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ON THE BOREL HIERARCHIES OF COUNTABLE PRODUCTS OF POLISH SPACES

1. Introduction. Let X be an uncountable Polish (complete, separable metric) space and let $H = X^{\omega}$. Equip H with the product of discrete topologies and also with the product of copies of the Polish topology. The former topology will be called the d-topology and the latter the p-topology (which is known to be Polish). Topological properties with respect to these topologies will carry the prefix d and p as the case may be. The d- topology on H gives rise to two hierarchies (of Borel sets) defined as follows.

Put $$\Sigma_0 = \Pi_0 = \left\{ A \subseteq H : A \text{ is } d\text{-clopen} \right\},$$

and inductively define for $\mu < \omega_1$,

$$\Sigma_\mu = \Big(\bigcup_{\nu<\mu} \Pi_\nu \Big)_\sigma;$$
$$\Pi_\mu = \{ A : A^c \in \Sigma_\mu \}.$$

Denoting the Borel σ-field on H with respect to p by $\mathfrak{B}$, we define

$$\Sigma_0^* = \Pi_0^* = \left\{ A \in \mathfrak{B} : A \text{ is } d\text{-clopen} \right\};$$
$$\Sigma_\nu^* = \Big(\bigcup_{\nu<\mu} \Pi_\nu^* \Big)_\sigma;$$
$$\Pi_\mu^* = \{ A : A^c \in \Sigma_\mu^* \}.$$

It is not hard to check (as pointed out to us by a referee) that

$$\mathfrak{B} = \bigcup_{\mu<\omega_1} \Sigma_\mu^* .$$

A.Maitra [2] asked whether

(I) $$\Sigma_\mu^* = \Sigma_\mu \cap \mathfrak{B} \qquad \text{for } 0 \leq \mu < \omega_1.$$

In connection with (I) he made the following conjecture (which he proved for the case $\mu = 1,2$).

(II) Suppose A and B are two analytic (Σ_1^1) subsets of H such that A can be separated from B by a Σ_μ set, $0 \leq \mu < \omega_1$. Then there is a Σ_μ^* set which separates A from B.

Observe that trivially (II) implies (I). In this short note we shall show that under a certain category- theoretic assumption, (II) is true for all $\mu < \omega_1$. We shall obtain this as a simple consequence of results and tecniques of Louveau developed in [1]. The relevent definitions and results are reviewed in the next section.

Note that the above assumption is not consistent with ZFC. However, it holds in the Lévy- Solovay model [4].

2. Regular and Separating Families of Sets. We shall use standard notation and terminology from effective descriptive set theory as found in Moschovakis [3]. All unexplained notation and terminology are from [3].

Definition 1. Let X be a recursively presentable (r.p.) space. By a coding pair we shall mean a pair $< W^X, C^X >$ (or $< W, C >$ when X is clear from the context) such that

(i) W is a Π_1^1 subset of $\omega^\omega \times \omega$.

(ii) C is a Π_1^1 subset of $X \times \omega^\omega \times \omega$ whose projection on $\omega^\omega \times \omega$ is W and such that the relation

$$(\alpha, n) \in W \;\&\; (x, \alpha, n) \notin C$$

is Π_1^1.

(*iii*) For each α, $\{ C_{\alpha,n} : n \in \omega, (\alpha, n) \in W \}$ is precisely the class of all $\Delta_1^1(\alpha)$ subsets of X. Observe that such a coding pair exists (*cf.*[1; p14]).

Definition 2 (*Louveau*). A family Φ of subsets of an r.p. space X is said to be *separating* with parameter $\alpha_0 \in \omega^\omega$, if it satisfies the following two conditions:

(i) The set $W_\Phi \overset{def}{=} \{ (\alpha,n) \in W : C\alpha,n \in \Phi \}$ is a $\Pi^1_1(\alpha_0)$ set .

(ii) If A_1 and A_2 are two $\Sigma^1_1(\alpha)$ subsets of X and if there is a $B \in \Phi$ which separates A_1 from A_2, then there exists a $\Delta^1_1(<\alpha_0,\alpha>)$ set in Φ which separates A_1 from A_2.

Definition 3 (*Louveau*). Let Φ be a family of subsets of a r.p. space X and $\alpha \in \omega^\omega$. The *separating kernel of* Φ *of order* α, written $S_\alpha(\Phi)$, is the family of $\Sigma^1_1(\alpha)$ subsets of X which can be separated from *every* disjoint $\Sigma^1_1(\alpha)$ set by a set which is $\Delta^1_1(\alpha)$ and in Φ

Clearly, $\Delta^1_1(\alpha) \cap \Phi \subseteq S_\alpha(\Phi)$; and if Φ is separating, then $\Sigma^1_1(\alpha) \cap \Phi \subseteq S_\alpha(\Phi)$.

Notation. For each r.p. space X , let $T^X(\alpha)$ denote the topology generated by the $\Sigma^1_1(\alpha)$ subsets of X. We shall often drop the superscript when it is clear from the context. *Note that if* X *is a product space the* $T^X(\alpha)$ *is not the product topology. In what follows , unless explicitly mentioned, we shall always use this topology and not the product topology when* X *is a product space.* For $A, B \subseteq X$, we write $A \sim_\alpha B$ iff $A \triangle B$ ($= (A-B) \cup (B-A)$) is $T(\alpha)$ - meager.

Definition 4 (*Louveau*). Let Φ be a family of subsets of an r.p. space X. Then Φ is said to be *regular* with parameter $\alpha_0 \in \omega^\omega$, if it satisfies the following two properties:

(i) The set W_Φ (*cf.* Definition 2) is $\Pi^1_1(\alpha_0)$.

(*ii*) *Property of regularity*: For every real α and for every set $E \in \Phi$, there is a sequence $\{ A_n : n \in \omega \} \subseteq S_{<\alpha_0,\alpha>}(\Phi)$ such that

$$E \sim_{<\alpha_0,\alpha>} (\cup_n A_n).$$

Theorem 1 (*Louveau*). For each $\mu < \omega_1$, define a family Φ_μ by the recursion:

$$\Phi_0 = \Phi ;$$
$$\Phi_{\mu+1} = (\Phi_\mu)_{\sigma c} ;$$
$$\Phi_\lambda = \bigcup_{\mu<\lambda} \Phi_\mu , \text{ if } \lambda \text{ is limit.}$$

Then, each Φ_μ is regular (and for $\mu > 0$, separating) if Φ is regular, with parameter $< \alpha_0,\alpha_\mu >$, where α_0 is the parameter of Φ and $\alpha_\mu \in WO$ is a real such that $|\alpha_\mu| = \mu$.

Moreover, let $\phi(\alpha, n, E)$ be the following set relation:

$$\phi(\alpha, n, E) \longleftrightarrow$$
$$\Big[(\alpha,n)\in W\Big]\&\Big[\exists\beta\in\Delta^1_1(<\alpha_0,\alpha>)(\forall m)\Big[(<\alpha_0,\alpha>,\beta(m))\in E \;\&\; C_{\alpha,n} = X - \bigcup_m C_{<\alpha_0,\alpha>,\beta(m)}\Big]\Big]$$

Plainly, ϕ is $\Pi^1_1(\alpha_0)$—monotone which defines inductively a sequence W^μ_Φ by

$$W^0_\Phi = W_\Phi ;$$
$$W^{\mu+1}_\Phi = \Big\{(\alpha, n) : \phi(\alpha, n, W^\mu_\Phi)\Big\};$$
$$W^\lambda_\Phi = \bigcup_{\mu<\lambda} W^\mu_\Phi , \; \textit{if } \lambda \text{ is limit.}$$

Then, for each ordinal μ,

$$W_{\Phi_\mu} = W^\mu_\Phi .$$

(For a proof of this theorem see [1]).

Proof of **Conjecture (II)** . *Without loss of generality, we assume that* $X = \omega^\omega$; the result for Polish spaces can be obtained by standard transfer theorems. We shall identify $X^n \times X^\omega$ with X^ω for each integer $n \geq 1$.

We first make the following easy but important observation.

Proposition. Any set $A \in \Sigma_1$ is of the form $A = \bigcup_{n\geq 1} A_n$, with each $A_n = A'(n) \times X^\omega$ with $A'(n) \subseteq X^n$.

Thus Σ_1 precisely consists of sets of the above type.

Proof. Let $A'(n) = \{ (x_1,\cdots, x_n) : \Sigma(x_1,\cdots, x_n) \subseteq A \}$, where $\Sigma(x_1,\cdots, x_n) = \left\{ \mathbf{x} \in X^\omega : (\mathbf{x})_1 = x_1, \ldots, (\mathbf{x})_n = x_n \right\}$. Clearly, $A'(n) \times X^\omega \subseteq A$ for every n. Now suppose $\mathbf{x} = (x_1, x_2,\cdots) \in A$. Since A is d-open, there is $n \geq 1$ such that

$$\mathbf{x} \in \Sigma(x_1,\cdots, x_n) \subseteq A.$$

Clearly $(x_1,\cdots, x_n) \in A'(n)$ and hence $\mathbf{x} \in A'(n) \times X^\omega$. □

Let $\Phi_0 = \{ A'(n) \times X^\omega : n > 0 \ \& \ A'(n) \subseteq X^n \}$ and inductively define Φ_μ as in **Theorem** 1. It is not hard to see that

$$\Pi_\mu = \begin{cases} \Phi_\mu & \text{if } 1 \leq \mu < \omega_0 ; \\ \Phi_{\mu+1} & \text{if } \mu \geq \omega_0 . \end{cases}$$

Now consider the following statement:

(𝔓) For each $n \geq 1$, every subset A of $(\omega^\omega)^n$ has the Baire property relative to $T(\alpha)$ for every α.

Observe that under the Axiom of Choice statement (𝔓) is not true. However, in the Lévy-Solovay model [4], it can be shown that (𝔓) holds (*cf.* [1]). As pointed out to us by a referee AD implies (𝔓)— this can be proved by playing the Banach - Mazur game with $\Sigma^1_1(\alpha)$ sets.

We now prove

Lemma. Assume that (𝔓) holds. Then the family Φ_0 is regular without parameter.

Proof. Let $< W^m, C^m >$ be (uniformly in m) a coding pair for X^m and let $<W, C>$ be a coding pair for X^ω. Observe that $A'_{(n)} \subseteq X^n$ is $\Delta^1_1(\alpha)$ in X^n iff $A'_{(n)} \times X^\omega$ is $\Delta^1_1(\alpha)$ in X^ω . Hence

$$(\alpha, n) \in W_{\Phi_0}$$

$$\longleftrightarrow \quad (\alpha, n) \in W \ \& \ C_{(\alpha, n)} \in \Phi_0$$

$$\longleftrightarrow \quad (\alpha, n) \in W \ \& \ (\exists m)(\exists k) \{ (\alpha, k) \in W^m \ \& \ C^m_{\alpha, k} \times X^\omega = C_{\alpha, n} \}.$$

It is easy to check that W_{Φ_0} is Π^1_1 . Next observe that if $E \subseteq X^n$ is $\Sigma^1_1(\alpha)$, then $E \times X^\omega$ is in $S_\alpha (\Phi_0)$. This follows from the Suslin-Kleene Theorem(*cf.* [3]).

Now, fix α and let $A \in \Phi_0$. Then for some n , $A = A'_{(n)} \times X^\omega$, where $A'_{(n)} \subseteq X^n$. Since ($\mathfrak{P}$) holds, there is a sequence $\{ E_k \}$ of $\Sigma^1_1(\alpha)$ subsets of X^n such that $A'_{(n)} \sim_\alpha (\bigcup_k E_k)$. Hence,

$$(A'_{(n)} \times X^\omega) \Delta (\bigcup_k (E_k \times X^\omega)) = (A'_{(n)} \Delta (\bigcup_k E_k)) \times X^\omega$$

is meager relative to $T(\alpha)$ by *Lemma* 2.13 of [1]. But each $E_k \times X^\omega \in S_\alpha (\Phi_0)$ by the observation above. Thus A has the regularity property.

This completes the proof.

Theorem 2. Assume ($\mathfrak{P}$). Suppose A_1 and A_2 are two Σ^1_1 subsets of H such that A_1 can be separated from A_2 by a $\mathbf{\Pi}_\mu$ set with $\mu \geq 1$. Then A_1 can be separated from A_2 by a $\mathbf{\Pi}^*_\mu$ set.

In other words, Conjecture (II) is true for all $\mu < \omega_1$.

Proof. We shall prove this by induction on μ. The result for $\mu = 1$ is known and can be easily proved. So assume $\mu > 1$ and fix z such that $\mu < \omega^z_1$ and A_1, A_2 are $\Sigma^1_1(z)$ subsets of H.

Now observe that, by the **Lemma** and **Theorem 1**, $\mathbf{\Pi}_\mu$ is a separating family with parameter α_μ (which can be chosen to be recursive in z). Hence there is a set B separating A_1 from A_2 such that B is $\Delta_1^1(<\alpha_\mu, z>)$ and in Π_μ. Fix n such that $(<\alpha_\mu, z>, n) \in W^H$ and $C^H_{<\alpha_\mu, z>, n} = B$. Plainly, $(<\alpha_\mu, z>, n) \in W_{\Pi_\mu} = W^{\mu+1}_{\Phi_0}$, by Theorem 1. (We assume for simplicity that $\mu \geq \omega$).

Hence there exists $\beta \in \Delta_1^1(<\alpha_\mu, z>)$ such that

$$(\forall m)\Big[(<\alpha_0,\alpha>,\beta(m))\in W^{\mu}_{\Phi_0}\Big] \;\&\; C^H_{<\alpha_\mu, z>, n} = H - \bigcup_m C^H_{<\alpha_\mu, z>, \beta(m)} .$$

Write $B_m = C^H_{<\alpha_\mu, z>, \beta(m)}$ for each m. Clearly, each B_m is $\Delta_1^1(<\alpha_\mu, z>)$ and in Π_{η_m} for some $\eta_m < \mu$. By induction hypothesis, $B_m \in \Pi^*_{\eta_m}$ and hence $\bigcup_m B_m \in \left(\bigcup_{\eta<\mu} \Pi^*_\eta\right)_\sigma = \Sigma^*_\mu$. Thus $B = H - \bigcup B_m$ is a Π^*_μ set which separates A_1 from A_2. This completes the proof.

As an immediate consequence we have

Corollary. Assume ($\mathfrak{P}$). Then, for $\mu < \omega_1$,

$$\Sigma^*_\mu = \Sigma_\mu \cap \mathfrak{B} .$$

Remark. As remarked earlier, the statement ($\mathfrak{P}$) holds in the Lévy-Solovay model. Consequently, both Conjectures (I) and (II) are true in that model.

Postscript. V.V. Srivatsa (unpublished) has proved (I) in ZFC.

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Martin's Axiom implies a stronger version of Blumberg's Theorem

Let **R** be the real line. In 1922, H. Blumberg proved the following theorem: **Blumberg's Theorem** [Bl]: If f:**R**→**R**, then there is a dense subset D of **R** such that f|D is continuous. Here, f|D is the real valued function on D with the subspace topology.

In any such theorem, it is of interest to ask how much the hypothesis can be weakened or the conclusion strengthened. The obvious way to weaken the hypothesis is to allow the domain of f to be some subset of **R** instead of **R**. A set $X \subseteq Y$ is **categorically dense** in Y if $X \cap U$ is of second category in Y for every nonempty open subset U of Y. Trivial modifications in the proof of Blumberg's Theorem then give the following strengthening:

Proposition: If X is a categorically dense subset of **R**, and f:X→**R**, then there is a dense $D \subseteq X$ such that f|d is continuous.

If every point of $X \subseteq$ **R** is isolated, then the same result holds trivially. For similar reasons, this is also true if X is scattered (just let D be the set of isolated points of X). However, if X is dense, it is easy to see that the hypothesis cannot be weakened any further, for if $X \subseteq$ **R** is dense and of first category, partition X into countably many sets X_n, $n<\omega$, each nowhere dense. Let $f(x)=n$ iff $x \in X_n$, and f obviously cannot be continuous on any dense subset. If X is dense and of second category, but not categorically dense, the same trick can be used on $X \cap I$ for some interval I, letting f be constant outside I.

Thus, for dense X, X being categorically dense is both necessary and sufficient (at least for subsets of **R**). It is perhaps somewhat surprising that the