PASSAGE TIME MOMENTS FOR MULTIDIMENSIONAL DIFFUSIONS

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Abstract

Let τ_r denote the hitting time of $B(0:r)$ for a multidimensional diffusion process. We give verifiable criteria for finiteness/infiniteness of $E_x(\tau_r^p)$. As an application we exhibit classes of diffusion processes which are recurrent but $E_x(\tau_r^p)$ is infinite $\forall p > 0$, $|x| > r > 0$; this includes the two-dimensional Brownian motion and the reflecting Brownian motion in a wedge with a certain parameter $\alpha = 0$.

Keywords: Hitting time; recurrent diffusions; generator; diffusion coefficients; sub/super-martingales; reflecting Brownian motion in a wedge

1. Introduction

Recently, Menshikov and Williams (1996) have given conditions for finiteness/infiniteness of pth moments ($p > 0$) of passage times of a continuous non-negative stochastic process in terms of sub/super-martingale inequalities for powers of the process. In this note we use these ideas to get conditions in terms of suitable Lyapunov-type functions for finiteness/infiniteness of $E(\tau_r^p)$ where τ_r denotes the hitting time of $B(0 : r)$ for a multidimensional diffusion process; and then use such functions in turn to obtain easily verifiable criteria in terms of the diffusion coefficients. No non-degeneracy assumption is made.

If a diffusion is transient it follows that $E_x(\tau_r^p) = \infty$ for any $p > 0$, $|x| > r$. However if the diffusion is recurrent, $E_x(\tau_r^p)$ can be finite only for certain p, r, x. (For a one-dimensional Brownian motion $E_x(\tau_r^p) < \infty$ (or $E_x(\tau_r^p) = \infty$) for $p < \frac{1}{2}$ (or $p > \frac{1}{2}$), $x > r$; this can be seen using Section 3 of Menshikov and Williams (1996).) In fact, as an application of our results, we exhibit a class of recurrent diffusions in $\mathbb{R}^d$ for which $E_x(\tau_r^p) = \infty$ for all $p > 0$, $|x| > r, r > 0$. This class includes the two-dimensional Brownian motion and the reflecting Brownian motion in a wedge with the Varadhan–Williams parameter $\alpha = 0$.

2. Criteria for multidimensional diffusions

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathrm{P})$ be a complete filtered probability space; let $\{Z(t) : t \geq 0\}$ be a d-dimensional $\mathcal{F}_t$-adapted diffusion process with generator

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}. \tag{1}$$

If L is non-degenerate we need to assume only continuity of $a_{ij}(\cdot)$, $b_i(\cdot)$; if L is degenerate we have to assume that $a(\cdot) := ((a_{ij}(\cdot)))$ has a Lipschitz-continuous square root and that $b_i(\cdot)$ are Lipschitz continuous so that the diffusion is well defined. In the context of the question we are investigating we can as well assume that the processes are non-explosive.

Denote by $\mathcal{G}$ the collection of all $u \in C^2(\mathbb{R}^d : \mathbb{R})$ such that
(a) $u \geq 0$, $u(0) = 0$, $u(x) \to \infty$ as $|x| \to \infty$;
(b) for each $r > 0$ there exist $0 < r_1 < r_2 < \infty$ with

$$u^{-1}([0, r_1]) \subseteq B(0 : r) \subseteq u^{-1}([0, r_2]). \tag{2}$$

Note. Suppose $u \in C^2(\mathbb{R}^d : \mathbb{R})$ satisfies (a) above and is of the form $u(x) = u_1(r)u_2(\theta)$, where $x = (r, \theta)$ is the polar decomposition: then $u \in \mathcal{G}$.

For $u \in \mathcal{G}$, $r > 0$ define the non-negative process $X(t) = u(Z(t))$, $t \geq 0$ and the stopping times $\tau_r = \inf\{t \geq 0 : |Z(t)| \leq r\}$, $\sigma_r = \inf\{t \geq 0 : X(t) \leq r\}$. For $u \in \mathcal{G}$, r, r_1, r_2 satisfying (2) note that

$$\sigma_{r_2} \leq \tau_r \leq \sigma_{r_1}. \tag{3}$$

For $u \in \mathcal{G}$, $q > 0$ by Itô's formula observe that

$$u^q(Z(t)) - u^q(Z(s)) = M(t) - M(s) + \int_s^t q u^{q-2}(Z(\alpha))[u(Z(\alpha))Lu(Z(\alpha)) + \tfrac{1}{2}(q-1)\langle a(Z(\alpha))\nabla u(Z(\alpha)), \nabla u(Z(\alpha))\rangle]\, \mathrm{d}\alpha\,, \tag{4}$$

where $M(\cdot)$ is a stochastic integral.

Theorem 1. *Let $r > 0$, $p > 0$ be fixed. Suppose there exist $u \in \mathcal{G}$, $\epsilon_0 > 0$ such that*

$$u(z)(Lu)(z) + \tfrac{1}{2}(2p-1)\langle a(z)\nabla u(z), \nabla u(z)\rangle < -\epsilon_0 \tag{5}$$

for all $z \in u^{-1}([r_1, \infty])$. Then for $z \in u^{-1}([r_1, \infty])$, $E_z(\tau_r^\beta) < \infty$ for all $0 < \beta < p$ if $p < 1$, and also for $\beta = p$ if $p \geq 1$.

Proof. Fix $z \in u^{-1}([r_1, \infty])$ and let $Z(0) = z$. In view of (3) it is enough to prove that $E(\sigma_{r_1}^\beta) < \infty$ for concerned β. Putting $\tilde{X}(t) = X(t \wedge \sigma_{r_1})$, by (4) and (5) we get for $0 \leq s \leq t$,

$$(\tilde{X}(t))^{2p} \leq (\tilde{X}(s))^{2p} + M(t \wedge \sigma_{r_1}) - M(s \wedge \sigma_{r_1}) - 2p\epsilon_0 \int_s^t I_A(\alpha)(\tilde{X}(\alpha))^{2p-2}\, \mathrm{d}\alpha\,,$$

where $A = [0, \sigma_{r_1}]$. Consequently,

$$E((\tilde{X}(t \wedge \eta_i))^{2p} \mid \mathcal{F}_s) \leq (\tilde{X}(s \wedge \eta_i))^{2p} - 2p\epsilon_0 E\left(\int_s^t I_{B_i}(\alpha)(\tilde{X}(\alpha))^{2p-2}\, \mathrm{d}\alpha \mid \mathcal{F}_s\right), \tag{6}$$

where $\{\eta_i\}$ is a sequence of localizing stopping times for the local martingale $\{M(t)\}$, and $B_i = [0, \sigma_{r_1} \wedge \eta_i]$. Letting $\eta_i \uparrow \infty$ in (6) we get

$$E((\tilde{X}(t))^{2p} \mid \mathcal{F}_s) \leq (\tilde{X}(s))^{2p} - 2p\epsilon_0 E\left(\int_s^t I_A(\alpha)(\tilde{X}(\alpha))^{2p-2}\, \mathrm{d}\alpha \mid \mathcal{F}_s\right). \tag{7}$$

The required result now follows in view of (7) and Theorem 2.1 of Menshikov and Williams (1996).

Corollary 1. *Let $r > 0$ be fixed. Suppose there exist $u \in \mathcal{G}$ such that* (i) $Lu^2(z) \leq 0, z \in u^{-1}([r_1, \infty))$, (ii) $\inf\{\langle a(z)\nabla u(z), \nabla u(z)\rangle : z \in u^{-1}([r_1, \infty))\} > 0$. *Then $E_z(\tau_r^p) < \infty$ for all $0 < p < 1, z \in u^{-1}([r_1, \infty))$.*

Proof. Clear as $p < 1$ and $2uLu + (2p-1)\langle a\nabla u, \nabla u\rangle = Lu^2 + (2p-2)\langle a\nabla u, \nabla u\rangle$.

Corollary 2. *Let $r > 0$ be fixed. Suppose there exist $u \in \mathcal{G}$, $p \geq 1$, $\epsilon > 0$ such that $Lu^{2p}(z) \leq -\epsilon u^{2p-2}(z)$, $z \in u^{-1}([r_1, \infty))$. Then $E_z(\tau_r^q) < \infty$ for all $q \leq p$, $z \in u^{-1}([r_1, \infty))$.*

Proof. Immediate as $Lu^{2p} = 2pu^{2p-1}Lu + p(2p-1)u^{2p-2}\langle a\nabla u, \nabla u\rangle$.

Theorem 2. *Let $r > 0$, $p > 0$ be fixed. Suppose there exist $u \in \mathcal{G}$, $0 < K < \infty$, $\lambda_0 < \infty$ such that*

$$0 \leq u(z)(Lu)(z) + \tfrac{1}{2}(2p-1)\langle a(z)\nabla u(z), \nabla u(z)\rangle < K \tag{8}$$

for all $z \in u^{-1}([r_2, \infty))$, and

$$\sup\{\langle a(z)\nabla u(z), \nabla u(z)\rangle : z \in u^{-1}([r_2, \infty))\} \leq \lambda_0. \tag{9}$$

Then $E_z(\tau_r^\beta) = \infty$ for all $\beta > p$, $z \in u^{-1}((r_2, \infty))$.

Proof. Fix $z \in u^{-1}((r_2, \infty))$ and let $Z(0) = z$. In view of (3) it is enough to prove that $E(\sigma_{r_2}^\beta) = \infty$ for $\beta > p$. Put $\hat{X}(t) = X(t \wedge \sigma_{r_2})$, $t \geq 0$. Using (4), the first inequality in (8) and an argument as in the proof of Theorem 1, we get that $\{(\hat{X}(t))^{2p} : t \geq 0\}$ is a local submartingale. Next observe that

$$uLu + \tfrac{1}{2}\langle a\nabla u, \nabla u\rangle = uLu + \tfrac{1}{2}(2p-1)\langle a\nabla u, \nabla u\rangle - (p-1)\langle a\nabla u, \nabla u\rangle.$$

Therefore using the first inequality in (8), (9) and a similar argument shows that $\{\hat{X}^2(t) + \mu(t \wedge \sigma_{r_2}) : t \geq 0\}$ is a local submartingle for any $\mu \geq \lambda_0[(p-1) \vee 0]$. Similarly for any $\gamma > (1 \vee p)$,

$$uLu + \tfrac{1}{2}(2\gamma-1)\langle a\nabla u, \nabla u\rangle = uLu + \tfrac{1}{2}(2p-1)\langle a\nabla u, \nabla u\rangle + (\gamma - p)\langle a\nabla u, \nabla u\rangle.$$

Hence (4), the second inequality in (8), (9) and an analogous argument give that for $\gamma > (1 \vee p)$, $\{\hat{X}(t)^{2\gamma} - \nu\int_0^t I_F(\alpha)(\hat{X}(\alpha))^{2\gamma-2}\,d\alpha : t \geq 0\}$ is a local supermartingale for any $\nu \geq K + (\gamma - p)\lambda_0$, where $F = [0, \sigma_{r_2}]$. Now apply Corollary 2.4 of Menshikov and Williams (1996) to get the result.

Corollary 3. *Let $r > 0$, $p > 0$ be fixed. Suppose there exist $u \in \mathcal{G}$, $\lambda_0 < \infty$, $0 < K < \infty$ such that (9) holds and*

$$-2(p-1)\langle a(z)\nabla u(z), \nabla u(z)\rangle \leq Lu^2(z) \leq K,$$

for $z \in u^{-1}((r_2, \infty))$. Then $E_z(\tau_r^\beta) = \infty$ for all $\beta > p$, $z \in u^{-1}((r_2, \infty))$.

Proof. Since $Lu^2 = 2uLu + \langle a\nabla u, \nabla u\rangle$ it is immediate.

Next for $x \neq 0$ set

$$A(x) = \sum_{i,j=1}^d a_{ij}(x)x_i x_j/|x|^2, \qquad B(x) = \sum_{i=1}^d a_{ii}(x), \qquad C(x) = 2\sum_{i=1}^d x_i b_i(x).$$

Theorem 3. *Let $r > 0$ be fixed.*

(a) *Suppose there exist $\epsilon > 0$, $p > 0$ such that*

$$B(x) + C(x) + 2(p-1)A(x) \leq -\epsilon$$

for all $|x| \geq r$. Then $E_x(\tau_r^\beta) < \infty$ for all $0 < \beta < p$ if $p < 1$, and also for $\beta = p$ if $p \geq 1$, for any $|x| \geq r$.

(b) *Suppose there exist $p > 0$, $\lambda_0 < \infty$, $0 < K < \infty$ such that*

$$0 \leq B(x) + C(x) + 2(p-1)A(x) \leq K$$
$$A(x) \leq \lambda_0$$

for all $|x| \geq r$. Then $E_x(\tau_r^\beta) = \infty$ for all $\beta > p$, $|x| > r$.

(c) *If eigenvalues of $a(\cdot)$ are bounded and bounded away from zero then for $|x| > r$,*

$$E_x(\tau_r^\beta) < \infty \quad \text{for all} \quad p < \inf\{1 - [(B(x) + C(x))/2A(x)] : |x| \geq r\},$$
$$E_x(\tau_r^p) = \infty \quad \text{for all} \quad p > \sup\{1 - [(B(x) + C(x))/2A(x)] : |x| \geq r\}.$$

Proof. If $u(x) = |x|$ outside a neighbourhood of the origin note that

$$Lu(x) = \frac{1}{2|x|}(B(x) + C(x) - A(x))$$

away from the origin. So assertions (a) and (b) are easy to see applying Theorems 1 and 2. Under the non-degeneracy hypothesis in (c) one can divide by $A(x)$; so (c) follows using (a), (b).

To illustrate our results we consider the following class of examples. Recurrence and transience of this class of diffusions have been studied by Friedman (1975).

Example 1. Let $b_i(\cdot) \equiv 0$, and

$$a_{ij}(x) = \delta_{ij} + \frac{g(|x|)}{|x|^2} x_i x_j,$$

where $g(\cdot)$ is a continuous function vanishing near 0, and

$$-1 < \mu \equiv \inf_r g(r) \leq \sup_r g(r) \equiv \nu < \infty. \tag{10}$$

Observe that $C(x) \equiv 0$, $B(x) = d + g(|x|)$, $A(x) = 1 + g(|x|)$; therefore

$$1 - \frac{B(x) + C(x)}{2A(x)} = \frac{1}{2} - \frac{d-1}{2(1 + g(|x|))}. \tag{11}$$

Also, by (10), $0 < 1 + \mu \leq A(x) \leq \nu + 1 < \infty$; and by (10) and (11)

$$\frac{1}{2} - \frac{d-1}{2(1+\mu)} \leq 1 - \frac{B(x) + C(x)}{2A(x)} \leq \frac{1}{2} - \frac{d-1}{2(1+\nu)}. \tag{12}$$

Using (10)–(12), in view of Theorem 3(c), the following are easily obtained.

(a) For a diffusion in this class $E_x(\tau_r^p) = \infty$ for any $p > \frac{1}{2}$, $|x| > r$.

(b) For a diffusion in this class with $\mu > (d-2)$, we have $E_x(\tau_r^p) < \infty$ for any $r > 0$,

$$p < \left[\frac{1}{2} - \frac{(d-1)}{2(1+\mu)}\right], \qquad |x| > r.$$

(In particular, for any one-dimensional diffusion in this class $E_x(\tau_r^p) < \infty$ for any $p < \frac{1}{2}$, $x > r$.)

(c) For a diffusion in this class with $-1 < \mu \leq \nu \leq (d-2)$ we get $E_x(\tau_r^p) = \infty$ for any $p > 0$, $r > 0$, $|x| > r$.

(d) Suppose $d \geq 2$, and $g(r) = (d-2-h(r))/(1+h(r))$, where h is a non-negative function with $h(r) \leq 1/\log r$ for all large r. In such a case the diffusion is known to be recurrent; see pp. 202–203 of Friedman (1975). Also it is easily seen that $-1 < \mu \leq \nu \leq (d-2)$. Thus $E_x(\tau_r^p) = \infty$ for any $p > 0$, $|x| > r > 0$ for such a diffusion. Observe that the two-dimensional Brownian motion is such a diffusion.

Example 2. Take $a_{ij} = \delta_{ij}$, $b_i(x) = -1/x_i$, for $|x| > r$. By Theorem 3(c), it is seen that for $|x| > r$,

$$E_x(\tau_r^p) < \infty \qquad \text{if } p < \tfrac{1}{2}(d+2),$$
$$E_x(\tau_r^p) = \infty \qquad \text{if } p > \tfrac{1}{2}(d+2).$$

3. Reflecting Brownian motion in a wedge

Let D denote the two-dimensional wedge given in polar coordinates by $D = \{(r, \theta) : r > 0, 0 < \theta < \xi\}$ where $\xi \in (0, 2\pi)$; the two arms of D are $\partial_1 D = \{(r, \theta) : r \geq 0, \theta = 0\}$, $\partial_2 D = \{(r, \theta) : r \geq 0, \theta = \xi\}$. For $i = 1, 2$ let v_i be a vector such that $\langle v_i, n_i \rangle = 1$ where n_i is the inward normal vector to $\partial_i D \backslash \{(0,0)\}$; let θ_i denote the angle v_i makes with n_i, with θ_i being positive if and only if v_i points towards the corner. Observe that $0 \leq \theta_i < \frac{1}{2}\pi$, $i = 1, 2$. Define $\alpha = (\theta_1 + \theta_2)/\xi$.

It is a fundamental result due to Varadhan and Williams (1985) that if $\alpha < 2$ then a unique reflecting Brownian motion $\{Z(t) : t \geq 0\}$ in $\bar{D}$ exists with directions of reflection on the boundary given by v_i on $\partial_i D \backslash \{(0,0)\}$, $i = 1, 2$; the process has been defined as the solution of the appropriate submartingale problem. Moreover, if $\alpha < 0$ the process never hits $(0, 0)$ and is transient; if $0 < \alpha < 2$ the process hits $(0, 0)$ with probability one and is recurrent; if $\alpha = 0$ the process is recurrent but does not hit the corner point $(0, 0)$; see Williams (1985).

We apply our analysis to the stopping time $\tau_r = \inf\{t \geq 0 : |Z(t)| \leq r\}$, $r > 0$ to get the following.

Theorem 4. *Let $r > 0$ be fixed. If $\alpha = 0$ then $E_z(\tau_r^p) = \infty$ for any $p > 0$, $|z| > r$.*

Proof. By the obvious modifications necessary to make the proof of Theorem 2 go through in the present context, for each $r > 0$, $p > 0$ we need a function $u \in \mathcal{G}$ such that u vanishes near $(0, 0)$ and

$$0 \leq u(z)\tfrac{1}{2}\Delta u(z) + \tfrac{1}{2}(2p-1)|\nabla u(z)|^2 \leq K, \qquad z \in u^{-1}([r_2, \infty)) \cap \bar{D} \qquad (13)$$

$$\sup\{|\nabla u(z)|^2 : z \in u^{-1}([r_2, \infty)) \cap \bar{D}\} < \infty, \qquad (14)$$

$$\langle v_i, \nabla u(z) \rangle = 0, \qquad z \in u^{-1}([r_2, \infty)) \cap \partial_i D, \qquad i = 1, 2. \qquad (15)$$

Let r_0 be arbitrary but fixed. Let ϕ, u be functions such that

$$\phi(r,\theta) = \log r + \theta\tan\theta_1,$$
$$u(r,\theta) = \exp(\phi(r,\theta)) = r\exp(\theta\tan\theta_1),$$

for $r \geq \frac{1}{2}r_0$, $0 < \theta < 2\pi$. Note that u can be extended to $\mathbb{R}^2$ so that $u \in \mathcal{G}$ and $u = 0$ near $(0,0)$. Observe that

$$\nabla u := \left(\frac{\partial u}{\partial r}, \frac{1}{r}\frac{\partial u}{\partial\theta}\right) = u\nabla\phi,$$

and

$$\Delta u := \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$
$$= u\left[\Delta\phi + \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2\right]$$

on $B(0:\frac{1}{2}r_0)^c$. Since $\langle v_i, \nabla\phi\rangle = 0$ on $\partial_i D$ it is clear that (15) is satisfied with $r_2 = r_0$. Also for any $p > 0$,

$$u\tfrac{1}{2}\Delta u + \tfrac{1}{2}(2p-1)|\nabla u|^2 = p(1+\tan^2\theta_1)\exp(2\theta(\tan\theta_1)) \tag{16}$$
$$|\nabla u|^2 = (1+\tan^2\theta_1)\exp(2\theta(\tan\theta_1)) \tag{17}$$

on $B(0:\frac{1}{2}r_0)^c$. As $0 \leq \theta_1 < \frac{1}{2}\pi$, (13) and (14) are now clear from (16), (17) for any $p > 0$. This completes the proof.

Remark 1. If $\alpha > 0$, using the function $u(r,\theta) = r(\cos(\alpha\theta - \theta_1))^{1/\alpha}$ analogously one can obtain Theorem 4.1 of Menshikov and Williams (1996); this is what is essentially being done in their proof.

References

FRIEDMAN, A. (1975). *Stochastic Differential Equations and Applications*, Vol. I. Academic Press, New York.

MENSHIKOV, M. AND WILLIAMS, R. J. (1996). Passage-time moments for continuous nonnegative stochastic processes and applications. *Adv. Appl. Prob.* **28**, 747–762.

VARADHAN, S. R. S. AND WILLIAMS, R. J. (1985). Brownian motion in a wedge with oblique reflection. *Commun. Pure Appl. Math.* **38**, 405–443.

WILLIAMS, R. J. (1985). Recurrence classification and invariant measure for reflecting Brownian motion in a wedge. *Ann. Prob.* **13**, 758–778.