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THEORETICAL PAPERS

OPTIMIZATION OF BICRITERION QUASI-CONCAVE FUNCTION SUBJECT TO LINEAR CONSTRAINTS*

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ABSTRACT

In this paper we provide algorithms for maximisation and minimisation of bicriterion quasi-concave function g (c_1x, c_2x) subject to linear constraints. The algorithm for maximisation is based on bisection approach. The algorithm for minimisation is an implicit enumeration method. With some minor modifications, this algorithm also enumerates all efficient solutions of bicriterion linear programs. Maximisation of system reliability of series-parallel and parallel-series systems (with two subsystems) through optimal assignment of components is treated as a special case.

1. Introduction

Suppose g (z_1, z_2) is a single valued function defined on R^2 which satisfies the properties : (i) g (z_1, z_2) is quasi-concave and (ii) g (z_1, z_2) increases with each argument. The problems, we study in this paper, are maximisation and minimisation of g $(c_1 x, c_2 x)$ subject to $Ax = b$, $x \geqslant 0$ where c_1 and c_2 are $1 \times n$ vectors, A is an $m \times n$ matrix and b an $m \times 1$

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vector. Practical justification and several examples of maximisation problem can be found in Geoffrion [6]. Special cases of these two problems can be found in Anand [1], Aneja et al. [3], Bector and Dahl [5] and Swarup [10, 11]. In all these special cases, $g(c_1 x, c_2 x)$ is of the form $(c_1 x + a_1)(c_2 x + a_2)$.

In Section 2, we consider the problem of maximising $g(c_1 x, c_2 x)$ subject to $Ax = b$, $x \geqslant 0$. It is assumed that $g(z_1, z_2)$ is continuously differentiable on R^2 and strictly increases with each argument. We first obtain some preliminary results and later develop an algorithm based on these results. Geoffrion [6] considered a more general maximisation problem in which $c_1 x$ and $c_2 x$ are replaced by two concave functions of x and gave algorithms separately for the general problem and the problem under consideration. These algorithms are based on parametric programming technique. We argue that our algorithm is more efficient than algorithm 2 of Goeffrion [6], which was developed to solve the problem of this section.

Finally we apply the algorithm to a problem of assigning components optimally to a series-parallel reliability system so as to maximise the system reliability.

In Section 3, we consider the minimisation problem. Here also, we derive some preliminary results and develop an algorithm to solve the problem on the basis of these results. If the matrix A is totally unimodular and b an integral vector, the algorithm yields an optimal solution even when x is restricted to be integral vector. Finally we apply the algorithm to maximise the reliability of parallel-series reliability system by optimally allocating the components.

2. Maximisation of Bicriterion Quasi-Concave Function

In this section, we consider the problem:

$$\text{Maximise} \quad g(c_1 x, c_2 x),$$

$$\text{subject to} \quad x \in K = \{x : Ax = b,\ x \geqslant 0\}. \qquad (P_1)$$

This is equivalent to the problem:

$$\text{Maximise} \quad g(z_1, z_2),$$

$$\text{subject to} \quad (z_1, z_2) \in Z = \{(z_1, z_2) : (z_1, z_2) = (c_1 x, c_2 x) \text{ for some } x \in K\}. \qquad (P_2)$$

We assume that $g(z_1, z_2)$ is continuously differentiable on R^2_+ and strictly increases with z_1 and z_2. In our notation, $(z_1, z_2) \geqslant (y_1, y_2)$ means $z_1 \geqslant$

$z_2 \geqslant y_2$ and $(z_1, z_2) \neq (y_1, y_2)$. A point (z_1, z_2) in Z is said to be efficient if there does not exist another point (y_1, y_2) in Z such that $(y_1, y_2) \geqslant (z_1, z_2)$. The optimal solution (z_1^*, z_2^*) of problem P_2 is necessarily an efficient point due to property *(ii)* of $g(z_1, z_2)$. So, there exists a positive number λ^* such that (z_1^*, z_2^*) maximises $z_1 + \lambda^* z_2$ on Z. Our approach is to first find λ^* and next (z_1^*, z_2^*) and an $x^* \in K$ such that $(z_1^*, z_2^*) = (c_1 x^*, c_2 x^*)$. Note that this x^* is an optimal solution of problem P_1.

We shall first present some preliminary results and using these results we develop an algorithm that gives λ^*, (z_1^*, z_2^*) and x^*. The algorithm starts with interval $[0, \infty)$ and partitions, in each iteration, an interval containing λ^* (selected in the previous iteration) into two subintervals and selects the one containing λ^*. It yields λ^*, (z_1^*, z_2^*) and x^* in the final iteration.

Preliminary Results

For the purpose of convenience, a point (z_1, z_2) in Z is denoted by z and $g(z_1, z_2)$ by $g(z)$. Let us denote by $L(\lambda)$ the problem of maximising $z_1 + \lambda z_2$ on Z. Throughout this section we consider the problem $L(\lambda)$ for $\lambda \geqslant 0$ only. Let us denote an optimal solution of $L(\lambda)$ by $z(\lambda) = (z_1(\lambda), z_2(\lambda))$.

LEMMA 1. $\lambda_1 < \lambda_2$ *and* $z(\lambda_1) \neq z(\lambda_2) \Rightarrow z_1(\lambda_1) > z_1(\lambda_2)$ *and* $z_2(\lambda_1) < z_2(\lambda_2)$.

Proof. We have $z_1(\lambda_1) \neq z_1(\lambda_2)$ and $z_2(\lambda_1) \neq z_2(\lambda_2)$ since $z(\lambda_1)$ and $z(\lambda_2)$ are optimal solutions of $L(\lambda_1)$ and $L(\lambda_2)$ and $z(\lambda_1) \neq z(\lambda_2)$. Suppose $z_1(\lambda_1) < z_1(\lambda_2)$. This implies $z_2(\lambda_1) > z_2(\lambda_2)$. We can write

$$z_1(\lambda_1) + \lambda_1 z_2(\lambda_1) \geqslant z_1(\lambda_2) + \lambda_1 z_2(\lambda_2),$$

i.e., $\{z_1(\lambda_2) - z_1(\lambda_1)\}/\{z_2(\lambda_1) - z_2(\lambda_2)\} \leqslant \lambda_1 < \lambda_2$,

i.e., $z_1(\lambda_2) + \lambda_2 z_2(\lambda_2) < z_1(\lambda_1) + \lambda_2 z_2(\lambda_1)$,

which contradicts the optimality of $z(\lambda_2)$ for $L(\lambda_2)$. Therefore $z_1(\lambda_1) > z_1(\lambda_2)$ and consequently $z_2(\lambda_1) < z_2(\lambda_2)$.

LEMMA 2. *Let* $\lambda_1 < \lambda_2$, $z(\lambda_1) \neq z(\lambda_2)$ *and*

$$\bar{\lambda} = [z_1(\lambda_1) - z_1(\lambda_2)]/[z_2(\lambda_2) - z_2(\lambda_1)] \tag{1}$$

Then $\lambda_1 \leqslant \bar{\lambda} \leqslant \lambda_2$. *Further if*

$$z_1(\bar{\lambda}) + \bar{\lambda}\, z_2(\bar{\lambda}) > z_1(\lambda_1) + \bar{\lambda}\, z_2(\lambda_1), \tag{2}$$

(*i*) $\lambda_1 < \bar{\lambda} < \lambda_2$

(*ii*) $g(z(\lambda_1)) \geqslant g(z(\bar{\lambda})) \Rightarrow g(z(\bar{\lambda})) > g(\hat{z})$ *if* $\hat{z} \in Z$, $\hat{z}_1 < z_1(\bar{\lambda})$ *and* $\hat{z}_2 > z_2(\bar{\lambda})$

(*iii*) $g(z(\lambda_2)) \geqslant g(z(\bar{\lambda})) \Rightarrow g(z(\bar{\lambda})) > g(\hat{z})$ *if* $\hat{z} \in Z$, $\hat{z}_1 > z_1(\bar{\lambda})$ *and* $\hat{z}_2 < z_2(\bar{\lambda})$.

Proof. We have $\bar{\lambda} > 0$ by Lemma 1. Suppose $\bar{\lambda} < \lambda_1$. Then $[z_1(\lambda_1) - z_1(\lambda_2)]/[z_2(\lambda_2) - z_2(\lambda_1)] < \lambda_1$, i.e., $z_1(\lambda_1) + \lambda_1 z_2(\lambda_1) < z_1(\lambda_2) + \lambda_1 z_2(\lambda_2)$, which contradicts the optimality of $z(\lambda_1)$ for $L(\lambda_1)$. Thus $\bar{\lambda} \geqslant \lambda_1$ and similarly $\bar{\lambda} \leqslant \lambda_2$.

(*i*) Inequality (2) means that $z(\bar{\lambda})$ does not lie on the line passing through $z(\lambda_1)$ and $z(\lambda_2)$. Using (1) and (2), one can easily see that $\lambda_1 < \bar{\lambda} < \lambda_2$.

(*ii*) Assume that $g(z(\lambda_1)) \geqslant g(z(\bar{\lambda}))$ and (2) holds. Consider a point $\hat{z} \in Z$ such that $\hat{z}_1 < z_1(\bar{\lambda})$ and $\hat{z}_2 > z_2(\bar{\lambda})$.

Since $\hat{z}_1 < z_1(\bar{\lambda}) < z_1(\lambda_1)$, we can find α, $0 < \alpha < 1$, such that $z_1(\bar{\lambda}) = \alpha\hat{z}_1 + (1-\alpha) z_1(\lambda_1)$. Now consider the point $\tilde{z} = \alpha\hat{z} + (1-\alpha) z(\lambda_1)$. It is obvious that $\tilde{z} \in Z$ and $\tilde{z}_1 = z_1(\bar{\lambda})$. Also $\tilde{z}_2 \leqslant z_2(\bar{\lambda})$ for otherwise optimality of $z(\bar{\lambda})$ for $L(\bar{\lambda})$ would be violated. Suppose $\tilde{z}_2 = z_2(\bar{\lambda})$. Then

$$z_1(\bar{\lambda}) + \bar{\lambda} z_2(\bar{\lambda}) = \alpha[\hat{z}_1 + \bar{\lambda}\hat{z}_2] + (1-\alpha)[z_1(\lambda_1) + \bar{\lambda} z_2(\lambda_1)]$$
$$< \alpha[\hat{z}_1 + \bar{\lambda}\hat{z}_2] + (1-\alpha)[z_1(\bar{\lambda}) + \bar{\lambda} z_2(\bar{\lambda})]$$

due to (2), i.e., $\hat{z}_1 + \bar{\lambda}\hat{z}_2 > z_1(\bar{\lambda}) + \bar{\lambda} z_2(\bar{\lambda})$ which contradicts the optimality of $z(\bar{\lambda})$ for $L(\bar{\lambda})$. Therefore $\tilde{z}_2 < z_2(\bar{\lambda})$. Now, by properties (*i*) and (*ii*) of $g(z)$, we can write

$$g(z(\bar{\lambda})) > g(\tilde{z}) \geqslant \min\{g(z(\lambda_1)),\ g(\hat{z})\},$$

which implies $g(z(\bar{\lambda})) > g(\hat{z})$ since $g(z(\lambda_1)) \geqslant g(z(\bar{\lambda}))$. (*ii*) also can be proved on the same lines.

LEMMA 3. *Let* $\lambda_1<\lambda_2$, $z(\lambda_1)\neq z(\lambda_2)$, $\bar{\lambda}=[z_1(\lambda_1)-z_1(\lambda_2)]/[z_2(\lambda_2)-z_2(\lambda_1)]$ *and* $\bar{z}_1+\bar{\lambda}\bar{z}_2>z_1(\lambda_1)+\bar{\lambda}z_2(\lambda_1)$ *for an optimal solution* $\bar{z}$ *of* $L(\bar{\lambda})$. *Then*

(*i*) $g(z(\lambda_1))\geqslant g(\bar{z})\Rightarrow g(\bar{z})\geqslant g(z(\lambda))$ *for* $\lambda>\bar{\lambda}$

(*ii*) $g(z(\lambda_2))\geqslant g(\bar{z})\Rightarrow g(\bar{z})\geqslant g(z(\lambda))$ *for* $\lambda<\bar{\lambda}$

Proof follows from Lemmas 1 and 2.

LEMMA 4. *Let* α *be the value of* $\dfrac{\partial g}{\partial z_2}\Big/\dfrac{\partial g}{\partial z_1}$ *at an optimal solution* $\bar{z}$ *of* $L(\bar{\lambda})$

Then

(*i*) $\bar{\lambda}\leqslant\alpha\Rightarrow g(\hat{z})\leqslant g(\bar{z})$ *if* $\hat{z}\in Z$, $\hat{z}_1>\bar{z}_1$ *and* $\hat{z}_2<\bar{z}_2$.

(*ii*) $\bar{\lambda}\geqslant\alpha\Rightarrow g(\hat{z})\leqslant g(\bar{z})$ *if* $\hat{z}\in Z$, $\hat{z}_1<\bar{z}_1$ *and* $\hat{z}_2>\bar{z}_2$.

Proof.

(*i*) Consider a $\hat{z}\in Z$ such that $\hat{z}_1>\bar{z}_1$ and $\hat{z}_2<\bar{z}_2$. Since $\bar{z}$ is optimal for $L(\bar{\lambda})$, we have

$$[\hat{z}_1-\bar{z}_1]/[\bar{z}_2-\hat{z}_2]\leqslant\bar{\lambda}\leqslant\alpha,$$

i.e.,

$$\hat{z}_1+\alpha\hat{z}_2\leqslant\bar{z}_1+\alpha\bar{z}_2. \tag{3}$$

Since $g(z)$ is continuously differentiable quasi-concave, $z_1+\alpha z_2=\bar{z}_1+\alpha\bar{z}_2$ is a supporting hyperplane of the closed convex set $S(\bar{z})=\{z:g(z)\geqslant g(\bar{z})\}$ at $z=\bar{z}$ and $S(\bar{z})\subseteq\{z:z_1+\alpha z_2\geqslant\bar{z}_1+\alpha\bar{z}_2\}$. Suppose $g(\hat{z})>g(\bar{z})$. Then $\hat{z}\in$ int $S(\bar{z})$. This implies $\hat{z}_1+\alpha\hat{z}_2>\bar{z}_1+\alpha\bar{z}_2$ which contradicts (3). Therefore $g(\hat{z})\leqslant g(\bar{z})$. (ii) also can be proved on the same lines.

LEMMA 5. *For* $\bar{\lambda}$ *and* α *of Lemma* 4,

(*i*) $g(z(\lambda))\leqslant g(\bar{z})$ *for* $0\leqslant\lambda<\bar{\lambda}$ *if* $\bar{\lambda}\leqslant\alpha$,

(*ii*) $g(z(\lambda))\leqslant g(\bar{z})$ *for* $\lambda>\bar{\lambda}$ *if* $\bar{\lambda}\geqslant\alpha$,

(*iii*) $g(z(\lambda))\leqslant g(\bar{z})$ *for* $\lambda\geqslant 0$ *if* $\bar{\lambda}=\alpha$.

Proof. (*i*) and (*ii*) follow from Lemmas 1 and 4. (*iii*) follows from (*i*) (*ii*) and Lemma 4 since any optimal solution $\tilde{z}\neq\bar{z}$ of $L(\bar{\lambda})$ satisfy either $\tilde{z}_1>\bar{z}_1$ and $\tilde{z}_2<\bar{z}_2$ or $\tilde{z}_1<\bar{z}_1$ and $\tilde{z}_2>\bar{z}_2$.

LEMMA 6. *Let* $\lambda_1 \leqslant \bar{\lambda} \leqslant \lambda_2$ *and* $z(\lambda_1)$ *and* $z(\lambda_2)$ *lie on the line* $\{z/z_1 + \bar{\lambda} z_2 = z_1(\bar{\lambda}) + \bar{\lambda} z_2(\bar{\lambda})\}$. *Then* $z(\lambda_1)$ *is the unique optimal solution of* $L(\lambda)$ *for* $\lambda_1 < \lambda < \bar{\lambda}$ *and* $z(\lambda_2)$ *is the unique optimal solution of* $L(\lambda)$ *for* $\bar{\lambda} < \lambda < \lambda_2$.

Proof. Note that $z(\lambda_1)$ and $z(\lambda_2)$ are also optimal for $L(\bar{\lambda})$. Consider a λ such that $\lambda_1 < \lambda < \bar{\lambda}$. Suppose $z(\lambda) \neq z(\lambda_1)$. Then we have $z_1(\lambda) < z_1(\lambda_1)$ and $z_2(\lambda) > z_2(\lambda_1)$ by Lemma 1. We also have

$$z_1(\lambda) + \lambda z_2(\lambda) \geqslant z_1(\lambda_1) + \lambda z_2(\lambda_1).$$

i.e.,
$$[z_1(\lambda_1) - z_1(\lambda)]/[z_2(\lambda) - z_2(\lambda_1)] \leqslant \lambda < \bar{\lambda}.$$

i.e.,
$$z_1(\lambda_1) + \bar{\lambda} z_2(\lambda_1) < z_1(\lambda) + \bar{\lambda} z_2(\lambda),$$

which contradicts the optimality of $z(\lambda_1)$ for $L(\bar{\lambda})$. Therefore $z(\lambda) = z(\lambda_1)$ and $z(\lambda_1)$ is the unique optimal solution of $L(\lambda)$. Similarly, $z(\lambda_2)$ is the unique optimal solution of $L(\lambda)$ for $\bar{\lambda} < \lambda < \lambda_2$.

The following algorithm developed on the basis of above results yields optimal solutions of problems P_1 and P_2 simultaneously.

Algorithm 1

Step 0: Find a point $z^{(1)}$ that maximises z_1 coordinate on Z. This can be done by maximising c_1x on K. Let $x^{(1)}$ be the point that maximises c_1x on K. Then $z^{(1)} = (c_1x^{(1)}, c_2x^{(1)})$. Set $x^* = x^{(1)}$, $z^* = z^{(1)}$ and $q^* = g(z^{(1)})$. Similarly obtain $x^{(2)}$ that maximises c_2x on K and find the image $z^{(2)}$ of $x^{(2)}$. If $q^* < g(z^{(2)})$, set $x^* = x^{(2)}$, $z^* = z^{(2)}$ and $q^* = g(z^{(2)})$.

Step 1: Find $\bar{\lambda} = [z_1^{(1)} - z_1^{(2)}]/[z_2^{(2)} - z_2^{(1)}]$ and set $F_0 = z_1^{(1)} + \bar{\lambda}\, z_2^{(1)}$.

Maximise $z_1 + \bar{\lambda}\, z_2$ on Z by solving LP : maximise $(c_1 + \bar{\lambda}\, c_2)\, x$ on K. Let $\bar{x}$ be the optimal solution of this LP and $\bar{z} = (c_1\bar{x}, c_2\bar{x})$ and $\bar{F} = \bar{z}_1 + \bar{\lambda}\bar{z}_2$. If $\bar{F} = F_0$, set $\lambda^* = \bar{\lambda}$ and go to Step 4. If $g(\bar{z}) > q^*$, go to Step 3. Otherwise go to Step 2.

Step 2 : If $x^* = x^{(1)}$, set $x^{(2)} = \bar{x}$ and $z^{(2)} = \bar{z}$. Otherwise set $x^{(1)} = \bar{x}$ and $z^{(1)} = \bar{z}$ and go to Step 1.

Step 3 : Set $x^* = \bar{x}$, $z^* = \bar{z}$ and $q^* = g(\bar{z})$. Evaluate $\alpha = \dfrac{\partial g}{\partial z_2} \Big| \dfrac{\partial g}{\partial z_1}$ at $z = \bar{z}$. If $\alpha < \bar{\lambda}$, set $x^{(2)} = \bar{x}$ and $z^{(2)} = \bar{z}$ and go to Step 1. If $\alpha > \bar{\lambda}$, set $x^{(1)} = \bar{x}$, $z^{(1)} = \bar{z}$ and go to Step 1. If $\alpha = \lambda$, stop. x^* and z^* are optimal solutions of the problems P_1 and P_2, respectively.

Step 4 : Define the function $\psi(\theta) = g(z^{(1)}+\theta(z^{(2)}-z^{(1)}))$ and maximise $\psi(\theta)$ over $0 \leqslant \theta \leqslant 1$. Let θ^* maximise $\psi(\theta)$ subject to $0 \leqslant \theta \leqslant 1$. Set $z^* = z^{(1)}+\theta^*(z^{(2)}-z^{(1)})$ and $x^* = x^{(1)}+\theta^*(x^{(2)}-x^{(1)})$ and stop. x^* and z^* are optimal solutions of the problems $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively.

Validity of Algorithm

The optimal solution of the problem $\mathbf{P}_2$ is an efficient point z^* and therefore there exists a finite positive λ^* such that z^* is an optimal solution of $L(\lambda^*)$. The above algorithm starts with the interval $[0, \infty)$ in search of λ^* and in each iteration it divides an interval containing a λ^* into two subintervals and selects the subinterval that contains λ^*. The algorithm obtains λ^* in Step 3 or Step 4 of the final iteration.

LEMMA 7. *Suppose* $z^{(1)}$ *and* $z^{(2)}$ *of Step* 1 *of the algorithm are obtained as optimal solutions of* $L(\lambda_1)$ *and* $L(\lambda_2)$, *respectively. Then* $\lambda_1 < \lambda_2$ *and the interval* $[\lambda_1, \lambda_2]$ *contains at least one value of* λ^*. *Further*,

(*i*) $g(\hat{z}) \leqslant g(z(\lambda_1))$ *if* $\hat{z} \in Z$, $\hat{z}_1 > z_1(\lambda_1)$ *and* $\hat{z}_2 < z_2(\lambda_1)$.

(*ii*) $g(\hat{z}) \leqslant g(z(\lambda_2))$ *if* $\hat{z} \in Z$, $\hat{z}_1 < z_1(\lambda_2)$ *and* $\hat{z}_2 > z_2(\lambda_2)$.

Proof. The proof is based on induction on the number of iterations. In the first iteration $\lambda_1 = 0$ and λ_2 is taken to be ∞ as a convention. Suppose the lemma holds for rth iteration and let $z^{(1)}$ and $z^{(2)}$ of Step 1 of rth iteration be solutions of $L(\lambda_1)$ and $L(\lambda_2)$ and $\bar{z}$ an optimal solution of $L(\bar{\lambda})$ where $\bar{\lambda}$ is as described in Step 1. If $\bar{F} \neq F_0$, we have, by Lemma 2, $\lambda_1 < \bar{\lambda} < \lambda_2$. Suppose $q^* \geqslant g(\bar{z})$ and $z^* = z^{(1)}$. Then $g(z^{(1)}) \geqslant g(\bar{z})$ and due to Lemmas 2 and 3, $g(z(\lambda)) \leqslant g(z^*)$ for $\lambda > \bar{\lambda}$ and $g(\hat{z}) < g(\bar{z})$ if $\hat{z} \in Z$, $\hat{z}_1 < \bar{z}_1$ and $\hat{z}_2 > \bar{z}_2$. It means that the interval $[\lambda_1, \bar{\lambda}]$ contains a value of λ^* and the lemma holds for $(r+1)$ th iteration when λ_2 and $z^{(2)}$ are updated as $\lambda_2 = \bar{\lambda}$ and $z^{(2)} = \bar{z}$. Similarly the cases (a) $q^* \geqslant g(\bar{z})$ and $z^* = z^{(2)}$ (*b*) $q^* < g(\bar{z})$ and $\bar{\lambda} < \alpha$, (*c*) $q^* < g(\bar{z})$ and $\bar{\lambda} > \alpha$ where $\alpha = \dfrac{\partial g}{\partial z_2} \Big/ \dfrac{\partial g}{\partial z_1}$ at $z = \bar{z}$ can also be proved using Lemmas 2, 3, 4 and 5.

THEOREM 1. *If* $\bar{F} = F_0$ *in Step* 1, *the optimal solution of the problem* $\mathbf{P}_2$ *lies on the line segment joining* $z^{(1)}$ *and* $z^{(2)}$ *of Step* 1.

Proof. If $\bar{F} = F_0$, $z^{(1)}$ and $z^{(2)}$ are on the line $z_1 + \bar{\lambda} z_2 = \bar{z}_1 + \bar{\lambda} \bar{z}_2$. Suppose $z^{(1)}$ and $z^{(2)}$ are obtained as optimal solutions of $L(\lambda_1)$ and $L(\lambda_2)$, respectively. By Lemmas 6 and 7, it is enough to consider the optimal solutions of $L(\lambda_1)$, $L(\bar{\lambda})$ and $L(\lambda_2)$. Consider an optimal solution $\hat{z}$ of $L(\lambda_1)$ which does not maximise $z_1 + \bar{\lambda} z_2$ on Z. We have either (a) $\hat{z}_1 < z_1^{(1)}$ and $\hat{z}_2 > z_2^{(1)}$ or b) $\hat{z}_1 > z_1^{(1)}$ and $\hat{z}_2 < z_2^{(1)}$ since both $\hat{z}$ and $z^{(1)}$ are optimal for $L(\lambda_1)$. Suppose (a) holds. Then we have $[z_1^{(1)} - \hat{z}_1]/[\hat{z}_2 - z_2^{(1)}]$ $= \lambda_1 \leqslant \bar{\lambda}$, i.e., $z_1^{(1)} + \bar{\lambda}\, z_2^{(1)} \leqslant \hat{z}_1 + \bar{\lambda}\, \hat{z}_2$ which contradicts that z is not optimal for $L(\bar{\lambda})$. Therefore (b) holds and due to Lemma 7, $g(\hat{z}) \leqslant g(z^{(1)})$. Similarly, if an optimal solution $\hat{z}$ of $L(\lambda_2)$ does not maximise $z_1 + \bar{\lambda}\, z_2$ on Z, then $g(\hat{z}) \leqslant g(z^{(2)})$. Now it is enough to consider only the optimal solutiens of $L(\bar{\lambda})$ in order to find the optimal solution of P_2.

Suppose $\hat{z}$ is an optimal solution of $L(\bar{\lambda})$ but is not on the line segment joining $z^{(1)}$ and $z^{(2)}$. Then either $\hat{z}_1 > z_1^{(1)}$ and $\hat{z}_2 < z_2^{(1)}$ or $\hat{z}_1 < z_1^{(2)}$ and $\hat{z}_2 > z_2^{(2)}$ and consequently by Lemma 7, $g(\hat{z}) \leqslant g(z^{(1)})$ or $g(\hat{z}) \leqslant g(z^{(2)})$. Hence the Lemma.

If the algorithm terminates in Step 3, then by Lemma 5, $\bar{\lambda}$ of final iteration is a value of λ^* and z^* is the optimal solution of P_2. If the algorithm terminates in Step 4, then by Theorem 1 the best point z^* on the line segment joining $z^{(1)}$ and $z^{(2)}$ is the optimal solution of P_2 and a point $x^* \epsilon K$ such that $z^* = (c_1x^*, c_2x^*)$ is an optimal solution of P_1.

Discrete Case

Consider the problem P_1 with x restricted to be an integral vector. Let I be the set of all integral points of K. A point x in I is said to be efficient with respect to I if there does not exist a y in I such that $(c_1x, c_2x) \leqslant (c_1y, c_2y)$. Note that a point which is efficient w.r.t. I need not be efficient in K. To maximise $g(c_1x, c_2x)$ on I, it is enough to consider points which are efficient w.r.t. I.

If A is totally unimodular and b is an integral vector, algorithm 1 can be made use of to solve the above problem. In this case, apply algorithm 1 ignoring the integrality restriction. If it terminates in Step 3, x^* is the required optimal solution. If it terminates in Step 4, enumerate all points which are efficient w.r.t. I and satisfy $c_1x > z_1^{(2)}$ and $c_2x > z_2^{(1)}$ and take the point among them which gives maximum value of g. If this point is better than x^*, then it is the required solution. Otherwise x^* is the required solution. Sometimes the structure of the matrix A enables us to develop implicit enumeration techniques such as branch and bound method to carry out the above mentioned enumeration, as illustrated in the following special case.

Special Case : Maximisation of Reliability of Series-Parallel System

Consider a series-parallel reliability system consisting of two parallel systems C_1 and C_2 in series. Suppose C_1 (C_2) consists of $n_1(n_2)$ positions and there are $n(=n_1+n_2)$ components any one of which can be assigned to any position. Let positions of C_1 be denoted by $1,2,\ldots,n_1$ and those of C_2 denoted by $n_1+1,\ldots,n$. Assume that reliability of component j is p_{ij} when it is assigned to position i. We represent an assignment by $n^2\times 1$ vector $x=(x_{11},\ldots x_{1n},x_{21},\ldots, x_{2n},\ldots,x_{n1},\ldots,x_{nn})^T$ where $x_{ij}=1$ if component j is assigned to position i and zero otherwise. For an assignment x, the system reliability can be written as

$$R_s(x) = [1-\prod_{i=1}^{n_1}\prod_{j=1}^{n} q_{ij}^{x_{ij}}]\,[1-\prod_{i=n_1+1}^{n}\prod_{j=1}^{n} q_{ij}^{x_{ij}}],$$

where $q_{ij}=1-p_{ij}$. This can be rewritten as

$$R_s(x) = [1-\exp(-s_1x)]\,[1-\exp(-s_2x)],$$

where

$$s_1=(s_{11},\ldots,s_{1n},s_{21},\ldots,s_{2n},\ldots,s_{n_11},\ldots,s_{n_1n},0,\ldots,0)$$

and $$s_2=(0,0,\ldots 0,s_{(n_1+1)1},\ldots,s_{(n_1+1)n},\ldots,s_{n1},\ldots s_{nn})$$

are two $1\times n^2$ vectors

and $s_{ij}=-\log q_{ij}$.

Now consider the problem (R_s) of assigning these n components to n positions of the system so as to maximise the system reliability. This problem is, in mathematical terms,

Maximise $\quad g(c_1x, c_2x)$, $\quad$ (R_s)

subject to $\quad x\in K$ and x being integral,

where $g(c_1x,c_2x) = [1-\exp(-c_1x)]\;[1-\exp(-c_2x)]$, $c_1= s_1$, $c_2=s_2$ and $K=\{x\colon \sum_{j=1}^{n} x_{ij}=1$ for $i=1$ to n, $\sum_{i=1}^{n} x_{ij} = 1$ for $j = 1$ to n and $x_{ij}\geqslant 0\}$.

Consider a function $f(x,y)= [1-\exp(-x)]\,[1-\exp(-y)]$ from R^2 to R. We now present a result concerning $f(x,y)$ which enables us to use algorithm 1 for solving the problem R_s.

LEMMA 8. *$f(x,y)$ is quasi-concave on $R^2_+ = \{(x,y) \mid x \geq 0, y \geq 0\}$.*

See Appendix for proof.

Since $g(z_1, z_2)$ is quasi-concave on R^2 by Lemma 8 and strictly increases with z_1 and z_2, we can make use of Algorithm 1 as suggested earlier for the discrete case. If Algorithm 1 terminates in Step 4, Algorithm 2 given below can be used to enumerate all assignments that satisfy $c_1x > z^2_{(1)}$ and $c_2x > z_2^{(1)}$. If an assignment x is represented by a permutation $(v_1, v_2, \ldots, v_n)$ of $1, 2, \ldots, n$, then $x_{iv_i} = 1$ for $i = 1$ to n and we can write $c_1x = \sum_{i=1}^{n_1} s_{iv_i}$ and $c_2x = \sum_{i=n_1+1}^{n} s_{iv_i}$. For a partial permutation $(u_1, u_2, \ldots, u_k)$ of $1, 2, \ldots, n$, let $c_1(u_1, \ldots, u_k)$ represent the maximum value of $\sum_{i=1}^{n_1} s_{iv_i}$ on the set of all permutations generated from $(u_1, \ldots, u_k)$. Similarly let $c_2(u_1, \ldots, u_k)$ represent the maximum value of $\sum_{i=n_1+1}^{n} s_{iv_i}$ on the same set of permutations. $c_i(u_1, \ldots, u_k)$, $i = 1, 2$, can be obtained using Hungarian method for the assignment problem. Let N denote the set $\{1, 2, \ldots, n\}$.

Algorithm 2

Step 0: Set $G = \{(1), (2), \ldots, (n)\}$ and $a_1 = z_1^{(2)}$ and $a_2 = z_2^{(1)}$

Step 1: Select a partial permutation $(u_1, \ldots, u_k)$ from G and set $G = G \setminus \{(u_1, \ldots, u_k)\}$. If $k = n-1$, go to Step 3. Otherwise evaluate $c_i(u_1, \ldots, u_k)$ for $i = 1, 2$. If $c_i(u_1, \ldots, u_k) > a_i$ for $i = 1, 2$, then set $G = \bigcup_{y \in N - \{u_1, \ldots, u_k\}} \{(u_1, \ldots, u_k, y)\} \cup G$

Step 2: If $G = \phi$, stop. x^* is optimal. Otherwise go to Step 1.

Step 3: If $(u_1, \ldots, u_n)$ is the permutation generation from $(u_1, \ldots, u_{n-1})$, evaluate $\sum_{i=1}^{n_1} s_{iu_i} - a_1$ and $\sum_{i=n_1+1}^{n} s_{iu_i} - a_2$. If one of these two values is non-positive, go to Step 2. Otherwise evaluate $g(\sum_{i=1}^{n_1} s_{iu_i}, \sum_{i=n_1+1}^{n} s_{iu_i})$. If it is greater than $g(c_1x^*, c_2x^*)$, set $x^*_{ij} = 1$ if $j = u_i$ and 0 otherwise for $i = 1$ to n. Go to Step 2.

3. Minimisation of Bicriterion Quasi-Concave Function

In this section, we consider the problem:

Minimise $g(c_1x, c_2x)$,

subject to $x \in K = \{x : Ax = b,\ x \geqslant 0\}$. (P_3)

This is equivalent to the problem:

Minimise $g(z_1, z_2)$,

subject to $(z_1, z_2) \in Z$, (P_4)

where Z is as described in Section 1.

Note that Z is a convex polyhedron. In this section, a point (z_1, z_2) in Z is said to be efficient if and only if there does not exist another point (y_1, y_2) in Z such that $(y_1, y_2) \leqslant (z_1, z_2)$. By properties (i) and (ii) of $g(z_1, z_2)$, there exists an efficient extreme point $(\overset{*}{z_1}, \overset{*}{z_2})$ that minimises $g(z_1, z_2)$ on Z. Since $(\overset{*}{z_1}, \overset{*}{z_2})$ is efficient, there exists a positive number λ^* such that $(\overset{*}{z_1}, \overset{*}{z_2})$ minimises $z_1 + \lambda^* z_2$ on Z. Our approach is to search for λ^* and find $(\overset{*}{z_1}, \overset{*}{z_2})$ and $x^* \in K$ such $(\overset{*}{z_1}, \overset{*}{z_2}) = (c_1x^*, c_2x^*)$ during the search. In this section also, a point (z_1, z_2) in Z is denoted by z and $g(z_1, z_2)$ by $g(z)$. We denote by $M(\lambda)$ the problem of minimising $z_1 + \lambda z_2$ on Z and denote by $z(\lambda)$ an optimal solution of $M(\lambda)$. Throughout this section we consider the problem $M(\lambda)$ for $\lambda \geqslant 0$ only.

Preliminary Results

LEMMA 9. *$\lambda_1 < \lambda_2$ and $z(\lambda_1) \neq z(\lambda_2) \Rightarrow z_1(\lambda_1) < z_1(\lambda_2)$ and $z_2(\lambda_1) > z_2(\lambda_2)$.*

Proof is similar to that of Lemma 1.

LEMMA 10. *Let $\lambda_1 < \lambda_2$, $z(\lambda_1) \neq z(\lambda_2)$ and $\bar{\lambda} = [z_1(\lambda_2) - z_1(\lambda_1)]/[z_2(\lambda_1) - z_2(\lambda_2)]$ Then $\lambda_1 \leqslant \bar{\lambda} \leqslant \lambda_2$, Further, if $z_1(\bar{\lambda}) + \bar{\lambda} z_2(\bar{\lambda}) < z_1(\lambda_1) + \bar{\lambda} z_2(\lambda_1)$, then $\lambda_1 < \bar{\lambda} < \lambda_2$.* Proof is similar to that of Lemma 2.

Suppose $\lambda_1 < \lambda_2$ and $z(\lambda_1)$ and $z(\lambda_2)$ are optimal solutions of $M(\lambda_1)$ and $M(\lambda_2)$ such that $z(\lambda_1) \neq z(\lambda_2)$. Let $\bar{\lambda} = [z_1(\lambda_2) - z_1(\lambda_1)]/[z_2(\lambda_1) - z_2(\lambda_2)]$ and h be the point of intersection of the lines $L_1 : z_1 + \lambda_1 z_2 = z_1(\lambda_1) + \lambda_1 z_2(\lambda_1)$ and $L_2 : z_1 + \lambda_2 z_2 = z_1(\lambda_2) + \lambda_2 z_2(\lambda_2)$. Let $\bar{c} = z_1(\lambda_1) + \bar{\lambda} z_2(\lambda_1)$. Note that the line $z_1 + \bar{\lambda} z_2 = \bar{c}$ intersects L_1 at $z(\lambda_1)$ and L_2 at $z(\lambda_2)$. If $\lambda_1 < \bar{\lambda} < \lambda_2$, the region $\{z : z_1 + \bar{\lambda} z_2 \leqslant \bar{c},\ z_1 + \lambda_1 z_2 \geqslant z_1(\lambda_1) + \lambda_1 z_2(\lambda_1)$ and $z_1 + \lambda_2 z_2 \geqslant z_1(\lambda_2) + (\lambda_2 z_2(\lambda_2)\}$

is a triangle formed by the points $z(\lambda_1)$, $z(\lambda_2)$ and h. Let us denote this triangle by $\Delta(\lambda_1, \lambda_2)$. This triangle is shown in Fig. 1 as shaded region. The piece-wise linear curve joining $z(\lambda_1)$ and $z(\lambda_2)$ in the figure is part of the boundary of Z.

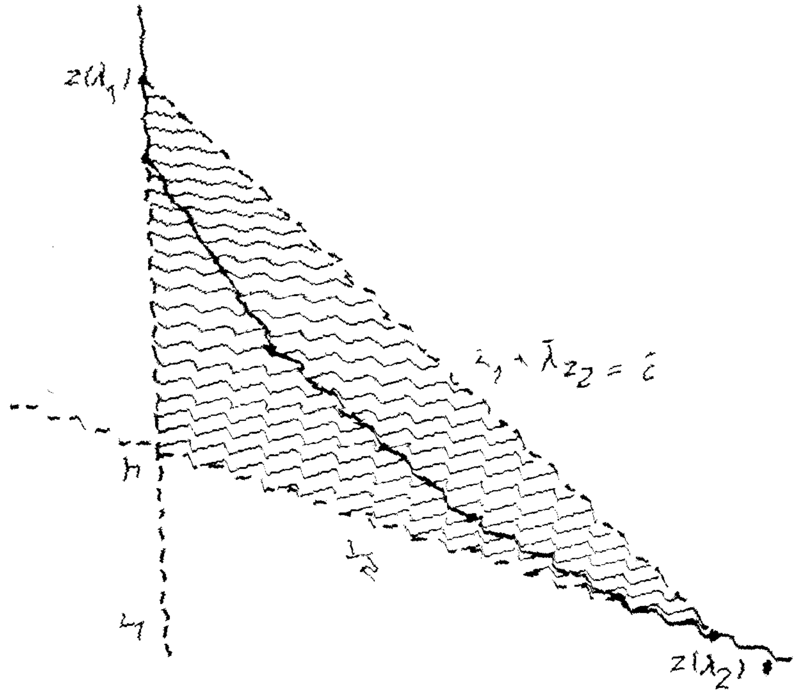

Fig. 1. Triangle $\Delta(\lambda_1, \lambda_2)$

LEMMA 11. $g(z(\lambda)) \geqslant \min \{g(z(\lambda_1)), g(z(\lambda_2)), g(h)\}$ for $\lambda_1 \leqslant \lambda \leqslant \lambda_2$.

Proof. By quasi-concave property of g, it is enough to show tha $z(\lambda) \in \Delta(\lambda_1, \lambda_2)$ for $\lambda_1 < \lambda < \lambda_2$. It is obvious that

$$z_1(\lambda) + \lambda_1 z_2(\lambda) \geqslant z_1(\lambda_1) + \lambda_1 z_2(\lambda_1),$$

and

$$z_1(\lambda) + \lambda_2 z_2(\lambda) \geqslant z_1(\lambda_2) + \lambda_2 z_2(\lambda_2) \quad \text{for } \lambda \in [0, \infty].$$

Suppose $z_1(\lambda) + \tilde{\lambda} z_2(\lambda) > \tilde{c}$ for some λ, $\lambda_1 < \lambda \leqslant \tilde{\lambda}$ and some optimal soluti $z(\lambda)$ of $M(\lambda)$, that is, $[z_1(\lambda) - z_1(\lambda_1)]/[z_2(\lambda_1) - z_2(\lambda)] > \tilde{\lambda} \geqslant \lambda$, i.e., $z_1(\lambda) + \lambda z_2(\lambda$ $z_1(\lambda_1) + \lambda z_2(\lambda_1)$ which contradicts the optimality of $z(\lambda)$ for $M(\lambda)$. T $z_1(\lambda) + \tilde{\lambda} z_2(\lambda) \leqslant \tilde{c}$ for $\lambda_1 < \lambda \leqslant \tilde{\lambda}$. Similarly, $z_1(\lambda) + \tilde{\lambda} z_2(\lambda) \leqslant \tilde{c}$ for $\tilde{\lambda} \leqslant \lambda <$ Therefore $z(\lambda) \in \Delta(\lambda_1, \lambda_2)$ for $\lambda_1 < \lambda < \lambda_2$.

LEMMA 12. *Let* $\lambda_1 < \bar{\lambda} < \lambda_2$. *Then*

(*i*) $g(\tilde{z}) \geqslant \min\{g(z(\lambda_1)), g(z(\lambda_2)), g(h)\}$ *for any optimal solution* $\tilde{z}$ *of* $M(\lambda_1)$ *satisfying* $\tilde{z}_1 > z_1(\lambda_1)$ *and* $\tilde{z}_2 < z_2(\lambda_1)$

(*ii*) $g(\tilde{z}) \geqslant \min\{g(z(\lambda_1)), g(z(\lambda_2)), g(h)\}$ *for any optimal solution* $\tilde{z}$ *of* $M(\lambda_2)$ *satisfying* $\tilde{z}_1 < z_1(\lambda_2)$ *and* $\tilde{z}_2 > z_2(\lambda_2)$.

Proof.

(*i*) We have $[\tilde{z}_1 - z_1(\lambda_1)]/[z_2(\lambda_1) - \tilde{z}_2] = \lambda_1 \leqslant \bar{\lambda}$,

i.e., $\tilde{z}_1 + \bar{\lambda}\tilde{z}_2 \leqslant \bar{c}$, which implies $\tilde{z} \in \Delta(\lambda_1, \lambda_2)$. Now, statement (*i*) follows from quasi-concave property of g.

(*ii*) also can be proved on the same lines.

LEMMA 13. *Suppose* $z(\lambda_1)$ *and* $z(\lambda_2)$ *lie on the line* $z_1 + \bar{\lambda} z_2 = z_1(\bar{\lambda}) + \bar{\lambda} z_2(\bar{\lambda})$. *Then* $z(\lambda_1)$ *is the unique optimal solution of* $M(\lambda)$ for $\lambda_1 < \lambda < \bar{\lambda}$ and $z(\lambda_2)$ *is the unique optimal solution of* $M(\lambda)$ *for* $\bar{\lambda} < \lambda < \lambda_2$.

This lemma can be proved on the same lines as Lemma 6 noting that $z(\lambda_1)$, $z(\bar{\lambda})$ and $z(\lambda_2)$ of this lemma minimise $z_1 + \lambda_1 z_2$, $z_1 + \bar{\lambda} z_2$ and $z_1 + \lambda_2 z_2$ on Z, respectively.

The following algorithm developed on the basis of above results yields optimal solutions of problems P_3 and P_4 simultaneously. The algorithm initially obtains $z^{(1)}$ and $z^{(2)}$ which minimise z_1 and z_2 on Z, respectively. If $z_1^{(1)} = z_1^{(2)}$ or $z_2^{(1)} = z_2^{(2)}$, then the algorithm stops giving one of $z^{(1)}$ and $z^{(2)}$ as an optimal solution of problem P_4. Otherwise the algorithm performs several iterations giving an efficient point $z^{(r+2)}$ in rth ($r \geqslant 1$) iteration.

In rth ($r \geqslant 1$) iteration, the algorthm starts with a collection W of pairs of indices. An element (i,j) in W represents the pair of efficient points $z^{(i)}$ and $z^{(j)}$. The algorithm selects a pair (i,j) in W and obtains, using $z^{(i)}$ and $z^{(j)}$, a new efficient point $z^{(r+2)}$ which results in the inclusion of pairs $(i, r+2)$ and $(r+2, j)$ in W and elimination of (i,j) from W. A point $x^{(r+2)}$ in K that corresponds to $z^{(r+2)}$ is simultaneously obtained along with $z^{(r+2)}$. Point z^* given in this iteration is a point in $\{z^{(1)}, \ldots z^{(r+2)}\}$

such that $g(z^*) = \min_{1 \leqslant l \leqslant r+2} g(z^{(l)})$. Some of the elements is W are eliminated by a criterion involving the value of g at this z^*. The algorithm stops when $W=\phi$. z^* of final iteration is an optimal solution of problem P_4 and the corresponding point x^* in K is optimal for problem P_3.

Algorithm 3

Step 0: Obtain $x^{(1)}$ $(x^{(2)})$ that minimises $c_1x(c_2x)$ on K and set $z^{(1)}=(c_1x^{(1)}, c_2x^{(1)})$ and $z^{(2)}=(c_1x^{(2)}, c_2 x^{(2)})$. Take $\lambda_1=0$ and $\lambda_2=\infty$ and, as a convention, take $0.z_1+z_2$ as the objective function of $M(\infty)$. Then $z^{(1)}$ and $z^{(2)}$ are optimal solutions of $M(\lambda_1)$ and $M(\lambda_2)$, respectively. If $z_1^{(1)} = z_1^{(2)}$, then $z^{(2)}$ and $x^{(2)}$ are optimal solutions of the problems P_4 and P_3. If $z_2^{(1)} = z_2^{(2)}$, then $z^{(1)}$ and $x^{(1)}$ are optimal solutions of P_4 and P_3. If $z^{(1)}$ and $z^{(2)}$ do not coincide in any coordinate, set $W=\{(1,2)\}$. $r=2$. $d_1=z_1^{(1)}$ and $d_2=z_2^{(2)}$.

Step 1: If $g(z^{(1)})\leqslant g(z^{(2)})$, set $z^*=z^{(1)}$, $q^* = g(z^{(1)})$ and $x^*=x^{(1)}$. Otherwise set $z^*=z^{(2)}$, $q^*=g(z^{(2)})$ and $x^*=x^{(2)}$.

Step 2: Choose any (i,j) from W. Set $W = W\diagdown\{(i,j)\}$ and $r=r+1$. Evaluate $\lambda_r = [z_1^{(j)} - z_1^{(i)}]/[z_2^{(i)} - z_2^{(j)}]$ and set $d=z_1^{(i)} + \lambda_r z_2^{(i)}$ and find $x^{(r)}$ that minimises $(c_1+\lambda_r c_2)x$ on K. Set $z^{(r)}=(c_1x^{(r)}, c_2x^{(r)})$. Then $z^{(r)}$ is an optimal solution of $M(\lambda_r)$. Set $d_r=z_1^{(r)} + \lambda_r z_2^{(r)}$. If $d_r=d$, go to Step 7. Otherwise find the solution $(h_1^{(i,r)}, h_2^{(i,r)})$ of $z_1+\lambda_i z_2=d_i$ and $z_1+\lambda_r z_2= d_r$ and the solution $(h_1^{(r,j)}, h_2^{(r,j)})$ of $z_1+\lambda_r z_2=d_r$ and $z_1+\lambda_j z_2=d_j$. Set $q(i,r)=g\,(h_1^{(i,r)}, h_2^{(i,r)})$ and $q(r,j)=g\,(h_1^{(r,j)}, h_2^{(r,j)})$. If $g(z^{(r)})\geqslant q^*$, go to Step 4.

Step 3: Set $z^*=z^{(r)}$, $x^*=x^{(r)}$, $q^*=g(z^{(r)})$.

Step 4: If $q(i,r)<q^*$, set $W=W \cup \{i,r)\}$.

Step 5: If $q(r,j)<q^*$, set $W=W\cup\{(r,j)\}$.

Step 6: Delete from W each (u,v) if $q(u,v) \geqslant q^*$.

Step 7: If $W\neq\phi$, go to Step 1. Otherwise stop. z^* and x^* are optimal solutions of the problems P_4 and P_3.

Validity of Algorithm

Consider an element (i, j) of W. Let

$$E_{ij} = \{ z : z \text{ is optimal for } M(\lambda) \text{ for some } \lambda,\ \lambda_i < \lambda < \lambda_j\}$$
$$\cup \{z : z \text{ is optimal for } M(\lambda_i) \text{ and } z_1 > z_1^{(i)},\ z_2 < z_2^{(i)}\}$$
$$\cup \{z : z \text{ is optimal for } M(\lambda_j) \text{ and } z_1 < z_1^{(j)},\ z_2 > z_2^{(j)}\}.$$

Let L_{ij} be the line segment joining $z^{(i)}$ and $z^{(j)}$. Defining E_{ij} as L_{ij} without end points $z^{(i)}$ and $z^{(j)}$ when $\lambda_i = \lambda_j$, we can write

$$E_{ij} = E_{ir} \cup E_{rj} \cup \{z^{(r)}\} \tag{4}$$

for an optimal solution $z^{(r)}$ of $M(\lambda_r)$ when $\lambda_i \leqslant \lambda_r \leqslant \lambda_j$.

LEMMA 14. *$E_{ij} \subseteq L_{ij}$ if $z_1(\bar{\lambda}) + \bar{\lambda} z_2(\bar{\lambda}) = z_1^{(i)} + \bar{\lambda} z_2^{(i)}$ where $\bar{\lambda} = [z_1^{(j)} - z_1^{(i)}] / [z_2^{(i)} - z_2^{(j)}]$.*

Proof. Case (i): $\lambda_i = \bar{\lambda} < \lambda_j$

The equality $z_1(\bar{\lambda}) + \bar{\lambda} z_2(\bar{\lambda}) = z_1^{(i)} + \bar{\lambda} z_2^{(i)}$ implies that $z^{(i)}$, $z(\bar{\lambda})$ and $z^{(j)}$ optimal for $M(\bar{\lambda})$ and lie on the line $z_1 + \bar{\lambda} z_2 = z_1(\bar{\lambda}) + \bar{\lambda} z_2(\bar{\lambda})$. In this case, we have $\{z : z$ is optimal for $M(\lambda)$ for some λ, $\lambda_i < \lambda < \lambda_j\} = z^{(j)}$ by Lemma 12. We also have $\{z : z$ is optimal for $M(\lambda_j)$ and $z_1 < z_1^{(j)}$, $z_2 > z_2^{(j)}\} = \phi$. Otherwise, we arrive at a contradiction that $z^{(j)}$ is not optimal for $M(\bar{\lambda})$. The set $\{z : z$ is optimal for $M(\lambda_i)$ and $z_1 > z_1^{(i)}$, $z_2 < z_2^{(i)}\}$ contains $z^{(j)}$ due to Lemma 9 and since $\lambda_i = \bar{\lambda}$ and $z^{(j)}$ is optimal for $M(\bar{\lambda})$. We claim that there does not exist an optimal solution $\hat{z}$ of $M(\lambda_i)$ such that $\hat{z}_1 > z_1^{(j)}$ and $\hat{z}_2 < z_2^{(j)}$. Otherwise we arrive at a contradiction that $z^{(j)}$ is not optimal for $M(\lambda_j)$. Therefore, the set $\{z : z$ is optimal for $M(\lambda_i)$ and $z_1 > z_1^{(i)}, z_2 < z_2^{(i)}\}$ is contained in L_{ij} and consequently $E_{ij} \subset L_{ij}$.

Case (ii): $\lambda_i < \bar{\lambda} = \lambda_j$.

This case can be verified on the same lines as case (*i*).

Case (iii): $\lambda_i < \bar{\lambda} < \lambda_j$.

We can easily see, following Lemma 13 and the arguments of case (*i*), that

$\{z : z$ is optimal for $M(\lambda)$ for some λ, $\lambda_i < \lambda < \lambda_j\} = \{z^{(i)}, z^{(j)}\} \cup \{z : z$ is optimal for $M(\bar{\lambda})\}$, (5)

$\{z : z$ is optimal for $M(\lambda_i)$ and $z_1 > z_1^{(i)}$, $z_2 < z_2^{(i)}\} = \phi$, (6)

and $\{z : z$ is optimal for $M(\lambda_j)$ and $z_1 < z_1^{(j)}$, $z_2 > z_2^{(j)}\} = \phi$. (7)

We also have

$\{z : z$ is optimal for $M(\bar{\lambda})\} = L_{ij}$.

Otherwise, we arrive at a contradication that either $z^{(i)}$ is not optimal for $M(\lambda_i)$ or $z^{(j)}$ is not optimal for $M(\lambda_j)$. From Equations(5)-(8) we can now conclude $E_{ij} = L_{ij}$.

THEOREM 2. *At the end of each iteration of the algorithm, the optimal solution of the problem* P_4 *belongs to* $\bigcup_{(i,j)\in W} E_{ij} \cup \{z^*\}$

Proof. We prove the theorem by induction on the number of iterations Suppose the theorem is true for l th iteration. Let $W^{(l)}$ and $W^{(l+1)}$ represent W in Step 7 of l th and $(l+1)$ th iterations. Assume that $(i,j) \in W^{(l)}$ is chosen in Step 2 of $(l+1)$ th iteration and suppose $d_r = d$ in that step. We have $E_{ij} \subseteq L_{ij}$ by Lemma 14 and $g(z) \geqslant g(z^*)$ for $z \in L_{ij}$ by quasi-concave of g. Thus $g(z) \geqslant g(z^*)$ for $z \in E_{ij}$ and the optimal solution of P_4 belongs to $\bigcup_{(i,j)\in W^{(l+1)}} E_{ij} \cup \{z^*\}$ since $W^{(l+1)} = W^{(l)} \setminus \{(i\ j)\}$ when $d_r = d$.

Suppose $d_r < d$. Consider the case $q(i,r) \geqslant q^*$ and $q(r,j) \geqslant q^*$. We have $g(z^{(r)}) \geqslant g(z^*)$ for the revised z^* and due to Lemmas 11 and 12, $g(z) \geqslant g(z^*)$ for $z \in E_{ir} \cup E_{rj}$, that is, $g(z) \geqslant g(z^*)$ for $z \in E_{ij}$ by Equation (4). We also have $g(z) \geqslant g(z^*)$ for $z \in E_{uv}$ if $(u,v) \in W^{(l)} \setminus \{(i,j)\}$ and $q(u,v) \geqslant q^*$. Now, an optimal solution of P_4 belongs to $\bigcup_{(u,v)\in W^{(l+1)}} E_{uv} \cup \{z^*\}$ by induction hypothesis since $W^{(l+1)} = (W^{(l)} \setminus \{(i,j)\}) \setminus \bigcup_{\substack{(u,v)\in W^{(l)} \\ q(u,v)\geqslant q^*}} \{(u,v)\}$. Similarly other three cases can also be verified.

Following the above arguments, one can easily see that the theorem holds for the first iteration also.

Theorem 2 implies that final z^* of the Algorithm 3 is optimal for the problem P_4 since final W is empty. Thus final x^* that satisfies $(c_1x^*, c_2x^*) = z^*$ (final) is an optimal solution of the problem P_3. In Step 2, any LP technique can be used to find $x^{(r)}$ that minimises $(c_1 + \lambda_r c_2)x$ on K.

Discrete Case

Consider the problem P_3 with x restricted to be an integral vector. Algorithm 3 yields an optimal solution for this discrete case also if A is totally unimodular and b is an integral vector provided that we use in Step 2 simplex method or any LP technique that yields an extreme point of K as an optimal solution. This is because all extreme points of K are integral when A is totally unimodular and b is an integral vector and x^* of Algorithm 3 is always an integral vector if we use in Step 2 any LP technique as mentioned above.

Special Case: Maximisation of Reliability of Parallel-Series System.

Consider a parallel-series reliability system consisting of two series systems S_1 and S_2 in parallel. Suppose $S_1(S_2)$ consists of $n_1(n_2)$ positions and there are $n(=n_1+n_2)$ components anyone of which can be assigned to any position. Assume that reliability of component j is p_{ij} if it is assigned to position i, that is, reliability of a component depends also on the position in which it is fixed. Now the problem is how to assign n components to n positions of the system in order to maximise the system reliability. Denote the positions of S_1 by $1,2,\ldots,n_1$ and those of S_2 by $n_1+1,\ldots,n$. We represent an assignment by an $n^2\times 1$ vector $x=(x_{11},\ldots,x_{1n},x_{21},\ldots,x_{2n},\ldots,x_{n1},\ldots,x_{nn})^T$ where $x_{ij}=1$ if component j is assigned to position i and zero otherwise. For an assignment x, the system reliability can be written as

$$R_p(x)=1-[1-\prod_{i=1}^{n_1}\prod_{j=1}^{n} p_{ij}^{x_{ij}}]\,[1-\prod_{i=n_1+1}^{n}\prod_{j=1}^{n} p_{ij}^{x_{ij}}].$$

In other words,

$$R_p(x) = 1-[1-\exp(-a_1x)][1-\exp(-a_2x)],$$

where

$$a_1 =(a_{11},\ldots,a_{1n},a_{21},\ldots,a_{2n},\ldots a_{n_1 1},\ldots,a_{n_1 n},0,\ldots,0)$$

and $a_2 = (0,\ldots,0,\ a_{(n_1+1)1}\ \ldots,a_{(n_1+1)n}\ ,\ldots,a_{n1},\ldots,a_{nn})$

are two $1\times n$ vectors

and $a_{ij} =-\log p_{ij}$.

The problem (R_p) of maximissing $R_p(x)$ over the set of all assignments can be posed as

Minimise $\quad g(c_1x,c_2x)$,

subject to $\quad x\in K$ and x being integral, where $g(c_1x, c_2x) = [1-\exp(-c_1x)]\,[1-\exp(-c_2x)]$, $c_1 = a_1$, $c_2 = a_2$,

and $K = \{x : \sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1 \text{ to } n,\ \sum_{i=1}^{n} x_{ij} = 1 \text{ for } j = 1 \text{ to } n \text{ and } x_{ij} \geqslant 0\}$.

Since $g(z_1, z_2)$ is quasi-concave on R^2_+ by Lemma 8 and the constraint matrix A is totally unimodular in this case, Algorithm 3 yields an optimal solution of the problem R_p. Any assignment technique can be used to solve the LP in Step 2 of the algorithm.

4. Discussion

Algorithm 1 of Section 2 yields an optimal solution of problem P_1. This algorithm is based on quasi-concave property and monotonic property (with respect to each argument) of $g(z_1, z_2)$. Geoffrion [6] also gave an algorithm to solve the problem P_1. However, he exploited the two properties of g in a different manner. He showed that as we move from one end of efficient frontier F(the set of efficient points) of Z to the other end, $g(z_1, z_2)$ is non-decreasing up to some point and non-increasing from that point onwards. Geoffrion's algorithm starts at one end of F and moves along F by parametric programming technique until it reaches an edge of Z that contains optimal z^*. Our algorithm divides in each iteration a part of F containing optimal z^* (selected in previous iteration) into two parts and select the one that contains z^*. This is done by solving an LP problem. If the algorithm ends in Step 4, it maximises $g(z_1, z_2)$ on the edge of Z that contains z^*. If it ends in Step 3, it directly gives z^*.

If the simplex method is used to solve LP of each iteration, the optimal basic feasible solution of LP in an iteration can be used as initial basic feasible solution of LP in the next iteration of the algorithm. We can see that no basic feasible solution is visited more than once throughout the algorithm. Since our algorithm is based on bisection approach, we claim that our algorithm would perform better than that of Geoffrion [6].

Algorithm 3 obtains optimal solutions of problems P_3 and P_4. It implicitly enumerates all efficient extreme points of K and finds an optimal one. The algorithm yields efficient frontier, if the following modifications are done:

(1) Ignore the points $h^{(i,r)}$ and $h^{(r,j)}$ of Step 2.

(2) In Step 3, update W also as $W = W \cup \{(i,r), (r, j)\}$

(3) Delete Steps 4,5, and 6.

(4) Store $z^{(r)}$ of each iteration including $z^{(1)}$ and $z^{(2)}$. Eliminate $z^{(1)}(z^{(2)})$ if there exists a $z^{(r)}$ such that $z^{(r)} \leqslant z^{(1)}(z^{(r)} \leqslant z^{(2)})$. Arrange all the remaining $z^{(r)}$'s in the increasing order of z_1 coordinate and take convex linear combinations of each pair of successive points in that order.

To generate all efficient points of Z of Section 2, we can further modify the Algorithm 3 by replacing minimisation by maximisation and eliminating initial $z^{(1)}$ and $z^{(2)}$ when they are not efficient.

An important advantage of Algorithm 3 is that it gives an optimal solution of problem P_3 even if x is restricted to be integral provided that the matrix A is totally unimodular and b is integral. We have taken this advantage to solve the problem of assigning components optimally to a series-parallel reliability system so as to maximise the system reliability.

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APPENDIX

Proof of Lemma 8

We prove the lemma by showing that for any two distinct arbitrary points (x_1,y_1) and (x_2,y_2) in R^2_+,

$$f(x_1+\lambda\bar{x},y_1+\lambda\bar{y}) \geqslant \min\{f(x_1,y_1), f(x_2,y_2)\},$$

for $0 \leqslant \lambda \leqslant 1$ where $\bar{x}=x_2-x_1$ and $\bar{y}=y_2-y_1$. Suppose

$$\min_{0\leqslant\lambda\leqslant 1} f(x_1+\lambda\bar{x}, y_1+\lambda\bar{y}) = f(x_1+\lambda^o\bar{x},y_1+\lambda^o\bar{y}) < \min\{f(x_1,y_1), f(x_2,y_2)\}, \quad (9)$$

for some λ^o, $0<\lambda^o<1$. Let $(x^o,y^o)=(x_1,y_1)+\lambda^o(\bar{x},\bar{y})$ and

$$h(\lambda) = f(x_1+(\lambda^o+\lambda)\bar{x}, y_1+(\lambda^o+\lambda)\bar{y}),$$

for $\lambda \in [-\lambda^o, 1-\lambda^o]$. We can write

$$h(0) = \min_{-\lambda^o<\lambda\leqslant 1-\lambda^o} h(\lambda) < \min\{h(-\lambda^o), h(1-\lambda^o)\}, \quad (10)$$

we have

$$h'(0)=\gamma(1-\delta)\bar{x}+\delta(1-\gamma)\bar{y},$$

and

$$h''(0)=-\gamma\bar{x}^2-\delta\bar{y}^2+\gamma\delta(\bar{x}+\bar{y})^2,$$

where $\gamma = \exp(-x^o)$ and $\delta = \exp(-y^o)$.

We know that $(x^o,y^o) \geqslant (0,0)$. Assume $(x^o,y^o)>(0,0)$. Then $0<\gamma<1$ and $0<\delta<1$. Since $h(\lambda)$ and $h'(\lambda)$ are continuous, equation (10) implies $h'(0)=0$ which in turn implies one of $\bar{x}$ and $\bar{y}$ is positive and the other is negative. Now one can easily see that

$$\gamma\delta(\bar{x}+\bar{y})^2 < \max(\gamma\bar{x}^2, \delta\bar{y}^2)$$

i.e., $h''(0)<0$. This contradicts the equality in (10). If one of x^o and y^o is 0 $h'(0)\neq 0$ which also contradicts the equality in (10). Therefore there cannot exist $\lambda^o, 0<\lambda^o<1$ such that (9) holds and hence the lemma.