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On $\mathbf{A}^*$-fibrations

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Abstract

In this paper we investigate minimal sufficient fibre conditions for a finitely generated flat algebra over a noetherian integral domain to be locally $\mathbf{A}^*$ or at least an $\mathbf{A}^*$-fibration. We also describe the structure of finitely generated locally $\mathbf{A}^*$-algebras.

1. Introduction

In [3, 3.4], the following result has been proved.

Theorem. *Let R be a noetherian normal domain with quotient field K and let A be a finitely generated faithfully flat R-algebra such that*
(i) *The generic fibre $K \otimes_R A$ is a polynomial ring in one variable over K.*
(ii) *For each prime ideal P of R of height one, the fibre ring $k(P) \otimes_R A$ is geometrically integral over $k(P)$ (where $k(P) = R_P/PR_P$).*
Then A is R-isomorphic to the Rees algebra $R[IT]$ of an invertible ideal I of R; in particular, A is an $\mathbf{A}^1$-fibration over R, i.e., the fibre at every point P of Spec R is a polynomial ring in one variable over $k(P)$.

The result is somewhat surprising as conditions on merely the generic and codimension one fibres imply that all fibres are $\mathbf{A}^1$. This phenomenon had also been observed earlier in [2, 3.10 and 3.12] for subalgebras of polynomial algebras. In this paper we show that an analogous result holds when the generic fibre is $\mathbf{A}^*$ (i.e., when $K \otimes_R A$ is a Laurent polynomial ring $K[T, T^{-1}]$). More precisely, we prove:

Theorem 3.11. *Let R be a noetherian normal domain with quotient field K and let A be a finitely generated flat R-algebra such that*
(i) *The generic fibre $K \otimes_R A$ is a Laurent polynomial ring in one variable over K.*
(ii) *For each prime ideal P of R of height one, the fibre ring $k(P) \otimes_R A$ is geometrically integral but is not $\mathbf{A}^1$ over $k(P)$.*
Then there exists an invertible ideal I in R such that A is a $\mathbf{Z}$-graded R-algebra isomorphic to the R-subalgebra $R[IT, I^{-1}T^{-1}]$ of $K[T, T^{-1}]$. In particular, A is locally $\mathbf{A}^*$ and hence an $\mathbf{A}^*$-fibration over R.

The crucial step in the proof is a patching Lemma 3.1. As an application of the patching lemma we shall also prove the following structure theorem for locally $\mathbf{A}^*$ algebras over noetherian domains.

Theorem 3.4. *Let A be a finitely generated algebra over a noetherian domain R such that for each maximal ideal M of R, A_M is a Laurent polynomial ring $R_M[T_M, T_M^{-1}]$. Then there exists an invertible ideal I in R such that A is a $\mathbf{Z}$-graded R-algebra isomorphic to $R[IT, I^{-1}T^{-1}]$.*

The above result is an analogue of a result of Eakin–Heinzer [4, 3.1] that affine domains which are locally $\mathbf{A}^1$ are the symmetric algebras of invertible ideals. (In fact, a little modification of our proof will give an alternative proof of the Eakin–Heinzer theorem for noetherian domains.) Finally, we investigate minimal sufficient conditions for a finitely generated flat algebra over an arbitrary noetherian domain to be an $\mathbf{A}^*$-fibration and prove the following analogue of [3, 3.5].

Theorem 3.13. *Let R be a noetherian domain with quotient field K and let A be a finitely generated flat R-algebra such that*
(i) *The generic fibre $K \otimes_R A$ is a Laurent polynomial ring in one variable over K.*
(ii) *For each prime ideal P of R of height one, the fibre ring $k(P) \otimes_R A$ is geometrically integral but is not an $\mathbf{A}^1$-form over $k(P)$.*
Then all the fibre rings are $\mathbf{A}^*$-forms. In fact, there exists a finite birational extension R' of R and an invertible ideal I of R' such that $R' \otimes_R A$ is a $\mathbf{Z}$-graded R'-algebra isomorphic to $R'[IT, I^{-1}T^{-1}]$. Further, if R contains a field of characteristic zero, and if all the fibres have more units than the respective residue fields, then A is an $\mathbf{A}^*$-fibration over R.

We also give examples to show that the conditions in our theorems cannot be relaxed.

2. Preliminaries

In this section we set up the notations, define the terms used in the paper, state a few elementary results and prove a result on $\mathbf{A}^*$-forms. Throughout our paper we will assume our rings to be commutative.

Notation. For a ring R, R^* will denote the multiplicative group of units of R. For a prime ideal P of R, $k(P)$ denotes the residue field R_P/PR_P. The notation $A = R^{[n]}$ will mean that A is a polynomial ring in n variables over R.

Definition. An R-algebra A is defined to be $\mathbf{A}^*$ if it is a Laurent poynomial ring in one indeterminate over R, i.e., if there exists an element T in A which is algebraically independent over R such that $A = R[T, T^{-1}]$.

An R-algebra A is defined to be *locally* $\mathbf{A}^*$ if A_M is $\mathbf{A}^*$ over R_M for every maximal ideal M of R.

A finitely generated flat R-algebra A is defined to be an $\mathbf{A}^*$-*fibration* over R if, at each point P of *SpecR*, the fibre ring $k(P) \otimes_R A$ is $\mathbf{A}^*$ over $k(P)$.

Let k be a field and let $\bar{k}$ denote the algebraic closure of k. A k-algebra B is said to be *geometrically integral* (*over* k) if $\bar{k} \otimes_k B$ is an integral domain. B is defined to be an $\mathbf{A}^*$-*form over* k if $\bar{k} \otimes_k B$ is $\mathbf{A}^*$ over $\bar{k}$. A k-algebra C is said to be an $\mathbf{A}^1$-*form over* k if $\bar{k} \otimes_k C = \bar{k}^{[1]}$.

Lemma 2.1. *Let $B \subseteq A$ be integral domains. Suppose that there exists a non-zero element π in B such that $B[1/\pi] = A[1/\pi]$ and the canonical map $B/\pi B \to A/\pi A$ is injective. Then $B = A$.*

Proof. Since the map $B/\pi B \to A/\pi A$ is injective, it is easy to see that $B \cap \pi^n A = \pi^n B$ for all $n \geq 1$. Let $a \in A$. Then $a = b/\pi^n$ for some $b \in B$ and non-negative integer n. Therefore $b = \pi^n a \in \pi^n B$. Hence $a \in B$. □

Lemma 2.2. *Let B be geometrically integral over the field k. Then k is algebraically closed in B.*

Proof. Let L be the algebraic closure of k in B. Then $L \otimes_k B$ is an integral domain. Suppose that $L \neq k$ and let $a \in L \setminus k$. Let f be the minimal polynomial of a over k and let $L_1 = k(a) \cong k[X]/(f(X))$. Then $L_1 \otimes_k B (\hookrightarrow L \otimes_k B)$ is an integral domain. On the other hand, $L_1 \otimes_k B (\cong B[X]/(f(X)))$ cannot be an integral domain since $(X - a)$ is a factor of $f(X)$ in $B[X]$. The contradiction shows that $L = k$. □

We now show that over a perfect field k, any $\mathbf{A}^*$-form having non-trivial units is $\mathbf{A}^*$.

Proposition 2.3. *Let k be a field and let B be a k-algebra such that $B^* \neq k^*$. Suppose that there exists a separable field extension L of k such that $L \otimes_k B$ is $\mathbf{A}^*$ over L. Then B is $\mathbf{A}^*$ over k.*

Proof. Let $L \otimes_k B = L[T, T^{-1}]$. We identify B with its image in $L \otimes_k B$. It is easy to see that B is finitely generated over k. Hence, there exists a finitely generated separable extension L_1 of k such that $L_1 \otimes_k B = L_1[T, T^{-1}]$. Thus, replacing L by L_1, we may assume L to be finitely generated over k to start with.

We first consider the case when L is finite algebraic over k. Replacing L by its splitting field, we may assume L to be finite Galois over k with Galois group G, say. Any $\sigma \in G$ can be extended to a B-automorphism of $L \otimes_k B (= L[T, T^{-1}])$ by defining $\sigma(x \otimes b) = \sigma(x) \otimes b$ for $x \in L$, $b \in B$. Let

$$T = a_0 \otimes 1 + a_1 \otimes e_1 + \cdots + a_r \otimes e_r,$$

where $1, e_1, \ldots, e_r$ form part of a k-basis of B and $a_i \in L$. Since L is Galois, the bilinear map $L \times L \to k$ given by $(x, y) \to Trace(xy)$ is non-degenerate. Hence, replacing T by aT $(a \in L)$ if necessary, we can assume that $Tr(a_i) \neq 0$ for some $i \geq 1$. Thus,

$$W = \sum \sigma(T) = Tr(a_0) \otimes 1 + Tr(a_1) \otimes e_1 + \cdots + Tr(a_r) \otimes e_r$$

is an element of $B \setminus k$; in particular, $W \neq 0$.

We now show that $B = k[W, 1/W]$. Let $f \in B^* \setminus k^*$. Since k is algebraically closed in B by (2.2), f is transcendental over k and hence over L. Therefore, $f = aT^m$ for some $a \in L^*$ and some non-zero integer m. Replacing f by $1/f$ if necessary, we may assume $m > 0$. Since B is invariant under every $\sigma \in G$, we have

$$aT^m = f = \sigma(f) = \sigma(a)(\sigma(T))^m.$$

Since $\sigma(a) \in L^*$, the above relation shows that $((\sigma(T))/T)^m \in L^*$, and hence $(\sigma(T))/T \in L^*$. Therefore, $\sigma(T) = a_\sigma T$ for some $a_\sigma \in L^*$. Hence, $W = aT$ for some $a \in L$. Since $W \neq 0$, $a \in L^*$. Therefore, $L[W, 1/W] = L[T, T^{-1}] = L \otimes_k B$. Now, L being a finite extension of k, $L \otimes_k B$ is integral over B. Hence, $B \cap (L \otimes_k B)^* = B^*$. Therefore, $1/W \in B$. Now, as $k[W, 1/W] \subseteq B$, by faithful flatness of L over k, it follows that $B = k[W, 1/W]$.

We now consider the case when L has positive transcendence degree over k. Now, since L is a finitely generated separable extension of k, there exists a purely transcendental extension $K = k(X_1, \ldots, X_n)$ of k such that L is a finite separable extension of K. Since $L \otimes_K (K \otimes_k B) = L[T, T^{-1}]$, by the previous case, it follows that $K \otimes_k B = K[W, 1/W]$ for some $W \in K \otimes_k B$. Since B is finitely generated over k, it is easy to see that there exists a polynomial $F(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$ such that

$$B[X_1, \ldots, X_n, 1/F(X_1, \ldots, X_n)] = k[X_1, \ldots, X_n, 1/F(X_1, \ldots, X_n), W, 1/W]. \qquad (*)$$

If k is an infinite field, then we can choose elements $c_1, \ldots, c_n \in k$ such that $F(c_1, \ldots, c_n) \neq 0$. Let N be the maximal ideal of $k[X_1, \ldots, X_n, 1/F(X_1, \ldots, X_n)]$ generated by $X_1 - c_1, \ldots, X_n - c_n, 1/F(X_1, \ldots, X_n) - 1/F(c_1, \ldots, c_n)$. From Eq. $(*)$, it follows, by taking quotient modulo the ideal N, that B is $\mathbf{A}^*$ over k.

If k is a finite field, let N be any maximal ideal of $k[X_1, \ldots, X_n, 1/F(X_1, \ldots, X_n)]$ and let $k' = k[X_1, \ldots, X_n, 1/F(X_1, \ldots, X_n)]/N$. Then k' is a finite vector space over k by Hilbert's Nullstellensatz and separable over k. Since $k' \otimes_k B$ is $\mathbf{A}^*$ over k' by Eq. $(*)$, it follows, by the previous case, that B is $\mathbf{A}^*$ over k. □

Remark 2.4. The assumption that $B^* \neq k^*$ is essential in the above result. For instance, consider the co-ordinate ring of the real circle, i.e., $B = \mathbf{R}[X,Y]/(X^2+Y^2-1)$. Then $\mathbf{C} \otimes_{\mathbf{R}} B$ is $\mathbf{A}^*$ over $\mathbf{C}$, though B is not $\mathbf{A}^*$ over $\mathbf{R}$.

3. Main theorems

In this section we shall prove our main results. We first prove a patching Lemma 3.1 and deduce a structure Theorem 3.4 for locally $\mathbf{A}^*$-algebras. Next we prove our result (3.11) on $\mathbf{A}^*$-fibration over Krull domains and finally we investigate the general case (3.13).

Lemma 3.1. *Let R be an integral domain with quotient field K and let A be a flat R-algebra. Suppose that there exists non-zero elements x, y in R such that*
(i) *x and y either form an R-sequence or are comaximal in R.*
(ii) *$A[1/x]$ is $\mathbf{A}^*$ over $R[1/x]$.*
(iii) *$A[1/y]$ is $\mathbf{A}^*$ over $R[1/y]$.*
Then there exists an invertible ideal I in R such that $A \cong \bigoplus_{n\in\mathbf{Z}} I^nT^n$ $(\subseteq K[T,T^{-1}])$ as a $\mathbf{Z}$-graded R-algebra.

Proof. Let

$$A_x = \bigoplus_{n\in\mathbf{Z}} R_xT^n \quad \text{and} \quad A_y = \bigoplus_{n\in\mathbf{Z}} R_yW^n.$$

Then

$$A_{xy} = \bigoplus_{n\in\mathbf{Z}} (R_{xy}T^n) = \bigoplus_{n\in\mathbf{Z}} (R_{xy}W^n).$$

Therefore, it is easy to see that W is either λT or λT^{-1} for some $\lambda \in R_{xy}^*$. Replacing T by T^{-1} if necessary, we assume that $W = \lambda T$. Let $\lambda = a/x^my^m$ where $a \in R$ and m is a non-negative integer. Again, replacing W by y^mW and T by T/x^m, we assume that

$$W = aT \quad \text{for some } a \in R \cap R_{xy}^*.$$

Since A is R-flat and $(R_xT^n)_y = (R_yW^n)_x$, using condition (i), it is easy to see that

$$A = A_x \cap A_y = \bigoplus_{n\in\mathbf{Z}} A_n,$$

where

$$A_n = R_xT^n \cap R_yW^n = (R_x \cap a^nR_y)T^n = (R \cap a^nR_y)T^n. \qquad (*)$$

Thus A_n is R-flat for every n. Note that, by condition (i), $A_0 (= R_x \cap R_y) = R$, showing that A is a $\mathbf{Z}$-graded R-algebra. Now let

$$I = R \cap aR_y \quad \text{and} \quad J = R \cap aR_x.$$

Therefore,

$$I_x = R_x, \quad I_y = aR_y, \quad J_x = aR_x \quad \text{and} \quad J_y = R_y.$$

We now show that $IJ = aR$. Since $R_x \cap R_y = R$, clearly $I \cap J = aR$ so that $IJ \subseteq aR$. Let M denote the module aR/IJ. Recall that $A_1 = IT$ is R-flat so that I is flat over R. Hence I/IJ is flat over R/J. From the construction of J it follows easily that x is a non-zero divisor in R/J. Hence, by flatness, x remains a non-zero divisor in I/IJ and hence in the submodule M. But $M_x = aR_x/I_xJ_x = 0$. Hence $M = 0$, i.e., $IJ = aR$. Thus I is an invertible ideal of R.

Let $B = \bigoplus_{n\in \mathbf{Z}} I^nT^n$. Since $I^n \subseteq R \cap a^nR_y$ for all n, by (*), $B \subseteq A$. I being invertible, B is R-flat. Hence, from condition (i), it follows that $B = B_x \cap B_y$. Now, since a is a unit in R_{xy} and $I_x = R_x$, it follows from (*) that $(A_n)_x = R_xT^n = I^nR_xT^n$, so that $A_x = B_x$. Similarly $(A_n)_y = a^nR_yT^n = I^nR_yT^n$, so that $A_y = B_y$. Therefore, $A = A_x \cap A_y = B_x \cap B_y = B = \bigoplus_{n\in \mathbf{Z}} I^nT^n$. □

Example 3.2. The assumption of flatness is essential in Lemma 3.1. For instance, let $R = \mathbf{C}[X,Y,Z,W]/(XY - ZW)$ and let x, y, z and w be the images in R of X, Y, Z and W, respectively. Let $I = (x,z)R$ and let $A = R[IT, I^{-1}T^{-1}]$. Then clearly A is not R-flat although A_x and A_y are $\mathbf{A}^*$ over R_x and R_y, respectively.

We shall now apply the patching Lemma (3.1) to prove a structure theorem for locally $\mathbf{A}^*$ algebras. For convenience, we first prove the structure theorem over semi-local noetherian domains.

Lemma 3.3. *Let R be a semi-local integral domain and let A be an R-algebra which is locally $\mathbf{A}^*$ over R. Then A is $\mathbf{A}^*$ over R.*

Proof. Clearly, A is finitely generated and flat over R. Let $P_1, \dots, P_n$ be the maximal ideals of R. We prove the result by induction on n, the number of maximal ideals of R.

If $n = 1$, there is nothing to prove. So let $n \geq 2$ and assume the result when the number of maximal ideals is $\leq n - 1$. Let $S_1 = R \setminus (P_1 \cup \cdots \cup P_{n-1})$ and $S_2 = R \setminus P_n$. By induction hypothesis, $S_1^{-1}A$ and $S_2^{-1}A$ are $\mathbf{A}^*$ over $S_1^{-1}R$ and $S_2^{-1}R$, respectively. Since A is finitely generated over R, it follows easily that there exists a pair of elements $x \in S_1, y \in S_2$ such that $A[1/x]$ and $A[1/y]$ are $\mathbf{A}^*$ over $R[1/x]$ and $R[1/y]$, respectively. Clearly, x and y are comaximal so that from (3.1) it follows that A is $\mathbf{A}^*$ over R. □

We now prove the structure theorem for locally $\mathbf{A}^*$ algebras.

Theorem 3.4. *Let R be an integral domain which is either noetherian or a Krull domain. Let A be a finitely generated R-algebra which is locally $\mathbf{A}^*$ over R. Then there exists an invertible ideal I in R such that A is isomorphic to $R[IT, I^{-1}T^{-1}]$ as a $\mathbf{Z}$-graded R-subalgebra of $K[T, T^{-1}]$, where K is the quotient field of R.*

Proof. Since A is finitely generated, from the given condition, it is easy to see that there exists $x \in R$ such that $A[1/x]$ is $\mathbf{A}^*$ over $R[1/x]$. If $x \in R^*$, we are through. If not, then since R is either noetherian or a Krull domain, xR has finitely many prime divisors. Let $P_1,\dots,P_n$ be the prime divisors of xR. Let $S=R\setminus(P_1\cup\cdots\cup P_n)$. Then $S^{-1}R$ being a semi-local integral domain, by (3.3), $S^{-1}A$ is $\mathbf{A}^*$ over $S^{-1}R$. Hence there exists $y\in S$ such that $A[1/y]$ is $\mathbf{A}^*$ over $R[1/y]$. By construction, x and y either form an R-sequence or are comaximal. Hence, A being flat, the result follows from (3.1). □

By a result of Asanuma [1, 3.4], an $\mathbf{A}^1$-fibration over a noetherian ring R is necessarily an R-subalgebra of a polynomial algebra over R; in particular, there is a retract from A to R (i.e., an R-algebra homomorphism from A to R). By contrast, the following corollary shows that even when A is locally $\mathbf{A}^*$ over a noetherian domain R, there would be a retract from A to R if and only if A is itself $\mathbf{A}^*$ over R.

Corollary 3.5. *Let A be a finitely generated locally $\mathbf{A}^*$ algebra over a noetherian domain R. Suppose that there exists a retract from A to R. Then A is $\mathbf{A}^*$ over R.*

Proof. By (3.4), $A=\bigoplus_{n\in\mathbf{Z}} I^nT^n$ for some invertible ideal I of R. Let ϕ be a retract from A to R. Let $J_1=\phi(IT)$ and $J_2=\phi(I^{-1}T^{-1})$. We show that the ideals J_1 and J_2 of R are actually the unit ideal. Let $a_1,\dots,a_n\in I$ and $b_1,\dots,b_n\in I^{-1}$ be such that $1=\sum a_ib_i=\sum(a_iT)(b_iT^{-1})$. Therefore,

$$1=\phi(1)=\sum \phi(a_iT)\phi(b_iT^{-1})\in J_1J_2$$

showing that $J_1J_2=R$ and hence $J_1=J_2=R$. Thus there is an R-surjection from I to R showing that I is principal. Therefore $A\cong R[T,T^{-1}]$. □

Example 3.6. The assumption of finite generation is essential in Theorem 3.4. For instance, consider $R=\mathbf{Z}$ and $A=\mathbf{Z}[X/2,2/X,X/3,3/X,\dots,X/p,p/X,\dots]$ where p varies over the set of prime integers. Then $\mathbf{Q}\otimes_{\mathbf{Z}}A=\mathbf{Q}[X,1/X]$ and $\mathbf{Z}_{(p)}\otimes_{\mathbf{Z}}A=\mathbf{Z}_{(p)}[X/p,p/X]$ for each prime integer p. Thus A is locally $\mathbf{A}^*$ over R. But A is not finitely generated over R.

We now investigate minimal sufficient conditions for a finitely generated overdomain of a discrete valuation ring to be $\mathbf{A}^*$.

Proposition 3.7. *Let R be a discrete valuation ring with uniformising parameter π and residue field k. Let A be a finitely generated overdomain of R such that*

(i) *The generic fibre $A[1/\pi]$ is $\mathbf{A}^*$ over $R[1/\pi]$.*
(ii) *The closed fibre $A/\pi A$ is geometrically integral over k.*

Then there are precisely two possibilities:

(a) *If $(A/\pi A)^*\neq k^*$, then A is $\mathbf{A}^*$ over R.*
(b) *If $(A/\pi A)^*=k^*$, then $A\cong R[X,Y]/(\pi^mXY+\alpha X+\beta Y+\gamma)$ for some $\alpha,\beta\in R^*$, $\gamma\in R$ and positive integer m. In particular, $A/\pi A=k^{[1]}$.*

Proof. Let $A=R[t_1,\dots,t_p]$. Since π is a prime element in A and $A[1/\pi]$ is factorial, it follows that A is factorial. From the factoriality of A, it is easy to see that there exists an element $T \in A$ such that $T^{-1} \in A$ and $A[1/\pi] = R[1/\pi][T, T^{-1}]$. Let $A_0 = R[T, T^{-1}]$. If $A_0 = A$, then A is $\mathbf{A}^*$ over R.

Suppose that $A_0 \neq A$. Let $x_0 = T$, $y_0 = T^{-1}$ and $F_0(X, Y) = XY - 1$. For an element $a \in A$, denote its image in $A/\pi A$ by $\bar{a}$. Since $A_0[1/\pi] = A[1/\pi]$ and $A_0 \neq A$ by hypotheses, the canonical map $A_0/\pi A_0 \to A/\pi A$ cannot be injective by (2.1). Therefore, $dim(k[\overline{x_0}, \overline{y_0}]) = 0$. Since k is algebraically closed in $A/\pi A$ by (2.2) and $\overline{x_0}\,\overline{y_0} = 1$, it follows that $\overline{x_0}, \overline{y_0} \in k^*$. Hence there exist $x_1, y_1 \in A$ and $\lambda_0, \mu_0 \in R^*$ such that $x_0 = \pi x_1 + \lambda_0$ and $y_0 = \pi y_1 + \mu_0$. Let $A_1 = R[x_1, y_1]$. Clearly $A_0 \subseteq A_1$. Since $\bar{\lambda}_0\bar{\mu}_0 = \bar{x}_0\bar{y}_0 = 1$, it follows that $\lambda_0\mu_0 - 1 = \pi\gamma_1$ for some element $\gamma_1 \in R$. Now,

$$F_0(\pi X + \lambda_0, \pi Y + \mu_0) = (\pi X + \lambda_0)(\pi Y + \mu_0) - 1$$

$$= \pi^2 XY + \pi\mu_0 X + \pi\lambda_0 Y + \pi\gamma_1 = \pi F_1(X, Y),$$

where $F_1 \in R^{[2]}$. Note that, by construction, $F_1(X, Y) = \pi XY + \alpha_1 X + \beta_1 Y + \gamma_1$, where $\alpha_1(=\mu_0) \in R^*, \beta_1(=\lambda_0) \in R^*$ and $\gamma_1 \in R$. Therefore, F_1 is irreducible and hence prime, and $F_1(x_1, y_1) = 0$ (since $F(x_0, y_0) = 0$). Hence it follows that $A_1 \cong R[X, Y]/(F_1(X, Y))$. If $A_1 = A$, then we are through (since in this case, $(A/\pi A)^* = (k^{[1]})^* = k^*$ and statement (b) is satisfied).

If $A_1 \neq A$, then we show that there exists a finite increasing chain of rings $A_0 \subset A_1 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots \subset A_m = A$ with $A_n = R[x_n, y_n]$, and a sequence of irreducible polynomials $F_n(X, Y) \in R[X, Y](=R^{[2]})$, $(1 \le n \le m)$, satisfying conditions (I) and (II) below for $1 \le n \le m$, and the recurrence relations (III) and (IV) for $1 \le n \le m - 1$.

(I) $F_n(X, Y) = \pi^n XY + \alpha_n X + \beta_n Y + \gamma_n$, where $\gamma_n \in R$ and $\alpha_n, \beta_n \in R^*$.

(II) $F_n(x_n, y_n) = 0$ and the map $R[X, Y]/F_n(X, Y) \to A_n$ defined by $X \to x_n$, $Y \to y_n$ is an isomorphism.

(III) $x_n = \pi x_{n+1} + \lambda_n$, $y_n = \pi y_{n+1} + \mu_n$ for some $\lambda_n, \mu_n \in R$.

(IV) $F_n(x_{n+1}, y_{n+1}) = \pi F_{n+1}(x_{n+1}, y_{n+1})$.

We have already defined A_1 and F_1 satisfying conditions (I) and (II) for $n = 1$. Assume that we have defined upto $A_n = R[x_n, y_n]$ and F_n, for some integer $n \ge 1$, such that conditions (I) and (II) hold. We show that if $A_n \neq A$, then it is possible to construct $A_{n+1} = R[x_{n+1}, y_{n+1}]$ and F_{n+1} using relations (III) and (IV), such that A_n is a proper subring of A_{n+1} and conditions (I) and (II) are satisfied by A_{n+1} and F_{n+1}.

Since $A_n[1/\pi] = A[1/\pi]$ and $A_n \neq A$, it follows, by arguing as in the case $n = 1$, that $\overline{x_n}, \overline{y_n} \in k$. Let $\lambda_n, \mu_n \in R$ be such that $\overline{x_n} = \overline{\lambda_n}$ and $\overline{y_n} = \overline{\mu_n}$. Hence there exist $x_{n+1}, y_{n+1} \in A$ such that relation (III) holds. Let $A_{n+1} = R[x_{n+1}, y_{n+1}]$. Clearly $A_n \subseteq A_{n+1}$. Since by induction hypothesis, condition (I) is valid for F_n, we have

$$F_n(\pi X + \lambda_n, \pi Y + \mu_n) = \pi^n(\pi X + \lambda_n)(\pi Y + \mu_n) + \alpha_n(\pi X + \lambda_n) + \beta_n(\pi Y + \mu_n) + \gamma_n$$

$$= \pi^{n+2} XY + \pi X(\alpha_n + \pi^n \mu^n) + \pi Y(\beta_n + \pi^n \lambda_n) + F_n(\lambda_n, \mu_n).$$

Now since (II) is valid for F_n, we have, $0 = F_n(x_n, y_n) = F_n(\pi x_{n+1} + \lambda_n, \pi y_{n+1} + \mu_n)$ which shows that $F_n(\lambda_n, \mu_n) \in \pi A \cap R = \pi R$. Let $\gamma_{n+1} = F_n(\lambda_n, \mu_n)/\pi (\in R)$. Now by the previous equations we can define

$$F_{n+1}(X,Y) = F_n(\pi X + \lambda_n, \pi Y + \mu_n)/\pi$$

$$= \pi^{n+1}XY + \alpha_{n+1}X + \beta_{n+1}Y + \gamma_{n+1},$$

where $\alpha_{n+1}(=\alpha_n + \pi^n\mu_n) \in R^*$, $\beta_{n+1}(=\beta_n + \pi^n\lambda_n) \in R^*$ and γ_{n+1} are all elements of R. Thus construction shows that F_{n+1} is an irreducible (and hence a prime) element of $R[X,Y]$ which satisfies condition (I) and the recurrence relation (IV). Moreover,

$$F_{n+1}(x_{n+1}, y_{n+1}) = F_n(x_n, y_n)/\pi = 0.$$

It follows that there is an R-isomorphism $R[X,Y]/(F_{n+1}(X,Y)) \to A_{n+1}$ mapping the images of X and Y to x_{n+1} and y_{n+1}, respectively. Thus (II) holds for the pair F_{n+1} and A_{n+1}.

We now show that $A_n \neq A_{n+1}$. Recall that $A = R[t_1,\dots,t_p]$. Let ℓ_n be the least integer such that $\pi^{\ell_n}t_j \in A_n$ $\forall j, 1 \leq j \leq p$. Such an integer exists since $A_n[1/\pi] = A[1/\pi]$. Moreover, $\ell_n > 0$ since $A_n \neq A$. Hence there exists $\phi \in R^{[2]}$ such that

$$\pi^{\ell_n}t_j = \phi(x_n, y_n)$$

$$= \phi(\pi x_{n+1} + \lambda_n, \pi y_{n+1} + \mu_n)$$

$$= \phi(\lambda_n, \mu_n) + \pi\theta(x_{n+1}, y_{n+1})$$

for some $\theta \in R^{[2]}$. Since $\ell_n > 0$, $\phi(\lambda_n, \mu_n) \in \pi A \cap R = \pi R$. Thus if

$$\psi(X,Y) = \phi(\lambda_n, \mu_n)/\pi + \theta(X,Y),$$

then

$$\pi^{\ell_n - 1}t_j = \psi(x_{n+1}, y_{n+1}) \in A_{n+1}$$

showing that

$$0 \leq \ell_{n+1} \leq \ell_n - 1 < \ell_n.$$

This shows that $A_n \neq A_{n+1}$.

Since the chain of integers

$$0 \leq \cdots \ell_{n+1} < \ell_n < \cdots < \ell_0$$

obviously cannot be infinite, there exists a positive integer m for which $\ell_m = 0$, i.e., $A_m = A$. In particular, $A/\pi A = k^{[1]}$ by construction of A_m.

We can now deduce conclusions (a) and (b).

(a) If $(A/\pi A)^* \neq k^*$, then $A = A_0$ (for otherwise, by our previous arguments, $A = A_m$ for a positive integer m and hence $(A/\pi A)^* = (k^{[1]})^* = k^*$, a contradiction). Thus A is $\mathbf{A}^*$ over R.

(b) If $(A/\pi A)^* = k^*$, then obviously $A \neq A_0$ and hence $A = A_m$ for some positive integer m and therefore by conditions (I) and (II), $A \cong R[X,Y]/(\pi^m XY + \alpha X + \beta Y + \gamma)$ for some $\alpha, \beta \in R^*$ and $\gamma \in R$. $\square$

Remark 3.8. Unlike the case of $\mathbf{A}^1$-fibration, the condition of geometric integrality on the closed fibre is not sufficient to ensure that a finitely generated overdomain of a discrete valuation ring, whose generic fibre is $\mathbf{A}^*$, is itself $\mathbf{A}^*$. In fact it is easy to see that if (R, π, k) is a discrete valuation ring, then for any $\alpha, \beta \in R^*$, $\gamma \in R$ and positive integer m, $R[X,Y]/(\pi^m XY + \alpha X + \beta Y + \gamma)$ is a finitely generated flat R-algebra whose generic fibre is $\mathbf{A}^*$ and closed fibre is $\mathbf{A}^1$.

Corollary 3.9. *Let R be a Principal Ideal Domain with quotient field K and suppose that A is a finitely generated overdomain of R such that*
(i) *The generic fibre $K \otimes_R A$ is $\mathbf{A}^*$ over K.*
(ii) *Each closed fibre A/PA is geometrically integral but is not $\mathbf{A}^1$ over R/P.*
Then A is $\mathbf{A}^*$ over R.

Proof. Let P be a maximal ideal of R. By the hypotheses, R_P is a discrete valuation ring and A_P is a finitely generated flat R_P-algebra whose generic fibre is $\mathbf{A}^*$, and whose closed fibre $k(P) \otimes_R A$ is geometrically integral, but $k(P) \otimes_R A \neq k(P)^{[1]}$. But then, by part (b) of Proposition 3.7, $(k(P) \otimes_R A)^* \neq k(P)^*$. Therefore, by part (a) of (3.7), A_P is $\mathbf{A}^*$ over R_P.

Thus, A is locally $\mathbf{A}^*$ over R. Since every invertible ideal of a PID is principal, by Theorem 3.4, it follows that A is $\mathbf{A}^*$ over R. □

Remark 3.10. Suppose that R is a PID and A is a finitely generated flat R-algebra such that the generic fibre is $\mathbf{A}^*$ and all the closed fibres are geometrically integral. Then, by (3.7), each closed fibre is either $\mathbf{A}^*$ or $\mathbf{A}^1$. It is possible that some are $\mathbf{A}^*$ and some $\mathbf{A}^1$. For instance, let R be a PID with two maximal ideals (π_1) and (π_2). Let $A = R[X,Y]/(\pi_2 XY + \pi_1 X + \pi_2 Y + 1)$. Then the generic fibre of A is $\mathbf{A}^*$, the closed fibre $A/\pi_1 A$ is $\mathbf{A}^*$ but the closed fibre $A/\pi_2 A$ is $\mathbf{A}^1$.

We now prove our main theorem over Krull domains.

Theorem 3.11. *Let R be a Krull domain with quotient field K and let A be a finitely generated flat R-algebra such that*
(i) *The generic fibre $K \otimes_R A$ is $\mathbf{A}^*$ over K.*
(ii) *For each prime ideal P of R of height one, the fibre ring $k(P) \otimes_R A$ is geometrically integral but is not $\mathbf{A}^1$ over $k(P)$.*
Then there exists an invertible ideal I in R such that A is isomorphic to $\bigoplus_{n\in\mathbf{Z}} I^n T^n$ as a $\mathbf{Z}$-graded R-algebra.

Proof. Since A is finitely generated over R, by condition (i), there exists a non-zero element $x \in R$ such that $A[1/x]$ is $\mathbf{A}^*$ over $R[1/x]$. If $x \in R^*$, we are through. If not, then let $P_1, \ldots, P_m$ be the prime divisors of xR and let $S = R \setminus (P_1 \cup \cdots \cup P_m)$. Since R is a Krull domain, $ht\, P_i = 1$ $\forall i, 1 \le i \le m$. Therefore $S^{-1}R$ is a semi-local Dedekind domain and hence a PID. It follows, by (3.9), that $S^{-1}A$ is $\mathbf{A}^*$ over $S^{-1}R$. Hence there exists $y \in S$ such that A_y is $\mathbf{A}^*$ over R_y. Since by construction x and y either form a sequence or are comaximal, the result now follows from (3.1). □

Remark 3.12. Note that the example in (3.10) shows that, in the statement of Theorem 3.11, it is necessary to impose the condition that the codimension one fibres are not $\mathbf{A}^1$. From Proposition 3.7, it follows (assuming all other hypotheses in (3.11)) that any codimension one fibre which is not $\mathbf{A}^1$, is automatically $\mathbf{A}^*$. Also note that, by (3.7), the codimension one fibres are $\mathbf{A}^1$ if and only if they do not have non-trivial units. Thus, condition (ii) in Theorem 3.11 will be satisfied, for instance, under either of the following hypotheses on the fibres at the prime ideals P of R of height one:
(ii)′ $(k(P) \otimes_R A)^* \neq (k(P))^*$.
(ii)″ $k(P) \otimes_R A$ are $\mathbf{A}^*$-forms.

We now investigate the general case.

Theorem 3.13. *Let R be a noetherian domain with quotient field K and let A be a finitely generated flat R-algebra such that*
(i) *The generic fibre $K \otimes_R A$ is $\mathbf{A}^*$ over K.*
(ii) *For each prime ideal P of R of height one, the fibre ring $k(P) \otimes_R A$ is geometrically integral but is not an $\mathbf{A}^1$-form over $k(P)$.*
Then the following results hold:
(a) *All the fibre rings are $\mathbf{A}^*$-forms.*
(b) *There exists a finite birational extension R' of R and an invertible ideal I of R' such that $R' \otimes_R A$ is a $\mathbf{Z}$-graded R-algebra isomorphic to $R'[IT, I^{-1}T^{-1}]$.*
(c) *If R contains a field of characteristic zero, and all the fibre rings have more units than the respective residue fields, then A is an $\mathbf{A}^*$-fibration over R.*

Proof. Suitable modifications in the arguments in [3, 3.5] would give a proof of (a). For the convenience of the reader we sketch the proof below.

Fix a prime ideal P of R. Replacing R by R_P, we assume that R is a local noetherian domain with maximal ideal P. We prove (a) by induction on $ht\,P = dim\,R$. The case $ht\,P = 0$ is trivial.

If $ht\,P(=dim\,R) = 1$, then, from the Krull–Akizuki theorem ([5, 33.2]), it would follow that the normalisation $\tilde{R}$ of R is a PID and $k(\tilde{P})$ are algebraic extensions of $k(P)$ for all maximal ideals $\tilde{P}$ of $\tilde{R}$. Therefore, from condition (ii), it follows that for all $\tilde{P} \in Max\,\tilde{R}$, $k(\tilde{P}) \otimes_{\tilde{R}} (\tilde{R} \otimes_R A)$ are geometrically integral but are not $\mathbf{A}^1$ over $k(\tilde{P})$. Moreover, by condition (i), the generic fibre of $\tilde{R} \otimes_R A$ is $\mathbf{A}^*$ over $\tilde{R}$. Hence, by (3.9), $\tilde{R} \otimes_R A$ is $\mathbf{A}^*$ over $\tilde{R}$. In particular, $k(\tilde{P}) \otimes_R A$ is $\mathbf{A}^*$ over $k(\tilde{P})$ $\forall \tilde{P} \in Max\,\tilde{R}$. Hence $k(P) \otimes_R A$ is an $\mathbf{A}^*$-form over $k(P)$.

If $ht\,P \geq 2$, then, by induction hypothesis, we assume that the fibre rings $k(Q) \otimes_R A$ are $\mathbf{A}^*$-forms for all non-maximal prime ideals Q of R. Let $\hat{R}$ denote the completion of R and let $\hat{A} = \hat{R} \otimes_R A$. Now $\hat{R}$ is a complete local ring with maximal ideal $\hat{P}$ such that $R/P \cong \hat{R}/\hat{P}$ and $\hat{A}$ is a finitely generated flat $\hat{R}$-algebra whose non-closed fibres are all

$\mathbf{A}^*$-forms. Moreover, if $\hat{Q}$ is a minimal prime ideal of $\hat{R}$, then, since $\hat{R}$ is R-flat, by the "going-down theorem", $\hat{Q}$ contracts to (0) in R. Hence, it follows from condition (i) that, the fibres of $\hat{A}$ at all minimal prime ideals of $\hat{R}$ are $\mathbf{A}^*$. Let $\hat{Q}_0$ be a minimal prime ideal of $\hat{R}$ such that $dim\,\hat{R} = dim(\hat{R}/\hat{Q}_0)$. Then, replacing R by $\hat{R}/\hat{Q}_0$ and A by $\hat{A}/\hat{Q}_0\hat{A}$, we may assume R to be a complete local noetherian domain to start with, and assume A to be a finitely generated flat R-algebra such that the generic fibre of A is $\mathbf{A}^*$ and the fibres at all non-maximal prime ideals of R are $\mathbf{A}^*$-forms. In ([5, 32.1]), the normalisation $\tilde{R}$ of R is a finite R-module and hence a noetherian normal local domain. Now, as before, it would follow that $\tilde{R} \otimes_R A$ is $\mathbf{A}^*$ over $\tilde{R}$ showing that $k(P) \otimes_R A$ is an $\mathbf{A}^*$-form over $k(P)$.

We now prove (b). Let $\tilde{R}$ denote the normalisation of R and let $\tilde{A} = \tilde{R} \otimes_R A$. By a theorem of Nagata [5, 33.10], $\tilde{R}$ is a Krull domain. Clearly $\tilde{A}$ is a finitely generated flat algebra over $\tilde{R}$ and its generic fibre is $\mathbf{A}^*$. Moreover, since the residue fields of $\tilde{R}$ are algebraic over the residue fields of R, by result (a), all fibres of $\tilde{A}$ are $\mathbf{A}^*$-forms over their respective residue fields. Hence, by Theorem 3.11, there exists an invertible ideal $\tilde{I}$ in $\tilde{R}$ such that

$$\tilde{R} \otimes_R A = \tilde{R}[\tilde{I}T, \tilde{I}^{-1}T^{-1}]. \qquad (*)$$

$\tilde{I}$, being invertible, is finitely generated, say, $\tilde{I} = (a_1, \dots, a_m)\tilde{R}$. Let $b_1, \dots, b_m \in K$ be such that $a_1b_1 + \cdots + a_mb_m = 1$, so that $\tilde{I}^{-1}$ is generated by $b_1, \dots, b_m$ as an $\tilde{R}$-module.

Since A is finitely generated over R, $A = R[t_1, \dots, t_p]$ for some $t_1, \dots, t_p \in A$. By Eq. $(*)$, $1 \otimes t_j = \sum_{-s_j \le i \le r_j} g_{ji}T^i$ for some $g_{ji} \in I^i$, $1 \le j \le p$. The coefficients g_{ji} may be expressed as

$$g_{ji} = \begin{cases} \displaystyle\sum_{i_1+\cdots+i_m=i} c_{ji_1\cdots i_m} a_1^{i_1} \cdots a_m^{i_m} & \text{for } i \ge 0 \\ \displaystyle\sum_{i_1+\cdots+i_m=-i} c_{ji_1\cdots i_m} b_1^{i_1} \cdots b_m^{i_m} & \text{for } i < 0, \end{cases}$$

where $c_{ji_1\cdots i_m} \in \tilde{R}$. Again, by Eq. $(*)$, we have

$$a_iT = \sum_{1 \le \ell \le q_i} u_{i\ell} \otimes v_{i\ell} \quad \text{and} \quad b_iT^{-1} = \sum_{1 \le \ell \le t_i} w_{i\ell} \otimes z_{i\ell}$$

for some $u_{i\ell}, w_{i\ell} \in \tilde{R}$ and $v_{i\ell}, z_{i\ell} \in A$.

Now let R' be the R-subalgebra of $\tilde{R}$ generated by the elements $a_1, \dots, a_m$; a_ib_ℓ (where $1 \le i, \ell \le m$); $c_{ji_1\cdots i_m}$ (where $i_1 + \cdots + i_m = |i|, -s_j \le i \le r_j, 1 \le j \le p$); $u_{i\ell}$ (where $1 \le \ell \le q_i, 1 \le i \le m$) and $w_{i\ell}$ (where $1 \le \ell \le t_i, 1 \le i \le m$). Let I be the ideal $(a_1, \dots, a_m)R'$. Then R' is a finite birational extension of R and I is an invertible ideal of R'.

Since A is flat over R, $R' \otimes_R A$ may be identified with its image in $\tilde{R} \otimes_R A$. Then it is easy to see that $R' \otimes_R A = R'[IT, I^{-1}T^{-1}]$.

Part (c) follows from (2.3). □

Remark 3.14. The above proof shows that in the statement of (3.13), in condition (i), it is enough to assume that the generic fibre is an $\mathbf{A}^*$-form. (In the proof take $\tilde{R}$ to be the integral closure of R in L, where L is a finite extension of K such that $L \otimes_R A$ is $\mathbf{A}^*$ over L.)

Suppose that R is a one-dimensional noetherian domain and A is a finitely generated flat R-algebra whose generic fibre is $\mathbf{A}^*$ and whose closed fibres are geometrically integral. We have seen in (3.10) that, in this situation, a closed fibre might be $\mathbf{A}^1$. Moreover, if R is a PID and a closed fibre is not $\mathbf{A}^1$, then, by (3.7), that closed fibre is necessarily $\mathbf{A}^*$. However, the following example shows that if R is not normal, then, under the above hypotheses, a closed fibre might be a non-trivial $\mathbf{A}^1$-form. Therefore we do need the stronger hypothesis in condition (ii) of Theorem 3.13 as compared to the corresponding condition in Theorem 3.11.

Example 3.15. Let k be a non-perfect field of characteristic p. Let $\beta \in k$ be such that $Z^p - \beta$ is irreducible in $k[Z]$. Let $L = k[Z]/(Z^p - \beta) = k(\alpha)$, where $\alpha^p = \beta$. Now let $R = k + (U)L[[U]]$, considered as a subring of $L[[U]]$. Then R is a one-dimensonal local domain with maximal ideal $M = (U)L[[U]]$, quotient field $K = L((U))$ and residue field k. Being a finite module over $k[[U]]$, R is noetherian.

Let $X_1 = X + \alpha Y$ and $Y_1 = Y - X_1^p$. Then it is easy to see that that $K[X_1, Y_1] = K[X, Y]$ and $UX_1, Y_1 \in R[X, Y]$. Let $F(X, Y) = UX_1 Y_1 + Y_1 + 1$ and $A = R[X, Y]/(F(X, Y))$. One can verify that A is R-flat, the generic fibre $K \otimes_R A$ is $\mathbf{A}^*$ over K and the closed fibre $k \otimes_R A$ is a non-trivial $\mathbf{A}^1$-form over k.

In [3, 3.5], it was shown that if R contains the field of rationals, then conditions on generic and codimension one fibres are enough to conclude that A is an $\mathbf{A}^1$-fibration over R. But below we give an example of a finitely generated flat algebra A over a two-dimensional noetherian local domain R, whose fibres at all non-closed points of $Spec R$ are $\mathbf{A}^*$, but whose closed fibre is a non-trivial $\mathbf{A}^*$-form. Thus in the non-normal situation, we need a condition on *all* fibres (i.e., the existence of non-trivial units) to conclude that all fibres are actually $\mathbf{A}^*$.

Example 3.16. Let $\mathbf{R}$ and $\mathbf{C}$ denote the field of real numbers and complex numbers, respectively. Let $R = \mathbf{R} + (U, V)\mathbf{C}[[U, V]]$ (considered as a subring of $\mathbf{C}[[U, V]]$). Then R is a two-dimensional local domain with maximal ideal $M = (U, V)\mathbf{C}[[U, V]]$, quotient field $K = \mathbf{C}((U, V))$ and residue field $\mathbf{R}$. Being a finite module over $\mathbf{R}[[U, V]]$, R is noetherian. Let $A = R[X, Y]/(X^2 + Y^2 - 1)$. Then A is a finitely generated R-algebra and being a free module over $R[X]$, it is also flat over R.

Now let $\tilde{R}$ denote the normalisation of R. Then $\tilde{R} = \mathbf{C}[[U, V]]$ and M is the conductor of $\tilde{R}$ in R. Clearly $\tilde{R} \otimes_R A$ is $\mathbf{A}^*$ over $\tilde{R}$ and hence $k(Q) \otimes_R A$ is $\mathbf{A}^*$ over $k(Q)$ for every prime ideal Q of $\tilde{R}$.

Since M is the conductor of $\tilde{R}$ in R, for every non-maximal prime ideal P of R, $R_P = \tilde{R}_P$ so that $k(P) \otimes_R A$ is $\mathbf{A}^*$ over $k(P)$. But $k(M) \otimes_R A = \mathbf{R}[X, Y]/(X^2 + Y^2 - 1)$ is an $\mathbf{A}^*$-form over $k(M)(=\mathbf{R})$ but is not $\mathbf{A}^*$ over $k(M)$. □

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