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SOME SIMPLE APPROXIMATIONS FOR THE DOUBLY NONCENTRAL *z* DISTRIBUTION

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Summary

We give two simple approximations for evaluating the cumulative probabilities of the doubly noncentral z distribution. These can easily be used for evaluating the cumulative probabilities of the doubly noncentral F distribution as well. We compare our results with those obtained by Tiku (1965) using series expansion. An industrial situation where a quality characteristic of interest follows the doubly noncentral z distribution is also cited. However, in this case the exact probabilities could be calculated using results on the ratio of two normal variables.

Key words: Doubly noncentral z distribution; cumulative probabilities; approximations; quality control problem; ratio of normal variables.

1. Introduction

The distribution of $F'' = (X_1'^2/m_1)/(X_2'^2/m_2)$ where $X_1'^2$ and $X_2'^2$ are two independent noncentral chi-square variables with degrees of freedom m_1, m_2 and noncentrality parameters ℓ_1, ℓ_2 respectively, is called the doubly noncentral F distribution. Then the variable $z'' = \frac{1}{2}\log F''$ is said to follow the doubly noncentral z distribution with parameters $(m_1, m_2; \ell_1, \ell_2)$.

Tiku (1965) has given series expansions for the tail probabilities of the distribution of F''. The computation of these functions, though straightforward, is laborious and requires the help of incomplete Beta functions. In this paper we give two simple approximations for the cumulative probabilities of the z'' distribution which are fairly accurate. We also compare

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our estimates with Tiku's. Tiku (1974) tabulated the doubly noncentral F distribution for even m_2.

We recently encountered an industrial situation where the quality characteristic of interest follows the doubly noncentral z distribution. Because of the special nature of this case, the exact probabilities could be calculated using results on the ratio of normal variables. The details of this quality control problem and its solution are given in §4.

2. Notation and Some Previous Results

Using the formulation given by Johnson & Kotz (1980, Chap.30, p.197) the probability density function $f(\cdot)$ of z'' can be written as

$$f(z) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} e^{-\frac{1}{2}(\ell_1+\ell_2)} \frac{(\frac{1}{2}\ell_1)^j(\frac{1}{2}\ell_2)^k}{j!k!} \frac{c(j,k)(e^{2z})^{\frac{1}{2}m_1+j}}{(m_2+m_1e^{2z})^{\frac{1}{2}(m_1+m_2)+j+k}}$$

where

$$c(j,k) = \frac{2m_1^{\frac{1}{2}m_1+j}\, m_2^{\frac{1}{2}m_2+k}}{B(\frac{1}{2}m_1+j, \frac{1}{2}m_2+k)}$$

and $B(\cdot,\cdot)$ is the Beta function.

Tiku (1965) gave four types of series expansions for the tail probabilities of F''. He recommended series I for ℓ_1 and ℓ_2, both small, series II and III for large ℓ_1 and small ℓ_2, and series IV for ℓ_1 and ℓ_2 both large. He used only series I and II for calculating approximations. He noted that the use of series IV is of little practical use since the labour in evaluating the incomplete beta function by three way interpolation is formidable.

Price (1964) showed how to evaluate the exact cumulative probabilities of F'' distribution when $m_1 = m_2 = 1$, using bivariate normal probabilities. However, for large values of ℓ_1 and ℓ_2 this requires approximations for the bivariate normal probability integral of the type

$$V(u,v) = \frac{1}{2\pi}\int_0^u e^{-x^2/2}\,dx \int_0^{vx/u} e^{-y^2/2}\,dy$$

and use its limiting value as u or v tends to infinity.

3. Approximations

3.1. Normal Approximation

Observe that $X_i'^2$ has mean $(m_i + \ell_i)$ and variance $2(m_i + 2\ell_i)$. Using Taylor's expansion we get

$$\mathrm{var}(z'') = \tfrac{1}{2}\left[\frac{m_1 + 2\ell_1}{(m_1 + \ell_1)^2} + \frac{m_2 + 2\ell_2}{(m_2 + \ell_2)^2}\right], \tag{3.1}$$

$$\begin{aligned} \mathrm{E}(z'') &= \mathrm{E}\left[\tfrac{1}{2}\log(X_1'^2/m_1) - \tfrac{1}{2}\log(X_2'^2/m_2)\right] \\ &= \tfrac{1}{2}\left[\log\left(\frac{m_2}{m_1}\right) + \log\left(m_1 + \ell_1 - \frac{m_1 + 2\ell_1}{2(m_1 + \ell_1)}\right)\right. \\ &\qquad \left. - \log\left(m_2 + \ell_2 - \frac{m_2 + 2\ell_2}{2(m_2 + \ell_2)}\right)\right]. \end{aligned} \tag{3.2}$$

We assume z'' to be approximately normal with the above mean and variance. Thus

$$\Pr\{z'' \le z_0\} \approx \Phi\left[(z_0 - \mathrm{E}(z''))/(\mathrm{var}(z''))^{\frac{1}{2}}\right], \tag{3.3}$$

where $\Phi(y) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{y} e^{-\frac{1}{2}y^2}\, dy$.

3.2. Laubscher Type Approximation

Laubscher (1960) found a simple type of approximation to be highly satisfactory for evaluating the cumulative probabilities of a noncentral F distribution. He considered $(2X_i'^2)^{\frac{1}{2}}$ to be approximately normal and used results on the ratio of two normal variables to obtain the cumulative probabilities. We refer to Hinkley (1969) for more general results on the ratio of two normal variables. Using this result, since $\Pr\{(2X_2'^2)^{\frac{1}{2}} \le 0\} \approx 0$ we get

$$\Pr\{z'' \le z_0\} \approx \frac{W_0 u_2 - u_1}{(W_0\sigma_2^2 - \sigma_1^2)^{\frac{1}{2}}} \tag{3.4}$$

where $u_i = \left[2(m_i+\ell_i) - (m_i+2\ell_i)/(m_i+\ell_i)\right]^{\frac{1}{2}}$, $\sigma_i^2 = (m_i+2\ell_i)/(m_i+\ell_i)$, $(i = 1, 2)$ and $W_0 = [\exp(2z_0)]^{\frac{1}{2}}(m_1/m_2)^{\frac{1}{2}}$.

Table 1 gives the values of $\Pr\{z'' \le z_0\}$ computed using the above two approximations together with some of the values obtained by Tiku. It can be seen that for the range of parameter values considered, both our approximations work fairly well, and are simple to compute.

TABLE 1

Values of $\Pr\{z'' \leq z_0\}$ *computed using our approximations and values obtained by Tiku*

m_1	m_2	ℓ_1	ℓ_2	z_0	Normal approx. (3.3)	Laubscher approx. (3.4)	Tiku's Series I	Tiku's Series II	Tiku's Exact*
6	8	0.25	6.25	−0.1438	0.6510	0.6480	0.6555	NA	0.6525
5	7	0.25	6.25	−0.1194	0.6932	0.6900	0.6976	NA	0.6944
10	12	0.25	0.5	−0.1116	0.3714	0.3752	NA	NA	
2	8	8	2	0.7474	0.5722	0.5693	NA	0.5747	
4	8	16	3	0.6724	0.5251	0.5243	NA	0.5227	
6	8	12	1	0.6378	0.6641	0.6576	NA	0.6551	
6	8	24	5	0.6378	0.5874	0.5847	NA	0.5818	
2	2	12	14	−0.1116	0.4542	0.4558	NA	NA	
3	3	50	60	−0.1116	0.4484	0.4497	NA	NA	

* The exact values considered by Tiku.

4. The Quality Control Problem and Its Solution

A telecommunication cable carries telecommunication signals. It is basically made of a number of pairs of copper conductor wires of specific sizes. One of the most important quality characteristics of such cables for satisfactory performance is called the resistance unbalance (RU). RU can be expressed as $\frac{1}{2}\log(d_1^2/d_2^2)$ where d_1 and d_2 are the diameters of the two conductors of a pair. Records show that the diameters follow the same normal distribution, as the conductor wires come from the same wire drawing process. Hence each d_i^2 follows a noncentral chi-square distribution and d_1^2/d_2^2 follows a doubly noncentral F distribution with $m_1 = m_2 = 1$ and $\ell_1 = \ell_2 = \mu^2/\sigma^2$ where $d_i \sim N(\mu, \sigma^2)$. Therefore RU follows a doubly noncentral z distribution. (We remark that Price (1962) encountered the doubly noncentral F distribution in engineering problems in the context of information theory.) It is required that for an n-pair cable, the RU for each pair must lie between −.04 and +.04, where n can take any of the values 20, 50, 100, 200 and 400.

Given μ and σ^2 the problem is therefore to find

$$\left[\Pr\{-.04 \leq RU \leq .04\}\right]^n.$$

In this case $\mu = .500$ and $\sigma = .002$. It is possible to use Price's result to calculate this probability. However the approximations for large ℓ (in this case 62500) introduce inaccuracies, e.g., $\Pr\{RU \leq -.04\} \approx -.0127$, so we do not use this.

Using Hinkley's results, write $d_1/d_2 = W$, where $d_i \sim N(\mu, \sigma^2)$ ($i = 1, 2$) and since $\Pr\{d_2 \leq 0\} = 0$,

$$\Pr\{z \leq z_0\} = \Pr\left\{\frac{d_1}{d_2} \leq \left[e^{2z_0}\right]^{\frac{1}{2}}\right\} \approx \Phi\left(\frac{\mu(W_0 - 1)}{\sigma(W_0^2 + 1)^{\frac{1}{2}}}\right)$$

where $W_0 = \left[\exp(2z_0)\right]^{\frac{1}{2}}$. In this case $\Pr\{z \leq .04\} \approx \Phi(7.07)$. Hence for all the values of n, the requirement on RU is satisfied.

Our approximations lead to the same value, namely, $\Phi(7.07)$.

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