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ON CHARACTERIZATIONS OF PITMAN CLOSENESS OF SOME SHRINKAGE ESTIMATORS

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ABSTRACT

For the multivariate normal mean (vector) estimation problem, some characterizations of the Pitman closest property of a general class of shrinkage (or Stein-rule) estimators (including the so called positive-rule versions) are studied. Further, for the same model when the parameter is restricted to a positively homogeneous cone, Pitman closeness of restricted shrinkage maximum likelihood estimators is established.

1. INTRODUCTION

Consider a p-variate normal distribution with an unknown mean vector $\underset{\sim}{\theta}$. Under a quadratic loss, the classical maximum likelihood estimator (MLE) is not admissible for $p \geq 3$, and shrinkage or Stein-rule versions dominate the MLE in the sense of having uniformly (in $\underset{\sim}{\theta}$) a smaller (or at most equal) risk. Sen, Kubokawa and Saleh (1989) have shown that a similar dominance result holds under a (generalized) Pitman closeness criterion (PCC), and further, for this, it suffices to take $p \geq 2$. The PCC is an intrinsic measure of the comparative behavior of two estimators without requiring the existence of their first or higher order moments. On the other hand, a characterization of the *Pitman closest property* (PCP) of an estimator (even within a class) may require an exhaustive pairwise comparisons with other estimators (belonging to the same class), and thereby may generally demand additional regularity conditions. Actually, such a PCP characterization may not universally hold. The usual transitiveness property which pertains to quadratic or other conventional loss functions may not hold under the PCC [viz., Blyth (1972)], and hence, some non-standard analysis (specifically tailored for specific models) may be necessary for such a PCP characterization. For some simple (mostly, univariate) models, some success along this line has been achieved, only very recently, by restricting to suitable class of *equivariant estimators* where *equivariance* is sought with respect to suitable group of transformations which map the sample space onto itself [viz., Ghosh and Sen (1989), Nayak (1990), and Sen (1990), among other]. In genuine multivariate estimation problems, the entire class of equivariant estimators may be too big to ensure the PCP characterization. For example, for the multivariate normal dispersion matrix model, there is a gradation of various equivariant estimators of the dispersion matrix, and a (generalized) PCP characterization may hold for certain subclasses, but not for the entire class [viz., Sen, Nayak and Khattree (1990)]. In this respect, the results in Sen, Kubokawa and Saleh (1989) are only partial, and there is a need to incorporate the PCP characterization results for a more comprehensive account. Motivated by this query, the first objective of the present study is to examine the PCP characterizations of estimators of the multivariate normal mean vector within a class of shrinkage or Stein-rule estimators. As will be seen that for some class of shrinkage estimators, such a PCP holds while it may not do so for some other class.

Sengupta and Sen (1991) have considered some multivariate normal mean models when the parameter is restricted to a *positively homogeneous cone*, and they have shown that in the light of the usual *quadratic risks*, the usual *restricted MLE* (RMLE) dominates the *unrestricted MLE* (UMLE), but is dominated by appropriate *restricted shrinkage* MLE (RSMLE). In view of the PC dominance results in the unrestricted model, treated in Sen, Kubokawa and Saleh (1985), it is quite natural to inquire whether parallel PC domiance results hold for the restricted parameter space model too? This is the dual objective of the current study.

Section 2 is devoted to the study of PCP characterization of shrinkage estimators of multivariate normal mean vectors in an unrestricted setup. Section 3 deals with the Pitman closeness dominance of RSMLE over the RMLE when the parameter belongs to a positively homogeneous cone. Some general remarks are appended in the concluding section.

2. PCP OF SHRINKAGE ESTIMATORS

Let $\underset{\sim}{X} \sim \mathcal{N}_p(\underset{\sim}{\theta}, \sigma^2 \underset{\sim}{V})$ where p is ≥ 2, $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_p)'$ and σ^2 are unknown parameters and $\underset{\sim}{V}$ is a known positive definite (p.d.) matrix. For two estimators $\underset{\sim}{\delta}_1$ and $\underset{\sim}{\delta}_2$ of $\underset{\sim}{\theta}$, and for a given positive definite (p.d.) matrix $\underset{\sim}{Q}$, defining the *norm* $\|\underset{\sim}{x}-\underset{\sim}{y}\|^2_Q$ as $(\underset{\sim}{x}-\underset{\sim}{y})'\underset{\sim}{Q}(\underset{\sim}{x}-\underset{\sim}{y})$, we say that $\underset{\sim}{\delta}_1$ is closer to $\underset{\sim}{\theta}$ than $\underset{\sim}{\delta}_2$ (in the norm $\|\cdot\|_{\underset{\sim}{Q}}$) in the Pitman sense if

$$P_{\underset{\sim}{\theta},\sigma}\{\|\underset{\sim}{\delta}_1 - \underset{\sim}{\theta}\|_{\underset{\sim}{Q}} \leq \|\underset{\sim}{\delta}_2-\underset{\sim}{\theta}\|_{\underset{\sim}{Q}}\} \geq 1/2, \quad \forall\ \underset{\sim}{\theta},\sigma, \tag{2.1}$$

with strict inequalities holding for some $\underset{\sim}{\theta}$. Note that (2.1) is an equivalent representation of

$$P_{\underset{\sim}{\theta},\sigma}\{\|\underset{\sim}{\delta}_1 - \underset{\sim}{\theta}\|_{\underset{\sim}{Q}} < \|\underset{\sim}{\delta}_2 - \underset{\sim}{\theta}\|_{\underset{\sim}{Q}}\} \geq P_{\underset{\sim}{\theta},\sigma}\{\|\delta_1 - \underset{\sim}{\theta}\|_{\underset{\sim}{Q}} > \|\underset{\sim}{\delta}_2 - \underset{\sim}{\theta}\|_{\underset{\sim}{Q}}\},\ \forall\ \underset{\sim}{\theta},\sigma.$$

However, in (2.1), the "less than or equal to" sign may not be generally replaceable by "less than" sign (unless the probability for the tie $\|\underset{\sim}{\delta}_1 - \underset{\sim}{\theta}\|_{\underset{\sim}{Q}} = \|\underset{\sim}{\delta}_2 - \underset{\sim}{\theta}\|_{\underset{\sim}{Q}}$ is null, for all $\underset{\sim}{\theta}$), and in the context of shrinkage or Stein-rule estimators, we shall find (2.1) more convenient than its variants. In the particular model, chosen above, one may take $\underset{\sim}{Q} = \underset{\sim}{V}^{-1}$, although $\underset{\sim}{Q}$ may often be chosen from other extraneous considerations. Sen, Kubokawa and Saleh (1989) considered some shrinkage (or Stein-rule estimators of $\underset{\sim}{\theta}$ of the form

$$\underset{\sim}{\delta}_\varphi = \underset{\sim}{X} - \varphi(X,S)S\|\underset{\sim}{X}\|^{-2}_{\underset{\sim}{Q},\underset{\sim}{V}}\ \underset{\sim}{Q}^{-1}\ \underset{\sim}{V}^{-1}\underset{\sim}{X}, \tag{2.2}$$

where S is distributed as $\sigma^2\ q^{-1}\ \chi^2_q$, independently of $\underset{\sim}{X}$, χ^2_q has the central chi square distribution with q degrees of freedom (DF), $0 \leq \varphi(\underset{\sim}{x},s) \leq (p-1)(3p+1)/2p$ for every $(\underset{\sim}{X},S)$ a.e., $p \geq 2$ and $\|\underset{\sim}{X}\|^2_{\underset{\sim}{Q},\underset{\sim}{V}} = \underset{\sim}{X}'\underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}\underset{\sim}{X}$. If σ^2 is known, in (2.2), S has to be replaced by σ^2.

Then, they have shown that

$$P_{\theta,\sigma}\{\|\delta_{\varphi} - \theta\|_Q \leq \|X-\theta\|_Q\} \geq 1/2, \ \forall \ (\theta, \sigma^2), \tag{2.3}$$

so that a SMLE δ_{φ} dominates the MLE X in the Pitman closeness measure for all $p \geq 2$ and all shrinkage factors $\varphi(\cdot)$ satisfying the bounds stated above. An important member of this class of SMLE is the simple James-Stein (1961) estimator for which $\varphi(X,S) = b : 0 < b < (p-1)(3p+1)/2p$, where the upper bound is modified in the light of the PC measure as in Sen, Kubokawa and Saleh (1989) and Keating and Mason (1988). The PC measure in (2.1) is a pairwise measure. If (2.1) holds for all $\delta_2 \in \mathscr{C}$ and $\delta_1 \in \mathscr{C}$, for a suitable class $\mathscr{C}$ of estimators, then δ_1 is said to be the *Pitman closest estimator* (PCE) of θ within the class $\mathscr{C}$. Let us denote by $\mathscr{C}$ the class of all SMLE $\{\delta_{\varphi}\}$ where φ satisfies the inequalities $0 < \varphi(X,S) \leq (p-1)(3p+1)/2p$. We are primarily interested in a characterization of the PCE of θ within the class $\mathscr{C}$ (or a suitable subset of it). In Sen, Kubokawa and Saleh (1989), positive rule versions of the SMLE δ_{φ}, (denoted by δ^+) were also considered, and it was shown that δ^+ dominates $\delta_{\varphi=b}$ in the PC sense. There is a natural question whether a PCE exists within the class of positive rule versions of SMLE? We shall study this problem too.

To set the inquires in proper perspectives, let us first consider the simplest situation where σ^2 is known, and without any loss of generality, we may set $V = I_p$. In this setup, without any loss of generality, we let $Q = I_p$, so that in (2.1), we need to use the Euclidean norm $\|\cdot\|$ and (2.2) reduces to

$$\delta_{\varphi} = \{1 - \varphi(X, \sigma^2)\ \sigma^2\|X\|^{-2}\}X \ . \tag{2.4}$$

For the usual James-Stein (1961) version, we set $\varphi = b\ (> 0)$, so that (2.4) reduces to

$$\underset{\sim}{\delta}(b) = \{1 - b\,\sigma^2\|\underset{\sim}{X}\|^{-2}\}\underset{\sim}{X},\ b > 0. \tag{2.5}$$

For the positive-rule version, we set

$$\varphi(\underset{\sim}{x},\sigma^2) = \begin{cases} \|\underset{\sim}{x}\|^2/\sigma^2, & \|\underset{\sim}{x}\|^2 \le b\,\sigma^2 \\ b, & \|\underset{\sim}{x}\|^2 > b\,\sigma^2, \end{cases} \tag{2.6}$$

so that

$$\underset{\sim}{\delta}^+(b) = \{1 - b\,\sigma^2\|\underset{\sim}{x}\|^{-2}\}^+\ \underset{\sim}{X},\quad b > 0, \tag{2.7}$$

where $a^+ = (a \vee 0)$ max(a,0). It follows from Sen, Kubokawa and Saleh (1989) that $\underset{\sim}{\delta}^+(b)$ dominates $\underset{\sim}{\delta}(b)$ in the Pitman closeness sense, for all permissible values of b (> 0) (for which $\underset{\sim}{\delta}(b)$ dominates $\underset{\sim}{X}$ in the same sense). This leads us to formulate the following classes of shrinkage estimators:

(i) $\mathscr{C}_{JS} = \{\underset{\sim}{\delta}(b) : 0 < b \le (p-1)(3p+1)/2p\}$

(ii) $\mathscr{C}^+_{JS} = \{\underset{\sim}{\delta}^+(b) : 0 < b \le (p-1)(3p+1)/2p\}$

and (iii) $\mathscr{C}_\varphi$ = general class in (2.4) with $\varphi \le (p-1)(3p+1)/2p$.

Even within each of these subclasses, there may not be a Pitman closest estimator. To illustrate this point, we denote by m_p = med(χ^2_p) and let

$$\mathscr{C}^{(1)}_{JS} = \{\underset{\sim}{\delta}(b) : m_p \le b \le (p-1)(3p+1)/2p\} \tag{2.8}$$

$$\mathscr{C}^{(2)}_{JS} = \{\underset{\sim}{\delta}(b) : 0 < b \le m_p\} \tag{2.9}$$

Then, we have the following.

<u>Theorem 2.1</u> $\underset{\sim}{\delta}(m_p) = \underset{\sim}{\delta}^0$ *is the Pitman closest estimator of* $\underset{\sim}{\theta}$ *within the class* $\mathscr{C}^{(1)}_{JS}$.

Proof. Note that

$$\|\underset{\sim}{\delta}(b) - \underset{\sim}{\theta}\|^2 = \|\underset{\sim}{X} - \underset{\sim}{\theta}\|^2 - 2b\,\sigma^2\|\underset{\sim}{X}\|^{-2}\,\underset{\sim}{X}'(\underset{\sim}{X}-\underset{\sim}{\theta}) + b^2\|\underset{\sim}{X}\|^{-2}\,\sigma^4,$$

for every $b > 0$. Therefore, by (2.1), we obtain that for every $b \geq m_p$, setting $\sigma^2 = 1$,

$$\begin{aligned}
&P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{\delta}^0 - \underset{\sim}{\theta}\| \leq \|\underset{\sim}{\delta}(b) - \underset{\sim}{\theta}\|\} \\
&= P_{\underset{\sim}{\theta}}\{(b^2 - m_p^2) \geq 2(b-m_p)\,\underset{\sim}{X}'(\underset{\sim}{X} - \underset{\sim}{\theta})\} \\
&= P_{\underset{\sim}{\theta}}\{\underset{\sim}{X}'(\underset{\sim}{X}-\underset{\sim}{\theta}) \leq \tfrac{1}{2}(b + m_p)\} \\
&= P_\lambda\{\chi^2_{p,\lambda} \leq \lambda + \tfrac{1}{2}(m_p + b)\}; \quad \lambda = \tfrac{1}{4}\|\underset{\sim}{\theta}\|^2,
\end{aligned} \tag{2.10}$$

where the last step follows by noting that $\underset{\sim}{x}'(\underset{\sim}{x}-\underset{\sim}{\theta}) = \|\underset{\sim}{x} - \frac{1}{2}\underset{\sim}{\theta}\|^2 - \lambda$ and $\underset{\sim}{X} - \frac{1}{2}\underset{\sim}{\theta} \sim \mathcal{N}_p(\frac{1}{2}\underset{\sim}{\theta}, \underset{\sim}{I}_p)$. Note that

$$\tfrac{1}{2}(b + m_p) > m_p, \text{ for every } b > m_p, \tag{2.11}$$

while by the subadditive property of $m_p^{(\lambda)}$ = median of $\chi^2_{p,\lambda}$ [viz., Sen (1989)], we have

$$m_p^{(\lambda)} \leq m_p^{(0)} + \lambda = m_p + \lambda, \quad \forall\ \lambda \geq 0. \tag{2.12}$$

Hence, from (2.10), (2.11) and (2.12), we have

$$P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{\delta}^0 - \underset{\sim}{\theta}\| \leq \|\underset{\sim}{\delta}(b) - \underset{\sim}{\theta}\|\} \geq \tfrac{1}{2}, \quad \forall\ \underset{\sim}{\theta},\ b \geq m_p, \tag{2.13}$$

so that within the class $\mathscr{C}^{(1)}_{JS}$, $\underset{\sim}{\delta}^0 = \underset{\sim}{\delta}(m_p)$ is the Pitman closest estimator of $\underset{\sim}{\theta}$. Q.E.D.

Before we proceed to consider the dual class $\mathscr{C}^{(2)}_{JS}$, we may note that if in (2.8), we replace m_p by any $a = m_p - \eta$, for some $\eta > 0$, then in (2.10) and (2.11), $\frac{1}{2}(b + a)$ can be made smaller than

m_p by choosing b sufficiently close to m_p. Thus, for $\underset{\sim}{\theta} = \underset{\sim}{0}$ (and a neighborhood of $\underset{\sim}{0}$), $\lambda + \frac{1}{2}(a + b)$ will be less than $m_p^{(\lambda)}$, and hence, (2.13) will be $< \frac{1}{2}$. Thus, (2.13) can not hold for every $\underset{\sim}{\theta}$, and hence, the desired PCP does not hold. This explains why the class $\mathscr{C}_{JS}^{(1)}$ has the vertex $\underset{\sim}{\delta}^0$ and that $\underset{\sim}{\delta}^0$ is the unique estimator of $\underset{\sim}{\theta}$ within this class to ensure the desired PC property.

Let us next examine the PCP within the class $\mathscr{C}_{JS}^{(2)}$. Proceeding as in (2.10), we have for every $0 < b < m_p$,

$$P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{\delta}^0 - \underset{\sim}{\theta}\| \le \|\underset{\sim}{\delta}(b) - \underset{\sim}{\theta}\|\}$$

$$= P_\lambda\{\chi^2_{p,\lambda} \ge \lambda + \frac{1}{2}(m_p + b)\}; \quad \lambda = \frac{1}{4}\|\underset{\sim}{\theta}\|^2 . \qquad (2.14)$$

Now, we can use (2.24) of Sen, Kubokawa and Saleh (1989) and conclude that $\underset{\sim}{\delta}^0$ dominates $\underset{\sim}{\delta}(b)$ in the Pitman sense whenever $\frac{1}{2}(m_p + b)$ is $\le (p-1)(3p+1)/4p$, i.e.,

$$0 < b \le (p-1)(3p+1)/2p - m_p = \frac{p^2-1}{2p} - \xi_p, \qquad (2.15)$$

where ξ_p is a decreasing function of p; at $p = 2$, $\xi_2 = 0.386$ and it monotonically converges to 0.334 as p increases. In a similar manner, it follows that on letting $\xi_p^0 = (p^2-1)/2p - \xi_p$, $\underset{\sim}{\delta}(b_p^0)$ is the Pitman closest estimator of $\underset{\sim}{\theta}$ within the subclass $\{\underset{\sim}{\delta}(b);\ 0 < b \le b_p^0\}$ ($\subset \mathscr{C}_{JS}^{(2)}$). However, $\underset{\sim}{\delta}(b_p^0)$ may not dominate $\underset{\sim}{\delta}^0$ (or any other $\underset{\sim}{\delta}(b)$, for b close to m_p) in the Pitman sense, and hence, $\underset{\sim}{\delta}(b_p^0)$ is not the Pitman closest estimator within the class $\mathscr{C}_{JS}^{(2)}$. Similarly, we shall see that $\underset{\sim}{\delta}^0$ fails to dominate (in the Pitman sense) any other $\underset{\sim}{\delta}(b)$ when b is very close (from below) to m_p. The proof of this rests on the following result. Let $M_p^{(\lambda)}$ and $m_p^{(\lambda)}$ be respectively the *mode* and *median* of $\chi^2_{p,\lambda}$, for $\lambda \ge 0$ and $p \ge 2$.

Lemma 2.1 For every $p \geq 2$ and $\lambda \geq 0$,

$$m_p^{(\lambda)} \leq M_{p+2}^{(\lambda)} . \tag{2.16}$$

Outline of the proof. Although (2.16) has been stated in Sen (1989), the proof remains obscured due to a multitude of typographical errors. As such, we provide here a more direct proof. As in Sen (1989), we denote the d.f. and the p.d.f. of $\chi^2_{q,\lambda}$ by $G_q^{(\lambda)}(x)$ and $g_q^{(\lambda)}(x)$, respectively, for $x \geq 0$, $\lambda \geq 0$ and $q \geq 1$. Also, let $\bar{G}_q^{(\lambda)}(x) = 1 - G_q^{(\lambda)}(x)$. Then, as in Sen (1989), we have

$$\begin{aligned}(\partial/\partial\lambda)\, \bar{G}_p^{(\lambda)}(M_{p+2}^{(\lambda)}) &= (\partial/\partial\lambda)\, P_\lambda\{\chi^2_{p,\lambda} \geq M_{p+2}^{(\lambda)}\} \\ &= -\frac{1}{2}\, G_p^{(\lambda)}(M_{p+2}^{(\lambda)}) + \frac{1}{2}\, G_{p+2}^{(\lambda)}(M_{p+2}^{(\lambda)}) - g_p^{(\lambda)}(M_{p+2}^{(\lambda)})(\partial/\partial\lambda)M_{p+2}^{(\lambda)} \\ &= [g_{p+2}^{(\lambda)}(M_{p+2}^{(\lambda)}) - g_p^{(\lambda)}(M_{p+2}^{(\lambda)})(\partial/\partial\lambda)M_{p+2}^{(\lambda)}] \\ &= [1 - (\partial/\partial\lambda)M_{p+2}^{(\lambda)}]\; g_{p+2}^{(\lambda)}(M_{p+2}^{(\lambda)},\end{aligned} \tag{2.17}$$

as $g_{p+2}^{(\lambda)}(M_{p+2}^{(\lambda)}) = g_p^{(\lambda)}(M_{p+2}^{(\lambda)})$, $\forall p$ and λ. Thus, we may proceed as in (2.23) through (2.28) of Sen (1989) and conclude that for our (2.16) to hold, it suffices to show that

$$M'_\lambda = (\partial/\partial\lambda)M_{p+2}^{(\lambda)} \leq 1, \quad \forall\, \lambda \geq 0,\ p \geq 2. \tag{2.18}$$

Writing $a_\lambda = g_{p+4}^{(\lambda)}(M_{p+2}^{(\lambda)})/g_{p+2}^{(\lambda)}(M_{p+2}^{(\lambda)})$, $b_\lambda = g_{p+6}^{(\lambda)}(M_{p+2}^{(\lambda)})/g_{p+4}^{(\lambda)}(M_{p+2}^{(\lambda)})$

(and noting that by the log-concavity of $g_0^{(\lambda)}(\cdot)$, $b_\lambda < a_\lambda$, $\forall\, \lambda \geq 0$), and making use of (2.13) of Sen (1989), we have

$$M'_\lambda = a_\lambda + \frac{\lambda}{2}(1 - a_\lambda)[a_\lambda + M'_\lambda] - \frac{\lambda}{2} a_\lambda(1 - b_\lambda)$$

i.e.,
$$M'_\lambda\{1 - \frac{\lambda}{2}(1 - a_\lambda)\} = \{1 - \frac{\lambda}{2}(1 - a_\lambda)\} - (1 - a_\lambda) + \frac{\lambda}{2} a_\lambda(1 - a_\lambda) - \frac{\lambda}{2} a_\lambda(1 - b_\lambda) < \{1 - \frac{\lambda}{2}(1 - a_\lambda)\}, \tag{2.19}$$

whence (2.18) follows by noting that $\frac{\lambda}{2}(1 - a_\lambda)$ is ≤ 1, $\forall\lambda \geq 0$.

Now, by virtue of the inequalities

$$m_p^{(\lambda)} \leq M_{p+2}^{(\lambda)} \leq p + \lambda; \quad m_p^{(\lambda)} \leq m_p + \lambda, \tag{2.20}$$

we note that for b sufficiently close to m_p, $b < m_p$,

$$\lambda + \frac{1}{2}(m_p + b) = m_p + \lambda - \frac{1}{2}(m_p - b)$$
$$= m_p^{(\lambda)} + \{m_p + \lambda - m_p^{(\lambda)}\} - \frac{1}{2}(m_p - b), \tag{2.21}$$

where the right hand side of (2.21) is less than m_p for $\lambda = 0$, but can be made greater than $m_p^{(\lambda)}$ for moderately large values of λ. Hence, although $\underset{\sim}{\delta}^0$ dominates (in the Pitman sense) $\underset{\sim}{\delta}(b)$ (for b close to m_p) near $\lambda = 0$, as λ moves away from 0, the opposite picture holds. Hence, within the class $\mathscr{C}_{JS}^{(2)}$, $\underset{\sim}{\delta}^0$ fails to be the Pitman closest estimator of θ. Nevertheless, as λ increases, $\{(m_p + \lambda - m_p^{(\lambda)}) - \frac{1}{2}(m_p - b)\}/\{2(p + 2\lambda)\}^{1/2} \to 0$, and hence, using the asymptotic (in λ) normality of $(\chi^2_{p,\lambda} - m_p^{(\lambda)})/\{2(p + 2\lambda)\}^{1/2}$, we conclude that (2.14) $\to$ ½ as $\lambda \to \infty$. Thus, even when $\underset{\sim}{\delta}^0$ may not be strictly closer than $\underset{\sim}{\delta}(b)$, the (generalized) Pitman closeness measure (distance) is quite small, so that even within the class

$\mathcal{C}_{JS}^{(2)}$, $\underset{\sim}{\delta}^0$ appears to be a natural choice. In this respect, $\mathcal{C}_{JS}^{(2)}$ can be replaced by $\mathcal{C}_{JS}$ as well.

Let us next consider the case of positive-rule shrinkage estimators defined by (2.7). For an arbitrary $b > 0$, note that (on letting $\sigma^2 = 1$)

$$\|\underset{\sim}{\delta}^+(b) - \underset{\sim}{\theta}\|^2 = \|\underset{\sim}{\theta}\|^2 \, I(\|\underset{\sim}{X}\|^2 \le b) + (\|\underset{\sim}{X}\| - \underset{\sim}{\theta}\|^2 + b^2\|\underset{\sim}{X}\|^{-2} - 2b\|\underset{\sim}{X}\|^{-2} \, \underset{\sim}{X}'(\underset{\sim}{X} - \underset{\sim}{\theta})\} \, I(\|\underset{\sim}{X}\|^2 > b), \tag{2.22}$$

(where $I(A)$ stands for the indicator function of the set A), so that for $0 < b_1 < b_2$, we have

$$\begin{aligned} &I(\|\underset{\sim}{\delta}^+(b_1) - \underset{\sim}{\theta}\| \le \|\underset{\sim}{\delta}^+(b_2) - \underset{\sim}{\theta}\|) \\ &= I(\|\underset{\sim}{X}\|^2 \le b_1) + I(b_1 < \|\underset{\sim}{X}\|^2 \le b_2) \, I(\underset{\sim}{X}'\underset{\sim}{\theta} \ge \tfrac{1}{2} (\|\underset{\sim}{X}\|^2 - b_1)) \\ &+ I(\|\underset{\sim}{X}\|^2 > b_2) \, I(\underset{\sim}{X}'(\underset{\sim}{X}-\underset{\sim}{\theta}) \le \tfrac{1}{2}(b_1 + b_2)), \end{aligned} \tag{2.23}$$

so that

$$\begin{aligned} &P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{\delta}^+(b_1) - \underset{\sim}{\theta}\| \le \|\underset{\sim}{\delta}^+(b_2) - \underset{\sim}{\theta}\|\} \\ &= P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{X}\|^2 \le b_1\} + P_{\underset{\sim}{\theta}}\{b_1 < \|\underset{\sim}{X}\|^2 \le b_2; \ \underset{\sim}{X}'\underset{\sim}{\theta} \ge \tfrac{1}{2}(\|\underset{\sim}{X}\|^2 - b_1)\} \\ &+ P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{X}\|^2 > b_2; \ \underset{\sim}{X}'(\underset{\sim}{X}-\underset{\sim}{\theta}) \le \tfrac{1}{2} (b_1 + b_2)\}. \end{aligned} \tag{2.24}$$

In order that $\underset{\sim}{\delta}^+(b_1)$ dominates $\underset{\sim}{\delta}^+(b_2)$, in the Pitman sense, (2.24) should be $\ge 1/2$ for all $\underset{\sim}{\theta}$. In particular at $\underset{\sim}{\theta} = \underset{\sim}{0}$, $\underset{\sim}{X}'\underset{\sim}{\theta} = 0$ with probability one, so that the second and third terms in (2.24) drop out. Thus,

$$\begin{aligned} &P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{\delta}^+(b_1) - \underset{\sim}{\theta}\| \le \|\underset{\sim}{\delta}^+(b_2) - \underset{\sim}{\theta}\| \ \ |\underset{\sim}{\theta} = \underset{\sim}{0}\} \\ &= P_{\underset{\sim}{0}}\{\|\underset{\sim}{X}\|^2 \le b_1\} = P\{\chi_p^2 \le b_1\}, \end{aligned} \tag{2.25}$$

which is $\geq 1/2$ only when b_1 is $\geq m_p$. In a similar manner, at $\underset{\sim}{\theta} = \underset{\sim}{0}$, $P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{\delta}^+(b_1) - \underset{\sim}{\theta}\| \leq \|\underset{\sim}{\delta}^+(b_2) - \underset{\sim}{\theta}\|\} \geq 1/2$ for $b_2 < b_1$ only when $b_1 \leq m_p$. From this perspective, an ideal choice of b_1 is m_p, i.e., the estimator

$$\underset{\sim}{\delta}^{0+} = \{1 - \sigma^2 m_p \|\underset{\sim}{X}\|^{-2}\}^+ \underset{\sim}{X}. \tag{2.26}$$

But, then the question is whether the PCP holds for $\underset{\sim}{\delta}^{0+}$ for the entire class of positive rule estimators [of the type 2.7)] or for suitable subclasses? For this, we consider the following:

$$\mathscr{C}_{JS}^{+(1)} = \{\underset{\sim}{\delta}^+(b) : m_p \leq b \leq b_0 = (p-1)(3p+1)/2p\},$$
$$\mathscr{C}_{JS}^{+(2)} = \{\underset{\sim}{\delta}^+(b) : 0 < b \leq m_p\}; \tag{2.27}$$

$$\mathscr{C}_{JS}^{+} = \mathscr{C}_{JS}^{+(1)} \cup \mathscr{C}_{JS}^{+(2)} . \tag{2.28}$$

<u>Theorem 2.2</u> $\underset{\sim}{\delta}^{0+}(= \underset{\sim}{\delta}^+(m_p))$ *is the Pitman closest estimator of* $\underset{\sim}{\theta}$ *within the class* $\mathscr{C}_{JS}^{+(1)}$.

<u>Proof</u>. Note that for every $b > m_p$, by (2.24),

$$P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{\delta}^{0+} - \underset{\sim}{\theta}\| \leq \|\underset{\sim}{\delta}^+(b) - \underset{\sim}{\theta}\|\}$$

$$= P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{X}\|^2 \leq m_p\} + P_{\underset{\sim}{\theta}}\{m_p < \|\underset{\sim}{X}\|^2 \leq b,\ \underset{\sim}{X}'\underset{\sim}{\theta} \geq \tfrac{1}{2}(\|\underset{\sim}{X}\|^2 - m_p)\}$$

$$+ P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{X}\|^2 > b,\ \underset{\sim}{X}'\underset{\sim}{\theta} \geq \|\underset{\sim}{X}\|^2 - \tfrac{1}{2}(m_p + b)\}$$

$$= P_{\underset{\sim}{\theta}}\{\underset{\sim}{X}'\underset{\sim}{\theta} \geq \|\underset{\sim}{X}\|^2 - \tfrac{1}{2}(m_p + b)\} + P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{X}\|^2 \leq m_p,\ \underset{\sim}{X}'\underset{\sim}{\theta} < \|\underset{\sim}{X}\|^2 - \tfrac{1}{2}(m_p+b)\}$$

$$- P_{\underset{\sim}{\theta}}\{m_p < \|\underset{\sim}{X}\|^2 \leq b,\ \|\underset{\sim}{X}\|^2 - \tfrac{1}{2}(m_p + b) \leq \underset{\sim}{X}'\underset{\sim}{\theta} < \tfrac{1}{2}(\|\underset{\sim}{X}\|^2 - m_p)\}$$

$$= P_{\underset{\sim}{\theta}}\{\underset{\sim}{X}'\underset{\sim}{\theta} \geq \|\underset{\sim}{X}\|^2 - m_p\} + P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{X}\|^2 - \tfrac{1}{2}(m_p + b) \leq \underset{\sim}{X}'\underset{\sim}{\theta} < \|\underset{\sim}{X}\|^2 - m_p\}$$

$$+ P_{\theta}\{\|X\|^2 \le m_p,\ X'\theta < \|X\|^2 - \tfrac{1}{2}(m_p + b)\}$$

$$- P_{\theta}\{m_p < \|X\|^2 \le b,\ \|X\|^2 - \tfrac{1}{2}(m_p + b) \le X'\theta \le \tfrac{1}{2}(\|X\|^2 - m_p)\}$$

$$= P_{\theta}\{X'\theta \ge \|X\|^2 - m_p\} + P_{\theta}\{\|X\|^2 \le m_p,\ X'\theta \le \|X\|^2 - m_p\}$$

$$+ P_{\theta}\{m_p < \|X\|^2 \le b,\ \tfrac{1}{2}(\|X\|^2 - m_p) \le X'\theta \le \|X\|^2 - m_p\}$$

$$+ P_{\theta}\{\|X\|^2 > b,\ \|X\|^2 - \tfrac{1}{2}(m_p + b) \le X'\theta \le \|X\|^2 - m_p\}$$

$$\ge P_{\theta}\{X'\theta \ge \|X\|^2 - m_p\}$$

$$= P_{\lambda}\{\chi^2_{p,\lambda} \le m_p + \lambda\} \quad (\lambda = \tfrac{1}{4}\|\theta\|^2)$$

$$= P_{\lambda}\{\chi^2_{p,\lambda} \le m_p^{(\lambda)} + [m_p + \lambda - m_p^{(\lambda)}]\}$$

$$\ge P_{\lambda}\{\chi^2_{p,\lambda} \le m_p^{(\lambda)}\} \quad (\text{as } m_p^{(\lambda)} \le m_p + \lambda,\ \forall\lambda)$$

$$= \tfrac{1}{2}. \tag{2.29}$$

This completes the proof of the theorem.

Theorem 2.3 δ^{0+} *is not the Pitman closest estimator of* θ *within class* $\mathscr{C}^{+(2)}_{JS}$.

Proof. Note that for every $0 < b < m_p$,

$$P_{\theta}\{\|\delta^{0+} - \theta\| \le \|\delta^{+}(b) - \theta\|\}$$

$$= P_{\theta}\{\|X\|^2 \le b\} + P_{\theta}\{b < \|X\|^2 \le m_p,\ \|X\|^2 - b > 2X'\theta\}$$

$$+ P_{\theta}\{\|X\|^2 > m_p,\ X'(X - \theta) \ge \tfrac{1}{2}(m_p + b)\}$$

$$= P_{\theta}\{X'(X-\theta) \ge \tfrac{1}{2}(m_p + b)\} + P_{\theta}\{\|X\|^2 \le b,\ X'(X-\theta) < \tfrac{1}{2}(m_p + b)\}$$

$$+ P_{\underset{\sim}{\theta}}\{b < \|\underset{\sim}{X}\|^2 \le m_p,\ \|\underset{\sim}{X}\|^2 - \frac{1}{2}(m_p + b) \le \underset{\sim}{X}'\underset{\sim}{\theta} < (\|\underset{\sim}{X}\|^2 - b)/2\}. \quad (2.30)$$

Now, the first term on the right hand side of (2.30) can be expressed as

$$P_\lambda\{\chi^2_{p,\lambda} \ge \lambda + \frac{1}{2}(m_p + b)\} \quad (\lambda = \frac{1}{4}\|\underset{\sim}{\theta}\|^2)$$

$$= P_\lambda\{\chi^2_{p,\lambda} \ge m_p^{(\lambda)} + (\lambda + m_p - m_p^{(\lambda)}) - \frac{1}{2}(m_p - b)\}, \quad (2.31)$$

where

$$m_p^{(\lambda)} \le \lambda + m_p,\ \forall \lambda \ge 0 \ \text{ and } \ \frac{1}{2}(m_p - b) > 0. \quad (2.32)$$

Consequently, noting that $m_p^{(0)} = m_p$, we conclude that for every $b < m_p$, there exists a λ, say $\lambda_b(> 0)$, such that $\lambda + \frac{1}{2}(m_p + b) \le m_p^{(\lambda)}$, $\forall \lambda \le \lambda_b$, and hence, (2.31) is $\ge \frac{1}{2}$, $\forall \lambda \le \lambda_b$. Actually, by virtue of Theorem 1 of Sen, Kubokawa and Saleh (1989), whenever $\frac{1}{2}(m_p + b) \le (p-1)(3p+1)/4p$ i.e.,

$$b \le (P-1)(3p+1)/2p - m_p = (p-1)(3p+1)/2p - (p-1+\eta_p)$$

$$= (p-1)(p+1)/2p - \eta_p = (p^2 - 1)/2p - \eta_p = b_0^*, \text{ say}, \quad (2.33)$$

(where $\eta_p \ge 1/3$, $\forall p \ge 2$ and $\eta_p \downarrow 1/3$ as $p \to \infty$; $\eta_2 = .45$), (2.33) exceeds 1/2, for all $\lambda \ge 0$. Thus, it suffices to consider only values of $b > b_0^*$. If $\underset{\sim}{\delta}^{0+}$ is the Pitman closest estimator of $\underset{\sim}{\theta}$ within the class $\mathscr{C}_{JS}^{+(2)}$, then (2.30) has to be $\ge \frac{1}{2}$, for every $b < m_p$ and $\lambda \ge 0$. We consider the case of b approaching m_p from below and denote this by m_p^-. Then the third term on the right hand side of (2.30) can be made arbitrarily small (and zero in the limit $b = m_p^-$). For the first term, using (2.16) we conclude that whenever $\lambda + m_p^- > M_{p+2}^{(\lambda)}$ $(\ge m_p^{(\lambda)})$, $P_\lambda\{\chi^2_{p,\lambda} \ge \lambda + m_p^-\} < P_\lambda\{\chi^2_{p,\lambda} \ge m_p^{(\lambda)}\} = 1/2$.

For this, proceeding as in (2.18)-(2.19), we have

$$M_{p+2}^{(\lambda)} = p + \lambda a_\lambda = p + \lambda - \lambda(1 - a_\lambda), \tag{2.34}$$

where $\lambda(1 - a_\lambda)$ is ≥ 0, $\forall\ \lambda \geq 0$, and as λ increases, $\lambda(1 - a_\lambda)$ converges to 2. Thus, there exists an $\lambda_0(> 0)$, such that $M_{p+2}^{(\lambda)} < m_p^- + \lambda$, for every $\lambda \geq \lambda_0$, so that the first term is $< \frac{1}{2}$, for every $\lambda \geq \lambda_0$. Further, using the asymptotic (in λ) normality of $(\chi^2_{p,\lambda} - p - \lambda)/\{2(p + 2\lambda)\}^{1/2}$ [viz., Johnson and Kotz (1970)], we obtain from the above that as λ increases,

$$P_\lambda\{\chi^2_{p,\lambda} > m_p^- + \lambda\} = \frac{1}{2} - O(\lambda^{-1/2}). \tag{2.35}$$

To complete the picture, we consider the second term of (2.30). Let $\underset{\sim}{Y} = (Y_1, \ldots, Y_p)' \sim N_p(\underset{\sim}{0}, \underset{\sim}{I})$ and let $Q = \Sigma_{j=2}^p Y_j^2$. Then we may rewrite (for $b = m_p^-$),

$$P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{X}\|^2 \leq m_p^-,\ \underset{\sim}{X}'(\underset{\sim}{X}-\underset{\sim}{\theta}) < m_p^-\}$$

$$= P\{Q + (Y_1 + \lambda)^2 < m_p^- + \lambda^2,\ Q + (Y_1 + 2\lambda)^2 \leq m_p^-\}. \tag{2.36}$$

Using Anderson's (1955) lemma and some standard arguments, it is easy to verify that (2.36) is nonincreasing in $\lambda(\geq 0)$; at $\lambda = 0$, it is equal to 1/2 and it converges to 0 as $\lambda \to \infty$. Moreover, (2.36) is bounded from above by $P\{Q + (Y_1 + 2\lambda)^2 \leq m_p\} \leq P\{Y_1 \leq -2\lambda + m_p^{1/2}\}$ $\leq P\{Y_1 \leq -\lambda\}$ (for $\lambda \geq m_p^{1/2}$), so that as λ increases, it converges to 0 at an exponential rate in λ. Comparing this with (2.35), we may therefore conclude that for large λ, (2.30) is strictly smaller than ½ (albeit very close to 1/2). Thus, while for small to moderate values of λ, $\underset{\sim}{\delta}^{0+}$ dominates $\underset{\sim}{\delta}^{+}(b)$ (for b close to m_p^-), in the PC sense, it fails to do so for larger values of λ. Hence, the PC dominance of $\underset{\sim}{\delta}^{0+}$ does not hold, and the proof of the theorem is complete.

Theorems 2.2 and 2.3 imply that within the class $\mathcal{C}_{JS}^{+}$ there is no Pitman closest estimator. Nevertheless, $\underset{\sim}{\delta}^{0+}$ emerges as a strong contender; it has the PCP over the subclass $\mathcal{C}_{JS}^{+(1)}$ as well as a greater domain of $\mathcal{C}_{JS}^{+(2)}$, and even on the complementary part (of $\mathcal{C}_{JS}^{+(2)}$), (4.30) is quite close to 1/2, indicating near attainability of the PCP. In the above comparison, we may note that within the class $C_{JS}^{+(1)}$, $\underset{\sim}{\delta}^{0+}$ is Pitman closest while within $C_{JS}^{+(2)}$, there is no estimator which is Pitman closer than $\underset{\sim}{\delta}^{0+}$ (for all λ), and hence, $\underset{\sim}{\delta}^{0+}$ is admissible in the PMC sense.

Let us note that with respect to a quadratic error risk, within the class of Stein-rule estimators of the form (2.5), an optimal choice of b is given by $b = p-2$ $(=M_p^{(0)}) < m_p^0$, and for this, we need $p \geq 3$. Note that for $p = 2$, $p - 2 = 0$, and hence, $\underset{\sim}{\delta}^{0}$ dominates the Stein-rule estimator (MLE) in the light of the PMC [viz., Sen, Kubokawa and Saleh (1989)]. A natural query is whether such a PMC-dominance is true for $p \geq 3$? The answer is in the negative. Neither δ^0 nor δ (p-2) dominates the other in the light of the Pitman closeness measure, albeit both of them being admissible in the same mode. To draw this conclusion, let us first note that $\underset{\sim}{\delta}(p-2)$ belongs to the class $\mathcal{C}_{JS}^{(2)}$ [see (2.9)]. As such, if we proceed as in (2.14), we have

$$P_{\underset{\sim}{\theta}}\{\|\underset{\sim}{\delta}^0 - \underset{\sim}{\theta}\| \leq \|\underset{\sim}{\delta}(p-2) - \underset{\sim}{\theta}\|\}$$
$$= P_{\lambda}\{\chi^2_{p,\lambda} \geq \lambda + \frac{1}{2}(m_p + p-2)\}. \tag{2.37}$$

Note that $m_p \geq p-1 + 1/3$ for every $p \geq 2$, so that $\frac{1}{2}(m_p + p-2)$ is $\geq p-1 - 1/3$. When $\lambda = 0$ (or is close to 0), $\frac{1}{2}(m_p + p-2) + \lambda$ is $\leq m_p^{(\lambda)}$ and hence, (2.37) is $\geq 1/2$, while as λ increases, $\frac{1}{2}(m_p + p-2) + \lambda$ exceeds $M_{p+2}^{(\lambda)}$ and, by (2.16), $m_p^{(\lambda)} \leq M_{p+2}^{(\lambda)}$, $\forall \lambda \geq 0$, so that (2.37) is $< 1/2$ for large values of λ (although it is very close to 1/2). A similar picture holds for $\underset{\sim}{\delta}^{0+}$ vs. $\underset{\sim}{\delta}^{+}(p-2)$. But looking at (2.37) we may gather that for a part of the domain of $\underset{\sim}{\theta}$, $\underset{\sim}{\delta}^{0}$ dominates $\underset{\sim}{\delta}(p-2)$ in the Pitman sense, while for the complementary part, the superiority of $\underset{\sim}{\delta}(p-2)$ to $\underset{\sim}{\delta}^{0}$ is very insignificant. From this picture, we may conclude that $\underset{\sim}{\delta}^{0}$ emerges on a good standing relative to the classical James-Stein estimator too; a similar picture holds for $\underset{\sim}{\delta}^{0+}$.

The PCP for general $\underset{\sim}{\delta}_{\varphi}$ in (2.2) depends heavily on the form of φ, and in general, a PCP characterization for $\varphi \in \{0 < \varphi(\underset{\sim}{X}, s) \leq (p-1)(3p+1)/2p\}$ may not hold. Even for the subclass of estimators of the form

$$\underset{\sim}{\delta}(b) = \{1 - b\, S\|\underset{\sim}{X}\|^{-2}\}\underset{\sim}{X}, \quad b > 0, \tag{2.38}$$

(where $qs/\sigma^2 \sim \chi_q^2$, independently of $\underset{\sim}{X}$), the choice of b depends on q as well as p, and within this class $b = m_p$ does not have the PCP. To see this, we proceed as in (2.10), and obtain by parallel arguments that for $b \geq m_p$,

$$\begin{aligned} &P_{\underset{\sim}{\theta},\sigma}\{\|\underset{\sim}{\delta}^{0} - \underset{\sim}{\theta}\| \leq \|\underset{\sim}{\delta}(b) - \underset{\sim}{\theta}\|\} \\ &\qquad = P_{\lambda}\{\chi_{p,\lambda}^2 \leq \lambda + \tfrac{1}{2}(m_p + b)q^{-1}\chi_q^2\}. \end{aligned} \tag{2.39}$$

If (2.39) has to be greater than or equal to $1/2$ $\forall\, \lambda \geq 0$, we must have

$$P_0\{q\ \chi_p^2/\chi_q^2 \le (m_p+b)/2\} \ge 1/2, \quad \forall\ b \ge m_p. \tag{2.40}$$

Note that $q\ p^{-1}\ \chi_p^2/\chi_q^2$ has the variance ratio (i.e., $F_{p,q}$ -) distribution with DF(p,q). For this $F_{p,q}$ distribution, the mean is $q/(q-2)$, the mode is $q(p-2)/p(q-2)$ and median which is larger than $\frac{1}{p}$ m_p. Hence, allowing b to be sufficiently close to m_p, we conclude that the median of $q\ \chi_p^2/\chi_q^2$ can be made larger than $\frac{1}{2}$ $(m_p + b)$, so that (2.40) does not hold. A similar treatment holds for the analogues of Theorems 2.2 and 2.3. In fact, it is quite intuitive to replace m_p by $m_{p,q}$ = med($q\ \chi_p^2/\chi_q^2$) and consider the estimator $\delta(m_{p,q})$. Note that on defining $F_{.5;p,q}$ by $\Pr[q\ \chi_p^2/p\ \chi_q^2 > F_{.5;p,q}] = .5$, we have $m_{p,q} = p\ F_{.5;p,q}$. However, the characterization of the PCP of this estimator within the entire class of $\underset{\sim}{\delta}(b)$ in (2.38) (or a subclass of it) requires more elaborate studies of some properties of non-central beta distributions, and hence, we shall consider then in a future communication. The simple proof of Theorem 2.1 or 2.2 or 2.3 may not work out in this case of unknown σ^2. Naturally, the case of arbitrary $\underset{\sim}{\Sigma}$ will be even more complicated to manipulate properly.

3. PC DOMINANCE OF RSMLE

As has been mentioned in Section 1, when the parameter $\underset{\sim}{\theta}$ (for the model $\underset{\sim}{X} \sim \mathcal{N}_p(\underset{\sim}{\theta},\underset{\sim}{\Sigma})$) belongs to a restricted domain (viz., positively homogeneous cone), the RMLE fares better than the MLE and the RSMLE dominates the RMLE in the light of the usual quadratic error risk. This has been studied in detail by Sengupta and Sen

(1991). A parallel picture in the light of the PCC will be depicted here.

For simplicity of presentation, we consider explicitly the *positive orthant model* for which

$$\Theta = \Theta_+ = R^{+p} = \{\theta \in R^p : \theta \geq 0\}. \tag{3.1}$$

Also, as in Section 2, we consider here the multi-normal model: $X \sim N_p(\theta, \sigma^2 I)$. For the case of $N_p(\theta, \Sigma)$, Σ arbitrary (p.d.), closed expressions for the RMLE and RSMLE are given in Sengupta and Sen (1990). For the specific model, $\Sigma = \sigma^2 I$, we have much more simplified expressions. For any $x \in R^p$, let $x^+ = x \vee 0 = (x_1 \vee 0, \ldots, x_p \vee 0)'$. Then the classical MLE of θ is X, while for θ confined to Θ_+, the RMLE is given by

$$\hat{\theta}_{RM} = X^+ = (X_1 \vee 0, \ldots, X_p \vee 0)'. \tag{3.2}$$

For every $x \in R^p$, let $a(x)$ be the number of coordinates of x which are positive, i.e.,

$$a(x) = \Sigma_{j=1}^p I(x_j > 0), \tag{3.3}$$

so that $a(x)$ assumes the values 0, 1, ..., p. Then, the RSMLE of θ, considered by Sengupta and Sen (1991), can be simplied as

$$\hat{\theta}_{RSM} = \{1 - [a(X) - 2]^+ \|X^+\|^{-2} \sigma^2\} X^+, \tag{3.4}$$

where $[a-2]^+ = \max[0, a-2]$, for $a = 0, 1, \ldots, p$. Note that for χ_p^2, the mode is $(p-2)$ and the median is m_p $(> p-1)$. In view of the emphasis placed on $\delta^0 = \delta(m_p)$ in Section 2, we shall consider, side by side, an alternative version of the RSMLE, given by

$$\hat{\theta}^*_{RSM} = \{1 - c_{a(X)} \sigma^2 \|X^+\|^{-2}\} X^+, \tag{3.5}$$

where

$$c_k = m_k, \ 2 \leq k \leq p; \quad c_0 = c_1 = 0. \tag{3.6}$$

Our contention is to compare (3.4) and (3.5) with each other and with (3.2), in the light of the Pitman closeness criterion. As in Sengupta and Sen (1990), we also consider the positive rule versions:

$$\hat{\theta}^{+}_{RSM} = \{1 - [a(\mathbf{x}) - 2]^{+} \|\mathbf{X}^{+}\|^{-2}\sigma^{2}\}^{+} \mathbf{X}^{+}, \tag{3.7}$$

$$\hat{\theta}^{*+}_{RSM} = \{1 - c_{a(\mathbf{x})} \|\mathbf{X}^{+}\|^{-2}\sigma^{2}\}^{+} \mathbf{X}^{+}, \tag{3.8}$$

and compare them with the other versions (under the PCC). In this setup, we shall confine ourselves to $\theta \in \Theta_+$.

There is a basic difference in the setup of Sections 2 and 3. In the unrestricted case, the MLE ($\mathbf{X}$) or its shrinkage version δ_{φ} is *equivariant* under the group of (affine) transformations:

$$\mathbf{X} \to Y = \mathbf{B}\,\mathbf{X}, \quad \mathbf{B} \ \text{non-singular}. \tag{3.9}$$

For this reason, we were able to choose $\mathbf{B}$ in such a way that $EY = \mathbf{B}\theta = \eta = (\eta, \mathbf{0}')$, where $\eta^2 = \|\theta\|^2$. Such a canonical reduction may not be possible for the restricted case; the main difficulty stems from the fact that the positive orthant Θ_+ does not remain invariant under such a non-singular (or even orthogonal) $\mathbf{B}$, although scalar transformations on the individual coordinates does not alter Θ_+. In the negation of this equivariance, it is not surprising to see that the performance characteristics (be it in the quadratic risk or the PC measure) of the RSMLE and RMLE may depend not only on $\|\theta\|$ but also on the direction cosines of the individual elements. As such, this picture when θ lies in the interior of θ_+ (i.e., $\theta > \mathbf{0}$) may not totally agree with the one when $\underset{\sim}{\theta}$ lies on the boundary of Θ_+ (i.e., $\theta_j = 0$ for some j, $1 \le j \le p$). However, the relative dominance picture remains the same, although the extent may differ from the edges to the interior of Θ_+. With these remarks, we consider the following.

<u>Theorem 3.1</u> *For every* $p \ge 2$, $\hat{\underset{\sim}{\theta}}^*_{RSM}$ *dominates* $\hat{\underset{\sim}{\theta}}_{RM}$ in the PCC, *and for* $p \ge 3$, $\hat{\underset{\sim}{\theta}}_{RSM}$ *dominates* $\hat{\underset{\sim}{\theta}}_{RM}$ *in the PCC.*

<u>Proof</u>. We provide a proof *for* $\hat{\underset{\sim}{\theta}}^*_{RSM}$ only, as a similar case holds for $\hat{\underset{\sim}{\theta}}_{RSM}$. Note that by (3.2),

$$\|\hat{\underset{\sim}{\theta}}_{RM} - \underset{\sim}{\theta}\|^2 = \|\underset{\sim}{X}^+ - \underset{\sim}{\theta}\|^2$$
$$= \Sigma_{j=1}^{p} \{I(X_j \le 0)\theta_j^2 + I(X_j > 0)(X_j - \theta_j)^2\} \qquad (3.9)$$

Similarly, by (3.5),

$$\|\hat{\underset{\sim}{\theta}}^*_{RSM} - \underset{\sim}{\theta}\|^2 = \|\underset{\sim}{X}^+ - \underset{\sim}{\theta}\|^2 + c^2_{a(\underset{\sim}{X})}\, \sigma^4 \|\underset{\sim}{X}^+\|^{-2}$$
$$- 2\, c_{a(\underset{\sim}{X})}\, \sigma^2 (\underset{\sim}{X}^+ - \underset{\sim}{\theta})'\underset{\sim}{X}^+\, \|\underset{\sim}{X}^+\|^{-2}. \qquad (3.10)$$

As a result,

$$P_{\underset{\sim}{\theta}}\{\|\hat{\underset{\sim}{\theta}}_{RSM} - \underset{\sim}{\theta}\| \le \|\hat{\underset{\sim}{\theta}}_{RM} - \underset{\sim}{\theta}\|\}$$
$$= P_{\underset{\sim}{\theta}}\{(\underset{\sim}{X}^+ - \underset{\sim}{\theta})'\underset{\sim}{X}^+ \ge \frac{1}{2}\, c_{a(\underset{\sim}{X})}\, \sigma^2\}, \qquad (3.11)$$

where we may note that whenever $a(\underset{\sim}{x}) = 0$ or 1, $c_{a(\underset{\sim}{x})} = 0$, so that (3.9) and (3.10) are equal. Hence, we may rewrite (3.11) a little but more explicitly as

$$P_{\underset{\sim}{\theta}}\{a(\underset{\sim}{X}) = 0 \text{ or } 1\} + P_{\underset{\sim}{\theta}}\{(\underset{\sim}{X}^+ - \underset{\sim}{\theta})'\underset{\sim}{X} \ge \frac{1}{2}\sigma^2\, c_{a(\underset{\sim}{X})},\ 2 \le a(\underset{\sim}{X}) \le p\}. \qquad (3.12)$$

Each of the terms in (3.12) depends on $\underset{\sim}{\theta}$ through the individual

$\theta_1, \ldots, \theta_p$, and therefore a complete working out of (3.12) for a general $\underset{\sim}{\theta}$ ($\in \Theta_+$) may require considerable manipulations. For reasons explained after (3.9), we take $\underset{\sim}{\theta} = (\eta, \underset{\sim}{0}')'$, $\eta \geq 0$, and for this edge we provide a complete proof. The simple proof holds for any of the other p-1 edges, while for higher dimensional subspaces of Θ_+, one may require much heavier manipulations.

Dealing with a quadratic risk, some of these manipulations are reported in Section 6 of Sengupta and Sen (1991), and in view of the similarity, we shall omit some of these details. Also, for simplicity of presentation, we take $\sigma = 1$.

Let $\Phi(x)$ stand for the standard normal d.f. Then note that

$$\begin{aligned} &P_{\underset{\sim}{\theta}}\{a(\underset{\sim}{x}) = 0 \text{ or } 1\} \\ &= P_{\underset{\sim}{\theta}}\{\underset{\sim}{X} \leq \underset{\sim}{0}\} + \Sigma_{i=1}^{p} P_{\underset{\sim}{\theta}}\{X_i > 0,\ X_j < 0,\ \forall\ j \neq i\} \\ &= \Phi(-\eta)\ 2^{-(p-1)} + \Phi(\eta)2^{-(p-1)} + \Phi(-\eta)(p-1)2^{-(p-1)} \\ &= 2^{-(p-1)}\{(p-1)\ \Phi(-\eta) + 1\}. \end{aligned} \tag{3.13}$$

Note that at $\eta = 0$, (3.13) equals to $(p+1)/2^p$, and it monotonically decreases with η (> 0) with $\lim_{n\to\infty}$ (3.13) $= 2^{-(p-1)}$. As such, if p ≤ 2, (3.12) is $\geq \frac{1}{2}$, $\forall\ \eta \geq 0$, and hence, the RSMLE dominates the RMLE in the PCC. As such, in the sequel, we only consider the case of $p \geq 3$. In this case, for $2 \leq a(\underset{\sim}{X}) \leq p$, we may identify the two situations:

(i) X_1 and (a-1) of the remaining (p-1) coordinates are positive, which the rest (i.e., p-a) negative, and (ii) X_1 and (p-a-1) of the coordinates are negative and the remaining a are positive.

As such, we can write the second term of (3.12) as

$$\Sigma_{a=2}^{p}\left\{\binom{p-1}{a-1} 2^{-(p-a)} P\{X_1(X_1-\eta) + \Sigma_{j=2}^{a} X_j^2 \geq \frac{1}{2} c_a,\ X_j \geq 0,\ 1 \leq j \leq a\}\right.$$

$$\left. + \binom{p-1}{a} 2^{-p-1-a} \Phi(-\eta)\ P\{\Sigma_{j=2}^{a+1} X_j^2 \geq \frac{1}{2} c_a,\ X_j \geq 0,\ 2 \leq j \leq a+1\}\right\}$$

$$= 2^{-(p-1)} \left[\sum_{a=2}^{p} \binom{p-1}{a-1} P\{\chi^2_{a-1,0} + X_1(X_1-\eta) \geq \frac{1}{2} c_a,\ X_1 > 0\} \right.$$

$$\left. + \sum_{a=2}^{p-1} \binom{p-1}{a} \Phi(-\eta)\ P\{\chi^2_{a,0} \geq \frac{1}{2} c_a\}\right]$$

$$= 2^{-(p-1)} \left[\sum_{a=1}^{p-1} \binom{p-1}{a} P\{\chi^2_{a,0} + (X_1-\eta)X_1 \geq \frac{1}{2} c_{a+1},\ X_1 > 0\} \right.$$

$$\left. + \sum_{a=2}^{p-1} \binom{p-1}{a} P\{\chi^2_{a,0} \geq \frac{1}{2} c_a\}\ \Phi(-\eta)\right], \qquad (3.14)$$

so that by (3.13) and (3.14), we rewrite (3.12) in the following way (where $c_1 = c_0 = 0$):

$$2^{-(p-1)}\left\{ 1 + \sum_{a=1}^{p-1} \binom{p-1}{a} \left[\Phi(-\eta)P\{\chi^2_{a,0} > \frac{1}{2} c_a\}\right.\right.$$

$$\left.\left. + P\{\chi^2_{a,0} + X_1(X_1-\eta) \geq \frac{1}{2} c_{a+1};\ X_1 > 0\}\right]\right\}. \qquad (3.15)$$

Let $\alpha_a = \bar{G}_a(\frac{1}{2} c_a) = P\{\chi^2_{a,0} > \frac{1}{2} c_a\}$ and $\alpha_a^* = \bar{G}_a(\frac{1}{2} c_{a+1})$, $a \geq 1$, so that $\alpha_1 = 1$, $\alpha_a > \frac{1}{2}$ for every $a \geq 2$, $\alpha_a > \alpha_a^*$, $\forall\ a \geq 1$ and $\alpha_a^* \geq \frac{1}{2}$, $\forall\ a \geq 2$. Then, it suffices to show that for every $a \geq 1$,

$$\Phi(-\eta)\alpha_a + P\{\chi^2_{a,0} + X_1(X_1-\eta) \geq \frac{1}{2} c_{a+1},\ X_1 > 0\} \geq \frac{1}{2},\ \forall\ \eta \geq 0. \qquad (3.16)$$

As $\alpha_1 = 1$, for a = 1, the proof of (3.16) is simpler, and hence, we consider only the case of $a \geq 2$. Note that for $\eta = 0$, the left hand side of (3.16) is

$$\frac{1}{2}\alpha_a + \frac{1}{2}\alpha_{a+1} = \frac{1}{2}(\alpha_a+\alpha_{a+1}) = \alpha_a^* + \frac{1}{2}(\alpha_a+\alpha_{a+1}-2\alpha_a^*) > \frac{1}{2}, \tag{3.17}$$

for every $a \geq 2$. Letting $\mu = \frac{1}{2}\eta$, we rewrite (3.16) as

$$\Phi(-\eta)\alpha_a + \int_0^\eta \bar{G}_a(\tfrac{1}{2}c_{a+1} + \mu^2 - (\mu-x)^2)\,\varphi(x)dx$$
$$+ \int_0^\infty \bar{G}_a(\tfrac{1}{2}c_{a+1} + \mu^2 - (\mu+x)^2)\varphi(x)dx$$

$$= \alpha_a^* + \Phi(-\eta)[\alpha_a - \alpha_a^*] - \int_0^\eta[\alpha_a^* - \bar{G}_a(\tfrac{1}{2}c_{a+1} + \mu^2 - (\mu-x)^2)]\varphi(x)dx$$
$$+ \int_0^\infty[\bar{G}_a(\tfrac{1}{2}c_{a+1} + \mu^2 - (\mu+x)^2) - \alpha_a^*]\varphi(x)dx. \tag{3.18}$$

The last term on the right hand side of (3.18) is ↑ in η (or μ); its lower bound, $\frac{1}{2}(\alpha_{a+1} - \alpha_a^*)$ (> 0) is attained at $\eta = 0$ (= μ) and its upper asymptote is $\frac{1}{2}(1 - \alpha_a^*)$. The third term, $\int_0^\eta[\alpha_a^* - \bar{G}_a(\frac{1}{2}c_{a+1} + \mu^r - (n-x)^2)]\ \varphi(x)dx$ is ↑ in η, is nonnegative and its upper asymptote (as $\eta \to \infty$) is $< \frac{1}{2}\alpha_a^*$. Hence, as $\eta \to \infty$, the right hand side of (3.18) converges to a limit $\geq \alpha_a^* + 0 - \frac{1}{2}\alpha_a^* + \frac{1}{2}(1 - \alpha_a^*) = \frac{1}{2}$. Let us consider the first order (partial) derivative of (3.18) with respect to η. It is equal to

$$-\varphi(\eta)[\alpha_a-\alpha_a^*] - g_a(\tfrac{1}{2}c_{a+1})[\varphi(0)-\varphi(\eta)]$$
$$+ \int_0^\mu y\, g_a'(\tfrac{1}{2}c_{a+1} + \mu^2-y^2)[\varphi(\mu+y)-\varphi(\mu-y)]dy$$
$$+ g_a(\tfrac{1}{2}c_{a+1})\varphi(0) - \int_0^\infty(\mu+x)g_a'(\tfrac{1}{2}c_{a+1} + \mu^2-(\mu+x)^2)\varphi(x)dx, \tag{3.19}$$

where $g_a(x) = -(d/dx)\bar{G}_a(x)$ and $g_a'(x) = (d/dx)g_a(x)$. For $a = 2$, $g_2'(x) = -\frac{1}{2} g_2(x)$ where $g_2(x) = \frac{1}{2} e^{-\frac{1}{2}x}$ is decreasing in x, and hence, (3.19) is bounded from below by

$$-\varphi(\eta)(\alpha_2 - \alpha_2^*) + \varphi(\eta)\, g_2(\tfrac{1}{2} c_3)$$

$$= -\varphi(\eta)\{e^{-\frac{1}{2}\alpha c_2} - e^{-\frac{1}{2}c_3} - \tfrac{1}{2} e^{-\frac{1}{2}c_3}\}$$

$$= -\varphi(\eta)e^{-\frac{1}{2}c_2}\{1 - \tfrac{3}{2} e^{-\frac{1}{2}(c_3-c_2)}\}$$

$$= -\varphi(\eta)e^{-\frac{1}{2}c_2}\{1 - .919\} = -.025\ \varphi(-\eta). \tag{3.20}$$

On the other hand, $\frac{1}{2}(\alpha_2 + \alpha_3) = 0.733$, so that for $\eta > 0$, $a = 2$, (3.18) is bounded from below by $0.733 - .025 \int_0^\eta \varphi(y)dy = 0.733 - .025\,[\Phi(\eta) - \frac{1}{2}] \geq 0.71 > \frac{1}{2}$, $\forall\, \eta \geq 0$. Also, note that $\frac{1}{2} c_{a+1}$ is $>$ a-2 for a = 2,3,4 and the opposite inequality holds for $a \geq 5$. As such, noting that $g_a'(\frac{1}{2} c_{a+1})$ is $\gtrless 0$ according as $\frac{1}{2} c_{a+1}$ is $\gtrless$ a-2, a similar proof works out for a = 3 and 4. Hence, in the sequel, we consider the case of $a \geq 5$ (for which $\frac{1}{2} c_{a+1} < a-2$). We define μ_a^0 by $(\mu_a^0)^2 = a-2 - \frac{1}{2} c_{a+1}$ (> 0), and note that for $0 \leq \mu \leq \mu_a^0$, $g_a'(\frac{1}{2} c_{a+1} + \mu^2 - y^2)$ is ≥ 0, $\forall\, y \leq \mu$, so that the third term in (3.19) is ≤ 0. On the other hand, in the last term, $g_a'(\frac{1}{2} c_{a+1} + \mu_2 - (\mu+x)^2) \geq 0$, $\forall\, x \geq 0$, and hence, (3.19) is easily shown to be negative. Since (3.18) is bounded from below by $\frac{1}{2}$ (as $\eta \to \infty$), we complete the proof by showing that (3.19) remains negative for all $\mu \geq \mu_a^0$ (i.e., $\eta \geq \eta_a^0 = 2\mu_a^0$). For this, we rewrite (3.19) as

$$-\varphi(\eta)\alpha_a-[\varphi(0)-\varphi(\eta)]\ g_a(\tfrac{1}{2}\ c_{a+1})\ \int_0^{\eta} x\ g_a(\tfrac{1}{2}\ c_{a+1}+\mu_2-(\mu-x)^2)\varphi(x)dx$$

$$+\int_0^{\infty} x\ g_a(\tfrac{1}{2}\ c_{a+1}+\mu^2-(\mu+x)^2)\ \varphi(x)dx. \qquad (3.21)$$

For every $\mu \geq 0$, let μ_a^* be defined by $\frac{1}{2}\ c_{a+1}+\mu^2-(\mu+\mu_a^*)=0$. Note that μ_a^*, for $\mu=0$, is equal to $(\frac{1}{2}\ c_{a+1})^{1/2}$ $(<\sqrt{a-2})$, and μ_a^* is decreasing in μ, with $\lim_{\mu\to\infty}\mu_a^*=0$. Then, in the last term in (3.21), the range $(0,\infty)$ can be replaced by $(0,\mu_a^*)$, and further

$$\int_0^{\infty} x\ g_a(\tfrac{1}{2}\ c_{a+1}+\mu^2-(\mu+x)^2)\ \varphi(x)dx$$

$$=\int_0^{\mu_a^*} g(\tfrac{1}{2}\ c_{a+1}+\mu^2-(\mu+x)^2)\ x\varphi(x)dx$$

$$\leq g(\tfrac{1}{2}\ c_{a+1})\int_0^{\mu_a^*} x\varphi(x)dx$$

$$= g(\tfrac{1}{2}\ c_{a+1})\ [\varphi(0)-\varphi(\mu_a^*)]. \qquad (3.22)$$

Finally, using the definitions of μ_a^0 and μ_a^*, it readily follows that

$$\mu_a^* \leq \eta \text{ for every } \eta \leq \eta_a^0 = 2\mu_\eta^0, \qquad (3.23)$$

so that from (3.21), (3.22) and (3.23) we conclude that (3.19) is ≥ 0, $\forall\ \eta \geq \eta_a^0$. Thus (3.19) is ≤ 0, $\forall\ \eta \geq 0$. Hence, (3.18) being $> \frac{1}{2}$ at $\eta = 0$ and $\geq \frac{1}{2}$ at $\eta \to +\infty$, is $\geq \frac{1}{2}$ for every $\eta \geq 0$. This completes the proof of the theorem.

We consider next the positive-rule versions in (3.7)-(3.8). By virtue of (3.5) and (3.8), we have

$$\hat{\underset{\sim}{\theta}}{}^{*+}_{RSM}-\underset{\sim}{\theta}=-\underset{\sim}{\theta}\ I(\|\underset{\sim}{X}^+\|^2<\sigma^2\ c_{a(\underset{\sim}{X})})+\{\hat{\underset{\sim}{\theta}}{}^{*}_{RSM}-\underset{\sim}{\theta}\}\ I(\|\underset{\sim}{X}^+\|^2\geq\sigma^2\ c_{a(\underset{\sim}{X})})\}, \qquad (3.24)$$

where (by virtue of $c_0 = c_1 = 0$) the first term on the right hand side vanishes for $a(\underset{\sim}{X}) = 0$ or 1. Hence, we have

$$P_{\underset{\sim}{\theta}}\{\|\hat{\underset{\sim}{\theta}}^{*+}_{RSM} - \underset{\sim}{\theta}\| \le \|\hat{\underset{\sim}{\theta}}^{*}_{RSM} - \underset{\sim}{\theta}\|\}$$

$$= P_{\underset{\sim}{\theta}}\{a(\underset{\sim}{X}) = 0 \text{ or } 1\} + P_{\underset{\sim}{\theta}}\{a(\underset{\sim}{X}) \ge 2,\ \|\underset{\sim}{X}^+\|^2 \ge c_{a(\underset{\sim}{X})}\sigma^2\}$$

$$+ P_{\underset{\sim}{\theta}}\{a(\underset{\sim}{X}) \ge 2,\ \|\underset{\sim}{X}^+\|^2 < \sigma^2 c_{a(\underset{\sim}{X})},\ \|\hat{\underset{\sim}{\theta}}^{*}_{RSM} - \underset{\sim}{\theta}\| \ge \|\underset{\sim}{\theta}\|\}. \quad (3.25)$$

As in the proof of Theorem 3.1, we consider there only the special case of $\underset{\sim}{\theta}' = (\eta, \underset{\sim}{0}')$, $\eta \ge 0$. Then the first term on the right of hand side of (3.25) is given by (3.13), so that (3.25) is $\ge \frac{1}{2}$, $\forall\, \eta$, for $p = 2$. For $p \ge 2$, the second term is given by (for $\sigma = 1$)

$$\sum_{a=2}^{p} \{2^{-(p-1)}[\tfrac{1}{2}\,\Phi(-\eta)\binom{p-1}{a} + \binom{p-1}{a-1}\int_0^\infty \bar{G}_{a-1}(c_a - x^2)\,\varphi(x-\eta)dx]\}. \quad (3.26)$$

Similarly, the last term on the right hand side of (3.25) is given by

$$\sum_{a=2}^{p} \{2^{-(p-1)}[\binom{p-1}{a}\tfrac{1}{2}\,\Phi(-\eta)$$

$$+ \binom{p-1}{a-1}\int_0^\infty G_{a-1}(c_a + \eta^2 - (x-\eta)^2)\varphi(x-\eta)dx]\} \quad (3.27)$$

As such, we may proceed as in (3.18) through (3.23) and conclude that (3.25) is $\ge \frac{1}{2}$, $\forall\, \eta \ge 0$. A very similar proof holds for the case of $\hat{\underset{\sim}{\theta}}^{+}_{RSM}$ vs. $\hat{\underset{\sim}{\theta}}_{RSM}$. Hence, we have the following.

<u>Theorem 3.2</u> *For* $p \ge 2$, $\hat{\underset{\sim}{\theta}}^{*+}_{RSM}$ *dominates* $\hat{\underset{\sim}{\theta}}^{*}_{RSM}$ *in the PCC, and for* $p \ge 3$, $\hat{\underset{\sim}{\theta}}^{+}_{RSM}$ *dominates* $\hat{\theta}_{RSM}$ *in the PCC.*

By an adaptation of Theorems 2.1, 2.2 and 2.3, it can also be shown that there is no PCE of $\underset{\sim}{\theta}$ within the class of RSMLE (or their positive-rule versions) where in (3.5), we allow c_a to be arbitrary.

4. SOME GENERAL REMARKS

The Theorems presented in Sections 2 and 3 place the PCC on a comparable standing with the conventional quadratic risk criterion. Moreover, the PCC leads to the desired dominance results even for $p = 2$, while in the other setup, we usually require that $p \geq 3$. In this context, we have confined ourselves to simple shrinkage estimators of the type (2.5) or (3.5). If instead of (2.5), we would have considered (2.4), then in (2.10) (and elsewhere), instead of the constant shrinkage factor b, we would have a $\varphi(\underset{\sim}{X}, \sigma^2)$, where $\varphi(\cdot)$ is arbitrary and $0 < \varphi(\underset{\sim}{x}, \sigma^2) < (p-1)(3p+1)/2p$. This arbitrariness of $\varphi(\cdot)$ eliminates the possibility of using the simple and direct proof of Theorem 2.2 (or the others), and a much more complicated approach may be needed. Moreover, if the PCE characterization does not hold within the class $\mathcal{C}_{JS}$ on $\mathcal{C}^+_{JS}$, it can not obviously hold for a larger class generated by such $\varphi(\cdot)$. The results of Sen, Kubokawa and Saleh (1989) can, of course, be used to strengthen the dominance results of Section 3 to the restricted parameter space model- however, the PCE characterization will be a trifle harder!

The results presented here are based on the fundamental properties of (noncentral) chi square distributions some of which were studied in Sen (1989). In a general context with possibly unknown σ^2 and/or arbitrary $\varphi(\cdot)$, the related distributional problems may become untractable. Moreover, the role of noncentral chi square distributions may have to be replaced by that of noncentral beta or variance ratio (i.e., $F_{p,q}$-) distributions. Some of these properties are under investigation now and will be reported in a future communication. Finally, the results presented here relate to underlying normal distributions, and they are exact in nature. In an asymptotic setup (i.e., granted the asymptotic normality of an estimator $\underset{\sim}{T}_n$ of $\underset{\sim}{\theta}$), the current results pertain to a much wider class, and in that sense, the results of the last section of Sen, Kubokawa and Saleh (1989) directly extend to the restricted parametric models in Section 3. However, we should then keep in mind that like in the case of quadratic risks, the PC dominance then remains perceptible only in a local neighborhood of the pivot. Of course, this is in conformity with the usual asymptotic setup.

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BIBLIOGRAPHY

Anderson, T.W. (1955). The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* 6, 170-176.

Blyth, C. (1972). Some probability paradoxes in choice from among random alternatives (with discussion). *J. Amer. Statist. Assoc.* 67, 366-373.

DasGupta, S. and Sarkar, S.K. (1984). On TP_2 and log-concavity. In *Inequalities in Statistics and Probability* (ed. Y.L. Tong). IMS Lecture Notes - Monograph Series, Vol.5, pp. 54-58.

Efron, B. (1975). Biased vs. unbiased estimation. *Adv. in Math.* 16, 259-266.

Ghosh, M. and Sen, P.K. (1989). Median unbiasedness and Pitman closeness. *J. Amer. Statist. Assoc.* 84, 1089-1091.

James, W. and Stein, C. (1961). Estimation with quadratic loss. *Proc. 4th Berkeley Symp. Math. Statist. Probab.* 1, 361-379.

Johnson, N.L. and Kotz, S. (1970). *Distributions in Statistics: Continuous Univariate distributions* 2. John Wiley, New York.

Keating, J.P. and Mason, R.L. (1988). James-Stein estimation from an alternative perspective. *Amer. Statist.* 42, 160-164.

Nayak, T. (1990). Estimation of location and scale parameters using generalized Pitman nearness criterion. *J. Statist. Plan. Inf.* 24, 259-268.

Pitman, E.J.G. (1937). The closest estimates of statistical parameters. *Proc. Cambridge Phil. Soc.* 33, 212-222.

Rao, C.R., Keating, J.P. and Mason, R.L. (1986). The Pitman nearness criterion and its determination. *Commun. Statist. Theor. Math.* 15, 3173-3191.

Sen, P.K. (1989). The mean-median-mode inequality and noncentral chi square distributions. *Sankhya, Ser.* A 51, 106-114.

Sen, P.K. (1990). The Pitman closeness of some sequential estimators. *Sequen. Anal.* 9, 383-400.

Sen, P.K., Kubokawa, T. and Saleh, A.K.M.E. (1989). The Stein-paradox in the sense of the Pitman measure of closeness. Ann. Statist. 17, 1375-1386.

Sen, P.K., Nayak, T. and Khattree, R. (1991). Comparison of equivariant estimators of a dispersion matrix under gene ralized Pitman nearness criterion. (To be published, *Communications in Statistics*, A20.)

Sengupta, D. and Sen, P.K. (1991). Shrinkage estimation in a restricted parameter space. *Sankhya, Ser.* A. 53, in press.

Stein, C. (1981). Estimation of the mean of a multivariate normal distribution. *Ann. Statist.* 9, 1135-1151.