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SEQUENTIAL ESTIMATION VIA REPLICATED PIECEWISE STOPPING NUMBER IN A TWO-PARAMETER EXPONENTIAL FAMILY OF DISTRIBUTIONS

A. BOSE
STAT-MATH DIVISION
INDIAN STATISTICAL INSTITUTE
203 B.T. ROAD
CALCUTTA 700035 INDIA

N. MUKHOPADHYAY
DEPARTMENT OF STATISTICS
UNIVERSITY OF CONNECTICUT
196 AUDITORIUM ROAD
STORRS CT 06269 U.S.A.

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Abstract

In a certain class of two-parameter exponential distributions, we consider minimum risk point estimation problems for one of the parameters. We propose to implement the sequential procedure of Bose and Boukai (1993) in smaller "pieces" along the lines of Mukhopadhyay and Sen (1993). Unlike the fully sequential procedure, one can obtain an unbiased estimator for the variance of the stopping number in a piecewise methodology. On top of this, the asymptotic second-order expansions of the *regret* functions for both the piecewise and fully sequential estimators are same up to $o(1)$ term and the piecewise sampling scheme is operationally more convenient. The effect on cost due to parallel processing is also discussed.

1. INTRODUCTION

The idea of a piecewise sequential methodology has been recently introduced in Mukhopadhyay and Sen (1993). Various associated second-order asymptotics in general and their applications in a few specific estimation problems had been incorporated in Mukhopadhyay and Sen (1993). Bose and Boukai (1993) discussed a purely sequential estimation procedure for one of the parameters in a fairly general class of two-parameter exponential family of distributions. To be specific, under a loss function that is given by a certain weighted squared error plus linear cost of sampling, Bose and Boukai (1993) introduced a purely sequential minimum risk point estimation methodology for one of the parameters in two-parameter exponential distributions and obtained asymptotic second-order characteristics of the average sample size and the "regret" function. Bose and Boukai (1993) had fully exploited and extended the machineries found in Woodroofe (1977, 1982). In this note, we examine how operational convenience can be brought inside such purely sequential techniques by invoking the ideas put forth in Mukhopadhyay and Sen (1993).

Let

$$f(x;\underset{\sim}{\theta}) = a\,(x)\exp\left[\theta_1 U_1\,(x) + \theta_2 U_2\,(x) + c(\underset{\sim}{\theta})\right], \qquad (1.1)$$

$\underset{\sim}{\theta}= (\theta_1,\theta_2)$, be a probability density function (p.d.f.), with respect to Lebesgue measure on $\mathbb{R}$, of a *regular* two-parameter exponential family of distributions. See Brown (1986). The natural parameter space Θ is defined by

$$\Theta = \left\{\underset{\sim}{\theta} \in \mathbb{R}^2 : e^{-c(\underset{\sim}{\theta})} = \int a\,(x)\exp\left[\theta_1 U_1\,(x) + \theta_2 U_2\,(x)\right] dx < \infty\right\},$$

so that Θ° =interior of Θ, which is assumed to be nonempty. It is well known that for any $\underset{\sim}{\theta} \in \Theta$, the random variable $\underset{\sim}{U}= (U_1,U_2)$ has moments of all orders. In particular, one writes

$$E_{\underset{\sim}{\theta}}(\underset{\sim}{U}) = (\mu_1,\mu_2)\,, \quad \mu_i = -\partial c(\underset{\sim}{\theta})/\partial\theta_i, \quad i = 1,2 \qquad (1.2)$$

and

$$V_{\underset{\sim}{\theta}}(\underset{\sim}{U}) = (\sigma_{ij})\,, \sigma_{ij} = -\partial^2 c(\underset{\sim}{\theta})/\partial\theta_i\partial\theta_j, \quad i,j = 1,2. \qquad (1.3)$$

Here, $V_{\underset{\sim}{\theta}}(\underset{\sim}{U})$ is the associated positive definite variance-covariance matrix.

Let X_1, ..., X_n, ... be a sequence of independent and identically distributed (i.i.d.) random variables having the p.d.f. given by (1.1). Let $T_{i:n} = \sum_{j=1}^{n} U_i\,(X_j)$ and denote by $\bar{T}_{i:n}, i = 1,2$, the usual averages. The joint distribution of $\underset{\sim}{T}_n = (T_{1:n}, T_{2:n})$ is a member of the same two-parameter exponential family (1.1) where

$$E_\theta(\underset{\sim}{T}_n) = (n\mu_1, n\mu_2), \quad V_\theta(\underset{\sim}{T}_n) = (n\sigma_{ij}), \quad i,j = 1,2. \tag{1.4}$$

Bar-Lev and Reiser (1982) considered a particular subfamily of (1.1) that is characterized by the following two conditions which we assume to hold throughout.

Assumption A1. *The parameter* θ_2 *can be represented as* $\theta_2 = -\theta_1\lambda'(\mu_2)$ *where* $\lambda'(\mu_2) = \partial\lambda(\mu_2)/\partial\mu_2$ *for some function* $\lambda(\cdot)$.

Assumption A2. $U_2(x) = h(x)$ *where* $h(x)$ *is a 1-1 function on the support of (1.1).*

The family of distributions (1.1) under the assumptions A1 and A2 is known to include the normal, gamma and inverse Gaussian distributions. The problem here is to estimate μ_2 in the presence of the nuisance parameter μ_1. In the three examples mentioned above, this problem reduces to one of estimating the mean in the presence of appropriate nuisance parameters. In what follows, we collect some of the pertinent properties associated with the density (1.1). See also Bose and Boukai (1993) for a few details. We mention that

(a) $$V(U_2(X)) = -[\theta_1\lambda'(\mu_2)]^{-1}; \tag{1.5}$$

(b) $$\begin{aligned} c(\theta_1,\mu_2) &= \theta_1[\mu_2\lambda'(\mu_2) - \lambda(\mu_2)] - G(\theta_1), \\ \mu_1 &= \lambda(\mu_2) + G'(\theta_1), \end{aligned} \tag{1.6}$$

where $G(\cdot)$ is an infinitely differentiable function for $\theta_1 \in \Theta_1$, the appropriate set;

(c) $\Theta_1 \subseteq \mathbf{R}^-$ or $\Theta_1 \subseteq \mathbf{R}^+$ and without loss of generality, we will assume that $\Theta_1 \subseteq \mathbf{R}^-$.

The minimum risk point estimation problem for μ_2 was introduced in the following way by Bose and Boukai (1993). Having recorded $X_1, \ldots, X_n$, $n \geq 2$, we have already defined $T_{i:n}$ and $\bar{T}_{i:n}$, $i = 1,2$. The maximum likelihood estimator of (θ_1, μ_2) is obtained as the solutions of

$$nG'(\theta_1) = Z_n^*, \quad \mu_2 = \bar{T}_{2:n} \tag{1.7}$$

where $Z_n^* = T_{1:n} - n\lambda\left(\bar{T}_{2:n}\right)$. Now, suppose that the loss function in estimating μ_2 by $\bar{T}_{2:n}$ is given by

$$L_n = \rho \left|\lambda''(\mu_2)\right| \left(\bar{T}_{2:n} - \mu_2\right)^2 + n \tag{1.8}$$

where the known factor $\rho (> 0)$ represents the importance of the relative squared error of estimation in comparison with the linear sampling cost. The corresponding risk is given by

$$R_n(\rho) = \rho \left(n \left|\theta_1\right|\right)^{-1} + n, \tag{1.9}$$

which is minimized if $n = n_0 = \left[\rho / \left|\theta_1\right|\right]^{\frac{1}{2}}$ and the corresponding minimum risk is given by

$$R(\rho) = R_{n_0}(\rho) = 2n_0, \tag{1.10}$$

had θ_1 been known. But, n_0 is indeed unknown and hence sequential methodologies are called for.

2. PURELY SEQUENTIAL PROCEDURE AND SOME PRELIMINARIES

For the minimum risk point estimation problem under the loss function (1.8), a purely sequential stopping rule was introduced in Bose and Boukai (1993) and here we mention only some of the highlights. One starts the experiment with $X_1, \ldots, X_m$, $m \geq 2$, and then proceeds purely sequentially by taking one sample at a time according to the stopping rule

$$N = N(\rho) = \inf\left\{n \geq m : Z_n^* a_n < nG'\left(-\rho/n^2\right)\right\}, \tag{2.1}$$

where $\{a_n : n \geq 2\}$ is a sequence of real numbers such that $a_n > 1$ and $a_n = 1 + a_0 n^{-1} + o(n^{-1})$, $a_0 \in \mathbb{R}$. The stopping variable is motivated from the expressions of n_0 and the maximum likelihood estimators. After stopping, μ_2 is estimated by $\bar{T}_{2:N}$.

Bose and Boukai (1993) proved that as $\rho \to \infty$, one has

$$N/n_0 \to 1 \ a.s., \quad E(N)/n_0 \to 1; \tag{2.2}$$

and

$$N^* = n_0^{-\frac{1}{2}}(N - n_0) \xrightarrow{\mathcal{L}} N\left(0, \left[4\theta_1^2 G''(\theta_1)\right]^{-1}\right). \tag{2.3}$$

Under additional assumptions, stated below, various asymptotic second-order characteristics of the sequential estimation rule (2.1) were also derived in Bose and Boukai (1993).

Assumption A3. For some $\alpha > \frac{1}{2}$, $\sup_{x \geq 4|\theta_1|} x^\alpha G'(-x) \leq M < \infty$.

Assumption A4. The initial sample size m is such that for some $\beta > 2(2\alpha-1)^{-1}$ and for all $\theta_1 \epsilon \Theta_1$, $E_{\theta_1}\left(Z_m^{*-\beta}\right) < \infty$.

Basically, Assumption A3 controls the left tail behavior of the underlying distribution function and Assumption A4 ensures an appropriate required initial sample size. As mentioned in Bose and Boukai (1993), in the normal, gamma and inverse Gaussian cases, one has $\alpha = 1$ and Assumption A4 is then satisfied with $m > 1 + 2\beta$.

Under Assumptions A1-A4, Bose and Boukai (1993) proved that as $\rho \to \infty$, one has

$$E(L_N)/R(\rho) \to 1; \tag{2.4}$$

$$E\left[n_0 N^{-1} I(N \le \epsilon\, n_0)\right] \to 0 \text{ for } 0 < \epsilon < 1. \tag{2.5}$$

Also known are the results such as:

If Assumption A4 is strengthened to $\beta > 3(2\alpha-1)^{-1}$, then, for some appropriate number δ,

$$E(N) = n_0 + \delta + o(1), \tag{2.6}$$

$$N^{*2} \text{ is uniformly integrable;} \tag{2.7}$$

If Assumption A4 is strengthened to $\beta > 5(2\alpha-1)^{-1}$, then

$$\begin{aligned} n_0^2 P(N \le \epsilon\, n_0) &\to 0 \quad \text{for} \quad 0 < \epsilon < 1; \\ E(L_N) - R(\rho) &= \tfrac{1}{4}\left[\theta_1^2 G''(\theta_1)\right]^{-1} + o(1). \end{aligned} \tag{2.8}$$

For specific distributions, it has been established in the literature that the distribution of N is fairly skewed to the right and hence practical implementation becomes awkward because, for certain sample paths, N can take a large value even if n_0 is of moderate magnitude. Also, one will readily notice that no unbiased estimator of variance of N is available. To circumvent these two difficulties, we propose to perform several smaller experiments in pieces and later combine these to arrive at the final estimator of μ_2. The following section provides the details of this methodology. Section 4 presents second-order characteristics of the effect on cost due to parallel processing.

3. PIECEWISE SEQUENTIAL PROCEDURE

Let $k\,(\ge 2)$ be a fixed integer and suppose that $X_{i1}, X_{i2}, \ldots$ are i.i.d. having the p.d.f. given by (1.1), $i = 1, \ldots, k$. Having observed $X_{i1}, \ldots, X_{in_i}$ for $n_i\,(\ge 2)$, let $\bar{T}_{2:n_i}$ and $Z_{in_i}^*$ be as defined in (1.7), merely depending on

$\{X_{i1}, \ldots, X_{in_i}\}$ alone, $i = 1, \ldots, k$. We write $\underset{\sim}{n} = (n_1, \ldots, n_k)$, $n = \sum_{i=1}^k n_i$, and the pooled estimator of μ_2 is

$$T_{2:\underset{\sim}{n}}^{(k)} = n^{-1} \sum_{i=1}^{k} n_i T_{2:n_i}. \tag{3.1}$$

Along the lines of (1.8), suppose that the loss function in estimating μ_2 by means of $T_{2:\underset{\sim}{n}}^{(k)}$ is given by

$$L_{\underset{\sim}{n}} = \rho \left|\lambda''(\mu_2)\right| \left(T_{2:\underset{\sim}{n}}^{(k)} - \mu_2\right)^2 + n. \tag{3.2}$$

under the standing assumptions A1-A2, $\rho\,(> 0)$ being a known constant. The associated risk $R_{\underset{\sim}{n}}(\rho)$ and the minimum risk $R(\rho)$ are respectively given by (1.9) and (1.10). Also, the optimal total fixed same size $n = n_0$ is given by the same expression $(\rho/|\theta_1|)^{\frac{1}{2}}$ as before.

We now define k separate stopping times which are all implemented at the same time, independently, as a parallel processing network. Each of the k components starts with $m\,(\geq 2)$ samples and then proceeds purely sequentially by taking one sample at a time. Define

$$N_i = N_i(\rho) = \inf\left\{n \geq m : n\left[-g\left(Z_{in}^{*} a_n\right)\right]^{\frac{1}{2}} > \rho^{\frac{1}{2}} k^{-1}\right\}, \tag{3.3}$$

$i = 1, \ldots, k$ where $g(u) = G'^{-1}(u)$. Note that the purely sequential stopping rule (2.1) is equivalent to N_1 with $k = 1$. Once all of the k separate and independent sequential processes given by (3.3) stops, the experimenter has available $X_{i1}, \ldots, X_{iN_i}$, $i = 1, \ldots, k$ and μ_2 is finally estimated by the combined estimator $T_{2:\underset{\sim}{N}}^{(k)}$ where $\underset{\sim}{N} = (N_1, \ldots, N_k)$. The total sample size required by this piecewise sequential methodology is given by $N = \sum_{i=1}^k N_i$. Since N_i's are i.i.d., variance of N can be unbiasedly estimated by

$$\Delta = k(k-1)^{-1} \sum_{i=1}^{k} \left(N_i - \bar{N}\right)^2 \tag{3.4}$$

where $\bar{N} = N/k$. Also, suppose that the ith stopping rule in (3.3) takes the time t_i' say, in certain unit, $i = 1, \ldots, k$. Here, because one exploits the notion of parallel processing, the piecewise methodology seems to come up with the final estimator in a time, $t^0 = \max\{t_1', \ldots, t_k'\}$. The existing literature shows that the original purely sequential procedure (2.1) will do this same job in a time that exceeds $t^1 = \sum_{i=1}^k t_i'$, and t^0 can indeed be substantially smaller than t^1. Hence, it will be quite pertinent to evaluate $E(L_{\underset{\sim}{N}})$ and other relevant characteristics of the piecewise methodology (3.3). We summarize our findings in the following subsection.

3.1. Asymptotic Properties

From Theorem 1 of Bose and Boukai (1993), note that $I\ (\underset{\sim}{N}=\underset{\sim}{n})$ is independent of $T_{2:\underset{\sim}{n}}^{(k)}$ for all fixed $\underset{\sim}{n}$ and hence the risk function associated with $T_{2:\underset{\sim}{N}}^{(k)}$ is given by

$$\begin{aligned} E(L_{\underset{\sim}{N}}) &= \rho\,|\theta_1|^{-1}\,E\,(N^{-1}) + E\,(N) \\ &= n_0^2 E\,(N^{-1}) + E\,(N) \qquad (3.5) \end{aligned}$$

where $N = \sum_{i=1}^{k} N_i$. The *regret* associated with the final estimator $T_{2:\underset{\sim}{N}}^{(k)}$ is then given by

$$E(L_{\underset{\sim}{N}}) - R\,(\rho) = E\left\{(N-n_0)^2/N\right\}. \qquad (3.6)$$

Now, we summarize the main results.

Theorem 1. *For the piecewise sequential methodology (3.3), under the loss function given by (3.2), we have:*

i) $n_0^{-\frac{1}{2}}\ (N-n_0) \xrightarrow{\mathcal{L}} N\left(0, [4\theta_1^2 G''(\theta_1)]^{-1}\right)$ *as* $\rho \to \infty$, *under Assumptions A1-A2;*

ii) $E(L_{\underset{\sim}{N}})/R\,(\rho) \to 1$ *as* $\rho \to \infty$, *under Assumptions A1-A4;*

iii) $E\,(N) = n_0 + k\delta + o\,(1)$ *as* $\rho \to \infty$, *under Assumptions A1-A3 and Assumption A4 strengthened to* $\beta > 3\,(2\alpha-1)^{-1}$;

iv) $E(L_{\underset{\sim}{N}}) - R\,(\rho) = \frac{1}{4}\,[\theta_1^2 G''(\theta_1)]^{-1} + o\,(1)$ *as* $\rho \to \infty$ *under Assumptions A1-A3 and Assumption A4 strengthened to* $\beta > 5\,(2\alpha-1)^{-1}$.

Proof. Part (ii) follows along the lines of Bose and Boukai (1993). From (2.6), it immediately follows that

$$E\,(N_i) = n_i^* + \delta + o\,(1) \qquad (3.7)$$

where $n_i^* = k^{-1}n_0$, under Assumptions A1-A3 and Assumption A4 strengthened to $\beta > 3\,(2\alpha-1)^{-1}$. Now, part (iii) follows from (3.7).

In order to prove part (i), let $N_i^* = (N_i - n_i^*)\,/n_i^{*\frac{1}{2}}$, $i = 1, \ldots, k$. Note that $N_1^*, \ldots, N_k^*$ are independent and $N_i^* \xrightarrow{\mathcal{L}} N\left(0, [4\theta_1^2 G''(\theta_1)]^{-1}\right)$ as $\rho \to \infty$, in view of (2.3). Hence, $(N_1^*, \ldots, N_k^*) \xrightarrow{\mathcal{L}} N_k\left(\underset{\sim}{0}, [4\theta_1^2 G''(\theta_1)]^{-1}\, I_{k\times k}\right)$ as $\rho \to \infty$. Thus, the multivariate central limit theorem implies that

$$\left(N - n_0\right)/n_0^{\frac{1}{2}} = k^{-\frac{1}{2}}\sum_{i=1}^{k} N_i^{\bullet} \xrightarrow{\mathcal{L}} N\left(0, \left[4\theta_1^2 G''(\theta_1)\right]^{-1}\right) \tag{3.8}$$

as $\rho \to \infty$, under "appropriate" assumptions.

In order to prove part (iv), we first show that $(N - n_0)^2/n_0$ is uniformly integrable. Observe that

$$\left(N - n_0\right)^2/n_0 = k^{-1}\sum_{i=1}^{k} N_i^{\bullet 2} + 2n_0^{-1}\sum_{1\le i<j\le k}\left(N_i - n_i^{\bullet}\right)\left(N_j - n_j^{\bullet}\right)$$

and hence, under "appropriate" assumptions, in view of (2.7) and (3.7), we write

$$\begin{aligned} E\left[\left(N - n_0\right)^2/n_0\right] &= k^{-1}k\left[4\theta_1^2 G''(\theta_1)\right]^{-1} + n_0^{-1}k(k-1)\left(\delta^2 + o(1)\right) + o(1) \\ &= \left[4\theta_1^2 G''(\theta_1)\right]^{-1} + o(1). \end{aligned} \tag{3.9}$$

From (3.8), we now conclude that $(N - n_0)^2/n_0$ is uniformly integrable. Now, one can utilize (3.6) and the basic techniques of Bose and Boukai (1993) to complete the proof of part (iv).

Remark 3.1. It is indeed most interesting to note that the regrets of the fully sequential estimator and the piecewise sequential estimator are exactly same, up to the second-order term, asymptotically. On the other hand, the piecewise methodology can be theoretically very attractive. Recall the discussions given after equation (3.4) in this context as well.

Remark 3.2. $\lim_{\rho\to\infty} E(N - n_0) = \delta$ or $k\delta$ provided that one uses a fully sequential estimator or the piecewise sequential estimator, respectively. Here, one sees some "negative" aspect associated with the piecewise scheme. However, "δ" is usually small in magnitude and hence "$k\delta$" will not be too large in magnitude if "k" is chosen moderately. The overall practical benefits obtained via piecewise methodology certainly outweighs this slight potential negative feature. We address this aspect a little further in the following section.

4. SECOND-ORDER APPROXIMATION FOR THE EFFECT DUE TO PARALLEL PROCESSING

Usually the cost of processing j units in a parallel network is smaller than j times the cost of processing a single unit. Motivated by this, we define

$$c_j = \text{ cost of processing } j \text{ units in a parallel network.}$$

Note that the possible values of j are 1, ..., k and it is assumed that c_j/j is a decreasing function of j. In order to carry out the cost analysis associated with our piecewise sequential methodology, let $N_{(1)} \leq \cdots \leq N_{(k)}$ be the order statistics corresponding to the stopping variables $N_1, \cdots, N_k$. For the moment, let us suppose that there are no ties. In many situations, it will be quite appropriate to quantify the total cost of processing (ignoring ties) as

$$C = c_k N_{(1)} + c_{k-1}\left(N_{(2)} - N_{(1)}\right) + \cdots + c_1\left(N_{(k)} - N_{(k-1)}\right).$$

Defining $\ell_j = c_j - c_{j-1}, c_0 = 0$, we may rewrite

$$C = \sum_{j=1}^{k} N_{(j)}\ell_j. \tag{4.1}$$

Recall that $(N_1^*, \ldots, N_k^*) \xrightarrow{\mathcal{L}} N_k\left(\underset{\sim}{0}, \sigma^2 I_{k\times k}\right)$ where $\sigma^2 = \left[4\theta_1^2 G''(\theta_1)\right]^{-1}$. Let $C^* = (C - n_0^* c_k)/n_0^{*\frac{1}{2}}$ and thus $C^* = \sum_{j=1}^{k}\left(N_{(j)} - n_0^*\right)\ell_j/n_0^{*\frac{1}{2}}$. Hence

$$C^* \xrightarrow{\mathcal{L}} Y_k \text{ as } \rho \to \infty \tag{4.2}$$

where $Y_k = \sum_{j=1}^{k} Z_{(j)}\ell_j$, Z's being i.i.d. $N(0, \sigma^2)$ and $Z_{(j)}$'s the corresponding order statistics. Let $\gamma_k(\sigma^2) = E(Y_k)$. Now, for $\beta > 3(2\alpha - 1)^{-1}$, we can claim that C^* is also uniformly integrable along the lines of Bose and Boukai (1993). In other words, if Assumption A4 is strengthened to $\beta > 3(2\alpha - 1)^{-1}$, then $E(C^*) = \gamma_k(\sigma^2) + o(1)$, that is

$$E(C) = k^{-1}c_k n_0 + \gamma_k(\sigma^2)(n_0 k^{-1})^{\frac{1}{2}} + o(n_0^{\frac{1}{2}}). \tag{4.3}$$

We note here that the effect of ties on the above expectation is of the order $o(n_0^{\frac{1}{2}})$, since the probability of ties is negligible as $\rho \to \infty$.

The corresponding second-order expansion in the case of the purely sequential scheme is given by

$$E(c_1 N) = c_1 n_0 + c_1\delta + o(1). \tag{4.4}$$

The point is that one should indeed compare this with (4.3) in order to get the proper perspectives.

From Remark 2 in Bose and Boukai (1993), it follows that (4.3) holds if $m \geq 12$, in the case of normal, gamma and inverse Gaussian distributions.

As such, our results are given for the general family (1.1). In specific special circumstances, the sufficient condition "$m \geq 12$" may possibly be improved substantially. For example, in the normal case, for the corresponding piecewise methodology, $|N_i^*|$ is uniformly integrable if $m \geq 2$ and hence (4.3) will hold if $m \geq 2$. This observation has direct bearing on the point estimation problem considered in Mukhopadhyay and Sen (1993). The expansion given in (2.6), however, holds in the normal case, if $m \geq 3$.

Remark 4.1. One may obtain expressions of $\gamma_k(\sigma^2)$ from Section 6.1.3 in Tong (1990). Other references can also be found in David (1981).

Bibliography

Bar-Lev, S.K. and Reiser, B. (1982). An exponential subfamily which admits UMPU test based on a single test statistic. *Ann. Statist.*, *10*, 979-989.

Bose, A. and Boukai, B. (1993). Sequential estimation results for a two-parameter exponential family of distributions. *Ann. Statist.*, *21*, 484-502.

Brown, L.D. (1986). *Fundamentals of Statistical Exponential Families*. IMS Lecture Notes, Vol. 9, S.S. Gupta, ed.

David, H.A. (1981). *Order Statistics*, 2nd edition. John Wiley & Sons, Inc., New York.

Mukhopadhyay, N. and Sen, P.K. (1993). Replicated piecewise stopping numbers and sequential analysis. *Sequential Anal.*, *12*, 179-197.

Tong, Y.L. (1990). *The Multivariate Normal Distribution*. Springer Verlag, Inc., New York.

Woodroofe, M. (1977). Second order approximations for sequential point and interval estimation. *Ann. Statist.*, *5*, 984-995.

Woodroofe, M. (1982). *Nonlinear Renewal Theory in Sequential Analysis*. SIAM Publications #39, Philadelphia.

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