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# $E$-Optimal Minimally Connected Block Designs Under Mixed Effects Model

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*Summary:* Considering a mixed effects model in a minimally connected block design set-up, we obtain designs which are $E$-optimal, uniformly in the ratio of the variance components, for inference on varietal contrasts which constitute the fixed effects in the model.

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*Key words:* Mixed effects model, optimal designs, saturated designs.

## 1 Introduction and Preliminaries

A connected block design is said to be *saturated* or *minimally connected* if the error degree of freedom is zero under a fixed effects model. This happens when the usual design parameters $v, b, k (\geq 2)$ satisfy the relation

$$v = b(k-1)+1 \quad . \tag{1.1}$$

The study of optimal minimally connected block designs is fairly recent and the following results are known under a fixed effects additive model without interaction:

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(a) all minimally connected designs are $D$-equivalent;
(b) any design having one treatment common to all the blocks is $A$- and $E$-optimal; such optimal designs are isomorphic to one another and they are uniquely $A$- and $E$-optimal.

Throughout this paper, two designs are said to be isomorphic to each other if any one of them can be obtained from the other by renaming the blocks and/or treatments. We refer to Mukerjee, Chatterjee and Sen (1986), Krafft (1988), Mandal, Shah and Sinha (1991) and Dey and Bapat (1991) for the above-mentioned and various other related results. Two related references are Mukerjee and Sinha (1990) and Birkes and Dodge (1991).

In this paper, we initiate a study of optimal minimally connected block designs under a mixed effects model where the block effects are random with variance $\sigma_b^2$ and the treatment effects are fixed. For fixed $v = b(k-1)+1$, $b, k(\geq 2)$, let $\mathscr{D} = \mathscr{D}(v, b, k)$ be the class of all minimally connected designs in the sense of (1.1) and, as usual, let $\sigma_\varepsilon^2$ denote the error variance. Define $\theta = \sigma_b^2/\sigma_\varepsilon^2$. Then for each design $d \in \mathscr{D}$, the information matrix for the treatment effects is (see Rao (1947), Shah and Sinha (1989, p. 86))

$$C_d = C_{1d} + (1+k\theta)^{-1} C_{2d} , \tag{1.2}$$

where

$$C_{1d} = R_d - k^{-1} N_d N_d' , \quad C_{2d} = k^{-1} N_d N_d' - (bk)^{-1} r_d r_d' , \tag{1.3}$$

$N_d$ is the incidence matrix of $d$, $R_d = \operatorname{diag}(r_{1d}, \ldots, r_{vd})$, $r_d = (r_{1d}, \ldots, r_{vd})'$, $r_{1d}, \ldots, r_{vd}$ being the replication numbers for the $v$ treatments in $d$. Note that $\theta = 0$ corresponds to no differential block effects while $\theta = \infty$ corresponds to the fixed effects model. Our study reveals that the design outlined in (b) above continue to be $E$-optimal uniformly in $\theta$.

## 2 $E$-Optimal Designs and their Uniqueness

*Lemma 2.1:* Each $d \in \mathscr{D}$ contains at least two blocks such that each of these blocks contains $k-1$ treatments which are replicated exactly once in $d$.

*Proof:* Let $T_d = \{i : 1 \leq i \leq v, r_{id} = 1\}$. If $m_{1d}$ is the cardinality of $T_d$ then one must have $bk \geq m_{1d} + 2(v - m_{1d})$, so that by (1.1),

$$m_{1d} \geq b(k-2)+2 \ . \tag{2.1}$$

If $m_{2d}$ is the number of blocks in $d$ containing $k-1$ members of $T_d$ then considering the occurrence of the members of $T_d$ in $d$,

$$m_{1d} \leq m_{2d}(k-1)+(b-m_{2d})(k-2) = b(k-2)+m_{2d} \ .$$

Hence by (2.1), $m_{2d} \geq 2$, completing the proof. #

In consideration of Lemma 2.1, for each $d \in \mathscr{D}$, rearranging the treatments and blocks, the incidence matrix $N_d$ can be expressed as

$$N_d = \begin{pmatrix} 1_{k-1} & 0 & 0 \dots 0 \\ 0 & 1_{k-1} & 0 \dots 0 \\ \hline & N_{1d} & \end{pmatrix} , \tag{2.2}$$

say, where for positive integral $q$, $1_q$ is a $q \times 1$ vector with all elements unity. Defining the $v \times 1$ vector $x = (1'_{k-1}, -1'_{k-1}, 0')'$ it follows, after some algebra, from (1.2), (1.3), (2.2) that for each $d \in \mathscr{D}$,

$$x' C_d x = (x'x)\{k^{-1} + (1+k\theta)^{-1}(1-k^{-1})\} \ .$$

Therefore, for each $d \in \mathscr{D}$, the minimum non-zero eigenvalue of $C_d$ satisfies

$$\lambda_{\min}(C_d) \leq k^{-1} + (1+k\theta)^{-1}(1-k^{-1}) \ . \tag{2.3}$$

Consider now the design $d^*(\in \mathscr{D})$ with blocks $\{1,2,\dots,k\}$, $\{1,k+1,\dots, 2k-1\}, \dots, \{1,(k-1)(b-1)+2,\dots,(k-1)b+1\}$. By explicit computation using (1.1)–(1.3), it can be seen that the non-zero eigenvalues of $C_{d^*}$ are

$$\lambda_1^* = 1 \ , \quad \lambda_2^* = k^{-1} + (1+k\theta)^{-1}(1-k^{-1}) \ , \quad \lambda_3^* = k^{-1} v \ , \tag{2.4}$$

with respective multiplicities $b(k-2)$, $b-1$ and 1. Since $\theta \geq 0$, one obtains

$$\lambda_3^* > \lambda_1^* \geq \lambda_2^* \ . \tag{2.5}$$

Hence $\lambda_{\min}(C_{d^*}) = \lambda_2^*$, and comparing this with (2.3), the following holds:

*Theorem 2.1:* For each $\theta$ $(0 \leq \theta \leq \infty)$, designs isomorphic to $d^*$ are $E$-optimal in $\mathscr{D}$.

In the rest of this section, we study the uniqueness of the $E$-optimal designs mentioned in Theorem 2.1. The following lemmas will be helpful.

*Lemma 2.2:* Suppose $d(\in \mathscr{D})$ is not isomorphic to $d^*$. Then $d$ contains two blocks such that

(i) each of these two blocks contains $k-1$ treatments which are replicated exactly once in $d$, and
(ii) the remaining treatments in these two blocks are distinct.

*Proof:* Let $T_d$ be as in the proof of Lemma 2.1, $B_d$ be the set of blocks in $d$ which contain exactly $k-1$ members of $T_d$, and $a = a_d$ be the cardinality of $B_d$. By Lemma 2.1, $2 \leq a \leq b$. If possible, suppose the conclusion of the lemma is false. Then there is some treatment common to each block of $B_d$. In that case, without loss of generality, let the $a$ blocks in $B_d$ be $\{1, 2, \ldots, k\}$, $\{1, k+1, \ldots, 2k-1\}, \ldots,$ $\{1, (k-1)(a-1)+2, \ldots, (k-1)a+1\}$.

Since $d$ is not isomorphic to $d^*$, one gets $a < b$, i.e., the set, $\bar{B}_d$, of blocks of $d$, which are not members of $B_d$, is non-empty. Define

$$\bar{T}_d = \{i\colon (k-1)a+2 \leq i \leq (k-1)b+1,\ r_{id} = 1\}\ , \quad \bar{m}_{1d} = \text{cardinality of } \bar{T}_d\ ,$$

$$\bar{m}_{2d} = \text{number of blocks in } \bar{B}_d \text{ containing exactly } k-1 \text{ members of } \bar{T}_d\ .$$

Then as in the proof of Lemma 2.1, noting that $\bar{B}_d$ contains $b-a$ blocks, that the treatments $(k-1)a+2, \ldots, (k-1)b+1$ do not occur in the set of blocks $B_d$, and that treatment 1 must be replicated at least once in the set of blocks $\bar{B}_d$ (for otherwise, $d$ is disconnected), one obtains

$$(b-a)k \geq \bar{m}_{1d} + 1 + 2\{(k-1)(b-a) - \bar{m}_{1d}\}\ ,$$

and

$$\bar{m}_{1d} \leq \bar{m}_{2d}(k-1) + (b-a-\bar{m}_{2d})(k-2)\ ,$$

so that, on simplification, $\bar{m}_{2d} \geq 1$. Thus there is at least one block in $\bar{B}_d$ containing exactly $k-1$ members of $\bar{T}_d$ and hence of $T_d$. But this is impossible by the definitions of $B_d$ and $\bar{B}_d$. This completes the proof. #

*Lemma 2.3:* Suppose $d(\in \mathscr{D})$ is not isomorphic to $d^*$. Then for each $\theta(0 < \theta \leq \infty)$,

$$\lambda_{\min}(C_d) < k^{-1} + (1+k\theta)^{-1}(1-k^{-1}) .$$

*Proof:* Since $d$ is not isomorphic to $d^*$, by Lemma 2.2, without loss of generality, $d$ has two blocks of the form $\{1,2,\ldots,k\}$, $\{\xi, k+1,\ldots,2k-1\}$, where $\xi \notin \{1,2,\ldots,2k-1\}$, and $r_{id} = 1$ $(2 \le i \le 2k-1)$. Let $w$ be a $v \times 1$ vector defined as

$$w = (-(k-1)(w_1+w_2), w_1 1'_{k-1}, w_2 1'_{k-1}, 0')' . \tag{2.6}$$

Then it can be seen that

$$w' R_d w = (k-1)^2 (w_1+w_2)^2 r_{1d} + (k-1)(w_1^2 + w_2^2) ,$$

$$w' N_d N'_d w = (k-1)^2 \{2 w_2^2 + (w_1+w_2)^2 (r_{1d}-1)\} ,$$

$$w' r_d r'_d w = (k-1)^2 (r_{1d}-1)^2 (w_1+w_2)^2 ,$$

so that by (1.2), (1.3),

$$w' C_d w - \{k^{-1} + (1+k\theta)^{-1}(1-k^{-1})\} w'w$$

$$= A_{1d}(\theta)(w_1^2 - w_2^2) + A_{2d}(\theta)(w_1+w_2)^2 , \tag{2.7}$$

where $A_{1d}(\theta) = (k-1)^2 \theta/(1+k\theta)$ and $A_{2d}(\theta)$ depends on $\theta$ and the characteristics of the design $d$ but is free from $w_1$, $w_2$. For $\theta > 0$, $A_{1d}(\theta) > 0$, and it is easy to see that one can choose $(w_1, w_2) \ne (0,0)$ such that the right-hand side of (2.7) is negative. Since by (2.6), $1'_v w = 0$, the result now follows from (2.7). #

From (2.4), (2.5) and Lemma 2.3, it follows that for each $\theta$ $(0 < \theta \le \infty)$, the designs isomorphic to $d^*$ are uniquely $E$-optimal in $\mathcal{D}$. For $\theta = 0$, however, uniqueness is not preserved since, as one can easily verify, designs non-isomorphic to $d^*$ can also be $E$-optimal in $\mathcal{D}$. Anyway, from the above discussion, it is clear that the only designs which are $E$-optimal in $\mathcal{D}$ over the *entire* range $0 \le \theta \le \infty$ are those which are isomorphic to $d^*$.

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