$.
(d) $1 > (1-t) = \operatorname{tr}\tilde{A}_2 = \operatorname{tr}\tilde{B}_2 > 0$.
(e) $\det\tilde{A}_i = \det\tilde{B}_i = 0$ for $i = 1, 2$.
(f) $A = \tilde{A}_1 + \tilde{A}_2$.
(g) $B = \tilde{B}_1 + \tilde{B}_2$.
(h) $C = \tilde{C}_1 + \tilde{C}_2$.
(i) $tI = \tilde{A}_1 + \tilde{B}_1$, t as in (c).
(j) $(1-t)I = \tilde{A}_2 + \tilde{B}_2$.

Proof.

$$\begin{aligned} \operatorname{tr}\tilde{C}_1 &= \operatorname{tr}(A^{\frac{1}{2}}Ve_{12}V^*B^{\frac{1}{2}}) \\ &= \operatorname{tr}(Ve_{12}V^*B^{\frac{1}{2}}A^{\frac{1}{2}}) \\ &= (\det A)^{\frac{1}{2}}(\operatorname{tr} Ve_{12}V^*) \text{ by (5)} \\ &= (\det A)^{\frac{1}{2}}(\operatorname{tr} e_{12}) = 0\,. \end{aligned}$$

Similarly $\operatorname{tr}\tilde{C}_2 = 0$, proving (a).
By definition,

$$\tilde{A}_1 = (A^{\frac{1}{2}}V)e_{11}(A^{\frac{1}{2}}V)^*$$

which is positive semidefinite since e_{11} is. Similarly for $\tilde{A}_2$, $\tilde{B}_1$, $\tilde{B}_2$, proving (b).

Since $\tilde{A}_1$ and $\tilde{B}_1$ are positive semidefinite, and not equal to zero (by the proof of (b), since $A^{\frac{1}{2}}V$, $B^{\frac{1}{2}}V$ are nonsingular), their traces are strictly positive. To show their equality, note

$$\begin{aligned} \operatorname{tr}\tilde{B}_1 &= \operatorname{tr}(B^{\frac{1}{2}}Ve_{22}V^*B^{\frac{1}{2}}) \\ &= \operatorname{tr}(Ve_{22}V^*B) = \operatorname{tr}(V(I-e_{11})V^*(I-A)) \\ &= 1 - \operatorname{tr}(VV^*A) + \operatorname{tr}(Ve_{11}V^*A) \\ &= 1 - 1 + \operatorname{tr}(A^{\frac{1}{2}}Ve_{11}V^*A^{\frac{1}{2}}) = \operatorname{tr}\tilde{A}_1 = t\,. \end{aligned}$$

Similarly $\operatorname{tr}\tilde{A}_2 = \operatorname{tr}\tilde{B}_2 > 0$. The fact that this is $(1-t)$ and hence that $t < 1$ will follow from $\operatorname{tr} A = \operatorname{tr} B = 1$ and (f) and (g). This proves (c) and (d).
By definition

$$\det\tilde{A}_1 = (\det A)(\det e_{11}) = 0 = (\det B)(\det e_{22}) = \det\tilde{B}_1\,.$$

Similarly for $\tilde{A}_2$, $\tilde{B}_2$, proving (e).

Computation shows that (using the Definition 8, and $|\alpha| = |\beta| = 1$):

$$Ve_{ii}V^* = We_{ii}W^* = e_{ii} \quad i = 1, 2$$

$$Ve_{12}V^* = \alpha e_{12}, \; We_{21}W^* = \beta e_{21} .$$

It follows that:

$$\tilde{A}_1 + \tilde{A}_2 = A^{\frac{1}{2}}(Ve_{11}V^* + We_{22}W^*)A^{\frac{1}{2}} = A .$$

Similarly,

$$\tilde{B}_1 + \tilde{B}_2 = B^{\frac{1}{2}}(Ve_{22}V^* + We_{11}W^*)B^{\frac{1}{2}} = B$$

$$\tilde{C}_1 + \tilde{C}_2 = A^{\frac{1}{2}}(Ve_{12}V^* + We_{21}W^*)B^{\frac{1}{2}}$$

$$= A^{\frac{1}{2}}UB^{\frac{1}{2}} = C$$

where the last line follows from Claim 3. This proves (f), (g), and (h). Since by [5],

$$A^{\frac{1}{2}}B^{\frac{1}{2}} = \det A^{\frac{1}{2}} = \det B^{\frac{1}{2}} = B^{\frac{1}{2}}A^{\frac{1}{2}}$$

it easily follows that

$$\tilde{A}_1\tilde{B}_1 = (A^{\frac{1}{2}}Ve_{11}V^*A^{\frac{1}{2}})(B^{\frac{1}{2}}Ve_{22}V^*B^{\frac{1}{2}})$$

$$= A^{\frac{1}{2}}Ve_{11}(\det A)e_{22}V^* = 0 = \tilde{B}_1\tilde{A}_1 .$$

Thus $\tilde{A}_1$ and $\tilde{B}_1$ are positive semidefinite commuting matrices. This means $\tilde{A}_1$ and $\tilde{B}_1$ can be simultaneously diagonalised. Since, by (b) $\operatorname{tr}\tilde{A}_1 = t = \operatorname{tr}\tilde{B}_1$ and by (5), $\det\tilde{A}_1 = \det\tilde{B}_1 = 0$, and since $\tilde{A}_1\tilde{B}_1 = 0$, we can write the simultaneous diagonal forms of $\tilde{A}_1$ and $\tilde{B}_1$ as $\operatorname{diag}(t, 0)$ and $\operatorname{diag}(0, t)$ respectively. So these simultaneous diagonal forms add up to tI. Thus $\tilde{A}_1$ and $\tilde{B}_1$ also add up to tI. Similarly $\tilde{A}_2 + \tilde{B}_2 = (1 - t)I$. This proves (i) and (j).

Now we are ready to prove the Lemma 2.8. In terms of the definitions of (9) and t as in (c) of Claim 4 above, let us define:

$$A_1 = t^{-1}\tilde{A}_1, \; B_1 = t^{-1}\tilde{B}_1, \; C_1 = t^{-1}\tilde{C}_1$$

$$A_2 = (1 - t)^{-1}\tilde{A}_2, \; B_2 = (1 - t)^{-1}\tilde{B}_2, \; C_2 = (1 - t)^{-1}\tilde{C}_2$$

$$L_i \stackrel{\text{def}}{=} \begin{pmatrix} A_i & C_i \\ C_i^* & B_i \end{pmatrix} \quad \text{for } i = 1, 2 .$$

First let us see that L_i for $i = 1, 2$ are Markov maps. From the last two equations in (9), it follows that L_i are positive semidefinite. From (a), (c) and (d) of Claim 4, it follows that $\operatorname{tr} A_i = \operatorname{tr} B_i = 1$, $\operatorname{tr} C_i = 0$. From (i), (j) of Claim 4, it follows that $A_i + B_i = I$. This shows $L_i \in D(M(2, \mathrm{K}))$ for $i = 1, 2$.

From (f), (g), (h) of Claim 4, we have

$$L = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix} = tL_1 + (1 - t)L_2 .$$

It only remains to show that $L_1 \neq L_2$. But if $L_1 = L_2$, we have $A_1 = A_2$ which implies

$$\begin{aligned}(1-t)\tilde{A}_1 &= t\tilde{A}_2\\ \Rightarrow (1-t)A^{\frac{1}{2}}Ve_{11}V^*A^{\frac{1}{2}} &= tA^{\frac{1}{2}}We_{22}W^*A^{\frac{1}{2}}\\ \Rightarrow (1-t)e_{11} &= te_{22} \Rightarrow t = 0,\ (1-t) = 0\end{aligned}$$

which is clearly a contradiction. This proves Lemma 2.8. □

Proof of Proposition 2.6. Let

$$L = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$$

be a Markov map which is extreme. By the preceding Lemma 2.8, we have that A is *singular* of trace one, as is B. Since $B = I - A$, we may assume, after a unitary basis change that

$$A = e_{11},\ B = e_{22}\ .$$

Thus, if U denotes this unitary

$$L = \begin{pmatrix} Ue_{11}U^* & UC'U^* \\ UC'^*U^* & Ue_{22}U^* \end{pmatrix}$$

where $C' = U^*CU$. Now since L is positive semidefinite extreme,

$$L' = \begin{pmatrix} e_{11} & C' \\ C'^* & e_{22} \end{pmatrix}$$

is also positive semidefinite extreme. Since 0 occurs in the 2-2 and 3-3 diagonal entries, this positivity implies that the whole second and third rows and columns are zero. Thus

$$C' = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$$

and $|\gamma| \leqq 1$. Now if $|\gamma| < 1$, it may be written as

$$\gamma = \frac{\omega_1 + \omega_2}{2}$$

where $|\omega_i| = 1$. This would force $L' = \dfrac{L_1 + L_2}{2}$, where

$$L_i = \begin{pmatrix} e_{11} & C_i' \\ C_i'^* & e_{22} \end{pmatrix} \quad i = 1, 2$$

where $C_i' = \omega_i e_{12}$ for $i = 1, 2$, contradicting the extremality of L'. Thus $|\gamma| = 1$. Now another unitary change preserves e_{11}, e_{22}, but converts $C' = \gamma e_{12}$ to e_{12}. This shows, by the Definition 2.3 that L is an inner automorphism. The Proposition 2.6 is thus proved. □

3 Extreme points of $C(M, N)$

Throughout this section, we assume that $N \subset M$ is an inclusion of finite factors such that the Jones index $\tau^{-1} = [M:N] < \infty$. We write E for the unique trace preserving conditional expectation of M onto N. Also we reserve the symbol e for the 'Jones projection' of $L^2(M, \mathrm{tr})$ onto $L^2(N, \mathrm{tr})$; recall that the von Neumann subalgebra M_1 of $\mathscr{L}(L^2(M, \mathrm{tr}))$ generated by M and e is again a finite factor, such that $E_M(e) = \tau . 1$ and that $eme = (Em)e$ for all m in M.

As in the introduction, we define

$$C = C(M, N) = \{x \in M_+ : Ex = \tau\} .$$

Since $Ex \geqq \tau x$ for all x in M_+, [6], it follows that $x \in C \Rightarrow 0 \leqq x \leqq 1$, and hence C is a compact convex set in the σ-weak topology. Since any projection in M is an extreme point of the set of positive contractions in M, the following inclusion is evident:

$$\mathscr{P}(M) \cap C \subset \partial_e C . \tag{10}$$

This section is devoted to the study of $\partial_e C$.

Lemma 3.1. *Let $p \in \mathscr{P}(M)$. Then,*
(i) *$p \wedge e = p_0 e$, where $p_0 = 1_{\{1\}}(Ep) \in \mathscr{P}(N)$;* further *$p_0 \leqq p$ and* $\mathrm{tr}\,(p \wedge e) = \tau \mathrm{tr}\, p_0$.
In particular, $\mathrm{tr}\, p > 1 - \tau \Rightarrow p_0 \neq 0$.
(ii) *If Ep is invertible, then*

$$p \wedge e^{\perp} = p(1 - (Ep)^{-1} e)p ;$$

in general - i.e. even if $0 \in sp\,(Ep)$,

$$\mathrm{tr}(p \wedge e^{\perp}) = \mathrm{tr}\, p + \tau \mathrm{tr}\, 1_{\{0\}}(Ep) - \tau .$$

(Here and elsewhere, we write $1_{\Lambda}(x)$ for the spectral projection of the normal operator x corresponding to the set Λ.)

Proof of (i)

$$\begin{aligned} p \wedge e &= s - \lim_{n \to \infty} (epe)^n \\ &= s - \lim_{n \to \infty} (Ep)^n e \\ &= p_0 e . \end{aligned}$$

Now,

$$\begin{aligned} p \geqq p \wedge e &\Rightarrow p_0 e = p p_0 e \\ &\Rightarrow p_0 = p p_0 \\ &\Rightarrow p_0 \leqq p . \end{aligned}$$

Finally,

$$\mathrm{tr}\, p > 1 - \tau \Rightarrow \mathrm{tr}(p \wedge e) > 0 .$$

This proves (i).

Proof of (ii). Compute thus:

$$pe^{\perp}p = p(1-e)p = p - pep$$
$$(pe^{\perp}p)^2 = p - pep - pep + pepep$$
$$= p - 2pep + p(Ep)ep\ .$$

An easy induction argument then shows that

$$(pe^{\perp}p)^n = p\left[1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k} (Ep)^{k-1}e\right]p \tag{11}$$
$$= p[1 + \psi_n(Ep)e]p\ ,$$

where

$$\psi_n(t) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} t^{k-1}$$
$$= \begin{cases} -n & t = 0 \\ \dfrac{(1-t)^n - 1}{t} & t \neq 0\ . \end{cases}$$

Since $0 \leqq Ep \leqq 1$, it would follow that $|1-t| < 1$ whenever $t \in \operatorname{sp} Ep$ and $t \neq 0$. In particular, if $0 \notin \operatorname{sp} Ep$, it follows that the sequence $\{\psi_n(Ep)\}$ converges in norm to $-(Ep)^{-1}$; hence

$$p \wedge e^{\perp} = w - \lim (pe^{\perp}p)^n = p[1 - (Ep)^{-1}e]p$$

In general, even if $0 \in \operatorname{sp} Ep$, we find on taking traces in (11), that

$$\operatorname{tr}(pe^{\perp}p)^n = \operatorname{tr} p + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \operatorname{tr}(p(Ep)^{k-1}e)$$
$$= \operatorname{tr} p + \tau \sum_{k=1}^{n} (-1)^k \binom{n}{k} \operatorname{tr}(p(Ep)^{k-1})$$
$$= \operatorname{tr} p + \tau \sum_{k=1}^{n} (-1)^k \binom{n}{k} \operatorname{tr}(Ep)^k$$
$$= \operatorname{tr} p + \tau \operatorname{tr}[(1-Ep)^n - 1]\ .$$

Now, $\{(1-Ep)^n\}$ converges weakly to $1_{\{0\}}(Ep)$, and so,

$$\operatorname{tr}(p \wedge e^{\perp}) = \lim \operatorname{tr}(pe^{\perp}p)^n$$
$$= \operatorname{tr} p + \tau \operatorname{tr} 1_{\{0\}}(Ep) - \tau\ . \qquad \square$$

(Actually, we will not really need (ii) in our further arguments, but gave it here as it is a natural complement to (i).)

Proposition 3.2. $x \in \partial_e C \Rightarrow \operatorname{tr} 1_{\{0\}}(x) \geqq \tau$.

Proof. Fix $\varepsilon > 0$, and put $p = 1_{(\varepsilon, 1]}(x)$ and $p_0 = 1_{\{1\}}(Ep)$. Suppose $\operatorname{tr} p > 1 - \tau$, so $p_0 \neq 0$. Then

$$p_0 \in \mathscr{P}(N) \Rightarrow [M_{p_0} : N_{p_0}] = [M : N] > 1$$
$$\Rightarrow \exists y = y^* \in M_{p_0} \text{ such that } y \neq 0 \text{ and } E_{N_{p_0}} y = 0\ .$$

Then $E_N y = 0$, because $z \in N \Rightarrow \operatorname{tr} yz^* = \operatorname{tr} p_0 y p_0 z^* = \operatorname{tr} y p_0 z^* p_0 = 0$. Without loss of generality assume that $\|y\| < \varepsilon$. Define $x_\pm = x \pm y$. Then

$$E_N x_\pm = \tau \text{ and } |y| \leqq \varepsilon p_0 \leqq \varepsilon p \leqq x .$$

So $x_\pm \geqq 0$. Therefore, $x_\pm \in C$, and $x = \dfrac{x_+ + x_-}{2}$. Thus, for all $\varepsilon > 0$, $\operatorname{tr} 1_{(\varepsilon, 1]}(x) \leqq 1 - \tau$. So, $\operatorname{tr} 1_{(0,1]}(x) \leqq 1 - \tau$. □

Corollary 3.3. *If* $[M:N] = 2$, *then* $\partial_e C(M, N) = C(M, N) \cap \mathscr{P}(M)$.

Proof. Here, $C = \{x \in M_+ : E_N x = \frac{1}{2}\}$. The map $x \mapsto 1 - x$ is an affine isomorphisms of C onto itself. (This is the only step where the hypothesis $[M:N] = 2$ is used.)

Thus

$$x \in \partial_e C \Leftrightarrow 1 - x \in \partial_e C .$$

Therefore, if $x \in \partial_e C$ and if $p = 1_{\{1\}}(x)$, we find $\operatorname{tr} p \geqq \tau$, since $p = 1_{\{0\}}(1 - x)$ and the Proposition 3.2 applies. However, $x \geqq p$ while

$$\operatorname{tr} x = \operatorname{tr} E_N x = \tau \leqq \operatorname{tr} p .$$

So, $x - p \geqq 0$ and $\operatorname{tr}(x - p) \leqq 0$. So $x = p$. □

Example 3.4. Let $M = M(n^2, \mathbb{C}) \simeq M(n, \mathbb{C}) \otimes M(n, \mathbb{C})$ and let $N = M(n, \mathbb{C}) \otimes 1 \subset M$. Thus,

$$N = \{[A_{ij}] : \exists A \in M(n, \mathbb{C}) \text{ with } A_{ij} = \delta_{ij} A\} \subset M(n, (M(n, \mathbb{C})) = M .$$

Then, $[M:N] = n^2$ and

$$E[A_{ij}] = [\delta_{ij} A]$$

where $A = \dfrac{1}{n} \sum_{i=1}^{n} A_{ii}$. Thus, in this example,

$$C = \left\{ [A_{ij}] \geqq 0 : \sum_{i=1}^{n} A_{ii} = \frac{1}{n} \cdot 1 \right\} .$$

Let $\{e_{ij} : 1 \leqq i, j \leqq n\}$ be the usual system of matrix units in $M(n, \mathbb{C})$, and define

$$e_0 = \frac{1}{n} [e_{ij}] .$$

Then $e_0 \in \mathscr{P}(M)$ and $E(e_0) = \dfrac{1}{n}$. It follows from [6] that if $p \in \mathscr{P}(M) \cap C$, then there exists a u in $U(n, \mathbb{C})$ such that

$$[p_{ij}] = \operatorname{diag}(u, u, \ldots, u) \left(\frac{1}{n} [e_{ij}] \right) \operatorname{diag}(u^*, u^*, \ldots, u^*)$$

which is to say

$$p_{ij} = \frac{1}{n} u e_{ij} u^* \quad \forall i, j .$$

Thus, $p = [p_{ij}] \in \mathscr{P}(M) \cap C$ implies $\operatorname{tr} p_{ij} = 0$ for $i \neq j$. [illegible] if $x = [x_{ij}]$ belongs to the convex hull of $\mathscr{P}(M) \cap C$, it must be the case that $\operatorname{tr} x_{ij} = 0$ for $i \neq j$. Since

$$a = \frac{1}{n^2} \mathbf{J} \in C$$

where $\mathbf{J}$ is the matrix with each entry being the identity matrix, the above observation, in conjunction with the Krein-Milman theorem, implies that

$$\partial_e C \underset{\neq}{\supset} C \cap \mathscr{P}(M) .$$

Thus the equality $\partial_e C(M, N) = \mathscr{P}(M) \cap C(M, N)$ need not hold in general. However, it is conceivable that perhaps this equality holds in general under the additional hypothesis that $N' \cap M = \mathbb{C}.1$. □

References

1. Arveson, W.B.: Subalgebras of C^*-algebras. Acta Math. **123**, 141–224 (1969)
2. Christensen, J.P.R., Vesterstrom, J.: A note on extreme positive definite matrices. Math. Ann. **244**, 65–68 (1979)
3. Grone, R., Pierce, S., Watkins, W.: Extremal correlation matrices. Linear Algebra Appl. **134**, 63–70 (1990)
4. Jones, V.F.R.: Index for subfactors. Invent. Math. **72**, 1–25 (1983)
5. Størmer, E.: Regular Abelian Banach algebras of linear maps of operators. J. Funct. Anal. **37**, 331–373 (1980)
6. Pimsner, M., Popa, S.: Entropy and index for subfactors. Ann. Sci. Éc. Norm. Supér., IV. Ser. **19**, 57–106 (1986)