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Fuzzy tools for the management of uncertainty in pattern recognition, image analysis, vision and expert systems

SANKAR K. PAL†

The paper presents different fuzzy tools that are useful for decision-taking in pattern recognition, image processing and vision problems, and in designing expert systems when the patterns are ill-defined or the input data does not have complete, precise or reliable information. Tools are explained with several examples. Algorithms for feature ranking, clustering evaluation, quantitative indices for image processing, fuzzy enhancement/segmentation of both grey tone image and colour image, the front end compiler, and for representing rules and facts in the knowledge base are also demonstrated with various examples of real-life problems. Some of the illustrations are taken from the existing literature.

1. Introduction

Machine recognition of patterns can be viewed as a two-fold task, consisting of learning the invariant and common properties of a set of samples characterizing a class, and of deciding a new sample as a possible member of the class by noting that it has properties common to those of the set of samples. In other words, the pattern recognition by computers can be described as a transformation from the measurement space M to the feature space F and finally to the decision space D (Duda and Hart 1973), i.e.

$$M \to F \to D$$

The mapping $\delta: F \to D$ is called the decision function. The elements $d \in D$ are called decisions. A loss function, depending on the decision d and the other probability of distribution on F, may be introduced in order to determine a preference for a certain decision $d_1 (\in D)$ over others. Minimization of the expected value of this loss (called 'risk') is the criterion for taking a decision.

When the input pattern is an image, the measurement space involves processing tasks such as enhancement, filtering, contour extraction and noise reduction, in order to extract salient features from the pattern. This is what is called image processing (Rosenfeld and Kak 1982, Pratt 1978). The ultimate aim of this is the understanding, recognition and interpretation from the processed information available from the image pattern. For example, the recognition of shape of a 3-D object pattern may be done from the contour extracted from its 2-D image, or from its shade, or from its motion. Similarly, a contour can correspond to the scene to a depth discontinuity, a surface orientation discontinuity, a reflectance discontinuity, an illumination discontinuity, or shadow. The task of understanding the scene in this context, i.e. the recovery of scene characteristics, comes under the heading 'computer vision' (Horn 1986).

Arficial intelligence is the field that aims to understand how computers can be made to exhibit intelligence in different aspects of thinking, reasoning, perception or

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† Electronics and Communication Sciences Unit, Indian Statistical Institute, Calcutta 700 035, India.

action in some limited sense. In other words, it involves the study of mental faculties using computational models (Charniak and McDermott 1985). A key contribution of AI is the observation that knowledge should be represented explicitly and not be heavily encoded, such as numerically, in ways that suppress structure and constraint. AI has developed a set of techniques such as the semantic network, frames and production rules, that are symbolic, highly flexible encoding of knowledge, but yet can be efficiently processed.

An expert system can be viewed as a rule-based AI application program which provides the user with the facility for posing and obtaining answers (which requires expertise) to questions relating to the information stored in its knowledge base (Hayes *et al.* 1983, Waterman 1985, Kohout and Bandler 1986). Typically, such systems possess a non-trivial inferential capability, and in particular have the capability to infer from premises which are imprecise, incomplete or not totally reliable.

Since Zadeh published his classic paper (Zadeh 1965), fuzzy set theory has been receiving more and more attention from researchers in a wide range of scientific areas, the most important of which is decision-making modes under different kinds of risk, uncertainty and ambiguity. Although we use the probabilistic theory of decision and estimation to design automatic decision-making systems under risk and uncertainty, it is felt that there exist some qualitatively different kinds of uncertainty—such as ill-definedness, vagueness and ambiguity, which have come to be known as fuzziness —which are not covered by statistical theories. Many situations are found in pattern recognition (PR) problems where the notion of probability alone is not adequate to describe the reality.

The application of fuzzy set theory in the problems of pattern recognition is found in various places (Zadeh *et al.* 1975, Kickert 1978, Gupta *et al.* 1989, Wang and Chang 1980, Dubois and Prade 1980, Bezdek 1981, Kandel 1982, 1986, Yager 1982, Gupta and Sanchez 1982, Pal and Dutta Majumder 1986). In some cases the performances of the algorithms are compared with those of conventional approaches. Pal and Dutta Majumder (1986) did not consider the fuzzy approaches as always a competitor to the statistical and syntactic approaches; rather, they considered this approach to be more general and a very useful supplement to the classical (syntactic and statistical) approaches, depending on the nature of the problems.

Similarly, a grey tone including a colour image possesses some ambiguity within the pixels, owing to the possible multivalued levels of brightness, and it is therefore justified to apply the concept and logic of a fuzzy set, rather than ordinary set theory, to image processing and vision problem. Keeping this in mind, an image can be considered as an array of fuzzy singletons, each with a value of membership function denoting the degree of having some property, say brightness, smoothness, edginess, semibrightness, or the degree of possessing some colour property. Some simple but effective pre-processing algorithms such as enhancement, edge extraction, primitive extraction, segmentation and coding were given by Pal and Dutta Majumder (1986), Pal and King (1981 a, b, 1983), Pal (1982 a, 1986), Pal *et al.* (1983 a, b), Nasrabadi *et al.* (1983).

Since the knowledge base of an expert system is a repository of human knowledge, and since much of human knowledge is imprecise in nature, it is usually the case that the knowledge base of an expert system is a collection of rules and facts which for the most part are neither totally certain nor totally consistent. The uncertainty of information in the knowledge base of any question-answering system thus induces some uncertainty in the validity of its conclusions (Negoita 1985, Gupta *et al.* 1985, Hart

1986). Therefore the answer to a question must be associated explicitly, or at least implicitly, with an assessment of its reliability. For this reason, a basic problem in the design of expert systems is how to analyse the transmission of uncertainty from the premises to the conclusion, and associate the conclusion with what is commonly called a certainty factor (Zadeh 1983).

In the existing systems, uncertainty is dealt with through a combination of predicate logic and probability-based methods. A serious shortcoming of these methods is that they are not capable of coming to grips with the pervasive fuzziness of information in the knowledge base, and as a result are mostly *ad hoc* in nature (Zadeh 1983). An alternative approach is suggested by Zadeh based on the logic of fuzzy sets. Details regarding the management of uncertainty in expert systems such as MYCIN (Shortcliffe 1976) and PROSPECTOR (Duda *et al.* 1979) were discussed by Zadeh (1983).

The present work discusses some fuzzy tools and their applications for the management of uncertainty (indeterminacy) in problems of pattern recognition, image processing and vision, and expert systems. The problems discussed here are:

(i) to provide quantitative measures for processed images;
(ii) to take decisions regarding the selection of thresholds, or the segmentation of an image into a meaningful region, without committing ourselves to a specific segmentation when the regions in an image are ill-defined;
(iii) to provide fuzzy transforms for the enhancement and segmentation of colour (including pseudo-colour) images;
(iv) to evaluate the importance of features in the PR problem;
(v) to extract the seed point in clustering a set of data and to give a performance measure for partitioning;
(vi) to equip an expert system with the computational capability to analyse the transmission of uncertainty in information from the knowledge base to the uncertainty in the validity of its conclusions.

The fuzzy tools considered here are measures of fuzziness—such as the index of fuzziness (Kaufmann 1975), entropy (De Luca and Termini 1972), index of non-fuzziness (Pal 1986), π-ness (Pal 1982 b) and dispersion (De Luca 1985), correlation between membership functions (Murthy 1985), fuzzy geometry (such as area, perimeter and compactness) (Rosenfeld 1984), fuzzy expected value and fuzzy expected intervals (Schneider and Kandel 1991, Kandel and Byatt 1978). The fuzzy measures are optimized in some of the PR and IP problems in order to take decisions for ill-defined patterns.

Some new operators (Murthy *et al.* 1987) for union, intersection and inclusion have also been mentioned. Fuzzy transforms for colour image processing and graphics are defined using the operator 'bounded difference' (Dubois and Prade 1980, Pal and Dutta Majumder 1986, Zadeh *et al.* 1975). Fuzzy expected value and intervals are used in designing expert systems when the input data do not have complete or precise information. The use of fuzzy and fractionally fuzzy grammars (Pathack and Pal 1986) for developing production rules is also discussed.

Various examples from real-life problems are taken into account in order to explain both the tools and the suitability of the algorithms developed.

2. Fuzzy sets

2.1. *Definitions*

A fuzzy set A in the universe of discourse $X = \{x\}$ is defined by its membership function $\mu_A(x)$, which assigns to each element $x \in X$ a real number in the interval

[0, 1]. The value of $\mu_A(x)$ represents the grade of membership of x in A. In other words, a fuzzy set A on X is denoted by its membership function

$$\mu_A : X \to [0, 1] \quad \text{or} \quad \mu_A(x), \quad \forall x \in X$$

The definitions of union, intersection and complementation originally proposed by Zadeh (1965) and used since then are as follows. Union of two fuzzy sets A and B in X:

$$\Rightarrow \mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \quad \forall x \in X \tag{1}$$

Intersection of two fuzzy sets A and B in X:

$$\Rightarrow \mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \quad \forall x \in X \tag{2}$$

Complementation of A in X:

$$\Rightarrow \mu_{\bar{A}}(x) = 1 - \mu_A(x), \quad \forall x \in X \tag{3}$$

The mathematical foundation to these ideas was given by Bellman and Giertz (1973) and Fung and Fu (1975).

There are also other definitions, such as bold union and bold intersection (Dubois and Prade 1980), in this regard. These are defined as

$$A \cup B \Rightarrow \mu_{A \cup B}(x) = \min[1, \mu_A(x) + \mu_B(x)], \qquad \forall x \in X \tag{4a}$$

$$A \cap B \Rightarrow \mu_{A \cap B}(x) = \max[0, \mu_A(x) + \mu_B(x) - 1], \quad \forall x \in X \tag{4b}$$

The above definitions of intersection and union are based only on the value of membership functions characterizing fuzzy sets. A different interpretation of these operators was given by Murthy *et al.* (1987) in the light of measure theory. They defined

$$A \cup B \Rightarrow \mu_{A \cup B}(x) = \lambda(A_x \cup B_x) \tag{5a}$$

$$A \cap B \Rightarrow \mu_{A \cap B}(x) = \lambda(A_x \cap B_x) \tag{5b}$$

where

$$\begin{aligned} A_x &= [0, \mu_A(x)] && \text{if } \mu_A(x) \text{ is non-decreasing at } x \\ &= [1 - \mu_A(x), 1] && \text{if } \mu_A(x) \text{ is non-increasing at } x \\ &= [0, 1] && \text{if } \mu_A(x) = 1 \\ &= \text{any finite set} && \text{if } \mu_A(x) = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} B_x &= [0, \mu_B(x)] && \text{if } g \text{ is non-decreasing at } x \\ &= [1 - \mu_B(x), 1] && \text{if } g \text{ is non-increasing at } x \\ &= [0, 1] && \text{if } \mu_B(x) = 1 \\ &= \text{any finite set} && \text{if } \mu_B(x) = 0 \end{aligned} \tag{7}$$

Therefore

$$\mu_A(x) = \lambda(A_x) \tag{8}$$

$$\mu_B(x) = \lambda(B_x) \tag{9}$$

where λ is the Lebesgue measure on $\mathbb{R}$.

Equations (5)–(9) are found to be generalized in the sense that Zadeh's definitions and bold union and intersection can be derived from them. These also satisfy the properties followed either by equations (1) and (2) or by equations (4).

Example 1

Let $\mu_A(x) = x$ and $\mu_B(x) = 1 - x^2$ then for $x = 0, 0{\cdot}3, 0{\cdot}5, 0{\cdot}8$ and 1, we have

$$A_x = [0, x] = [0, 0], \quad [0, 0{\cdot}3], \quad [0, 0{\cdot}5], \quad [0, 0{\cdot}8], \quad [0, 1]$$

$$B_x = [x^2, 1] = [0, 1], \quad [0{\cdot}09, 1], \quad [0{\cdot}25, 1], \quad [0{\cdot}64, 1], \quad [1, 1]$$

$$\mu_{A \cup B}(x) = \lambda[0, 1] = 1{\cdot}0, \quad 1{\cdot}0, \quad 1{\cdot}0, \quad 1{\cdot}0, \quad 1{\cdot}0$$

$$\mu_{A \cap B}(x) = \lambda[x^2, x] = x - x^2 = 0, \quad 0{\cdot}21, \quad 0{\cdot}25, \quad 0{\cdot}16, \quad 0$$

On the other hand, we have using equations (1) and (2)

$$\mu_{A \cup B}(x) = 1{\cdot}0, \quad 0{\cdot}91, \quad 0{\cdot}75, \quad 0{\cdot}8, \quad 1{\cdot}0$$

$$\mu_{A \cap B}(x) = 0, \quad 0{\cdot}3, \quad 0{\cdot}5, \quad 0{\cdot}36, \quad 0$$

and using equations (4)

$$\mu_{A \cup B}(x) = 1{\cdot}0, \quad 1{\cdot}0, \quad 1{\cdot}0, \quad 1{\cdot}0, \quad 1{\cdot}0$$

$$\mu_{A \cap B}(x) = 0, \quad 0{\cdot}21, \quad 0{\cdot}25, \quad 0{\cdot}16, \quad 0$$

Thus the generalized definition of equations (5)–(8) follows the results obtained by bold union and intersection. Similarly, if $\mu_B(x) = x^2$, then equations (5)–(8) would follow equations (1) and (2).

2.2. *Membership functions when $x \in \mathbb{R}$*

The standard S function is defined (Zadeh *et al.* 1975) as

$$\begin{aligned} S(x; a, b, c) &= 0, & x \leqslant a \\ &= 2\left(\frac{x-a}{c-a}\right)^2, & a \leqslant x \leqslant b \\ &= 1 - 2\left(\frac{x-c}{c-a}\right)^2, & b \leqslant x \leqslant c \\ &= 1, & x \geqslant c \end{aligned} \tag{10}$$

with $b = (a + c)/2$. The standard π function is defined as

$$\begin{aligned} \pi(x; b, c) &= S\left(x; c - b, c - \frac{b}{2}, c\right), & x \leqslant c \\ &= 1 - S\left(x; c, c + \frac{b}{2}, c + b\right), & x \geqslant c \end{aligned} \tag{11}$$

In $S(x; a, b, c)$ b is the cross-over point, i.e. $S(b; a, b, c) = 0{\cdot}5$. In $\pi(x; b, c)$ b is the bandwidth, i.e. the separation between the cross-over points of π-function. c is the central point at which $\pi = 1$.

The S and π functions represent the compatibility functions corresponding to the fuzzy sets 'x is large' and 'x is c', respectively. Besides these standard functions, there are several other forms of these functions (Bezdek 1981, Pal and Dutta Majumder 1986, Kaufmann 1975) as used for practical problems.

2.3. *Membership functions when* $x \in \mathbb{R}^n$

The standard S and π functions when $x \in \mathbb{R}^n$ are defined by Pal and Pramanik (1986) by extending equations (10) and (11). These are as follows:

$$\hat{S}(x; b, \lambda) = \tfrac{1}{2}(1 - \|x - b\|/\lambda)^2$$

or

$$\begin{aligned} \hat{S}(x; b, \lambda) &= 1 - \tfrac{1}{2}(1 - \|x - b\|/\lambda)^2, \quad \|x - b\| \leqslant \lambda \\ &= 0 \quad \text{or} \quad 1, \quad \text{otherwise} \end{aligned} \tag{12}$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$, $\lambda > 0$ is the radius of $\hat{S}(x; b, \lambda)$ and b is the cross-over point. It is to be noted that $\hat{S}(x; b, \lambda)$ is a two-valued (values being complementary) function.

$$\begin{aligned} \hat{\pi}(x; c, \lambda) &= \min \hat{S}\left(x; y, \frac{\lambda}{2}\right), \quad \frac{\lambda}{2} \leqslant \|x - c\| \leqslant \lambda \\ &= \max S\left(x; y, \frac{\lambda}{2}\right), \quad 0 \leqslant \|x - c\| \leqslant \frac{\lambda}{2} \end{aligned} \tag{13}$$

where $\|y - c\| = \lambda/2$, and $\min \hat{S}(x; y, \lambda/2)$ implies the minimum of the two values of the $\hat{S}$ function at the point x. Similarly, $\max \hat{S}(x; y, \lambda/2)$ implies the maximum of the two values of the $\hat{S}$ function at the point x. c is the central point, i.e. $\hat{\pi}(c; c, \lambda) = 1$, and λ is the bandwidth.

On simplification we have

$$\begin{aligned} \hat{\pi}(x; c, \lambda) &= \tfrac{1}{2}(1 - 2\|x - y\|/\lambda)^2, \quad \frac{\lambda}{2} \leqslant \|x - c\| \leqslant \lambda \\ &= 1 - \tfrac{1}{2}(1 - 2\|x - y\|/\lambda)^2, \quad 0 \leqslant \|x - c\| \leqslant \lambda/2 \end{aligned} \tag{14}$$

with $\|y - c\| = \lambda/2$. Considering the euclidean norm

$$\begin{aligned} \|x - y\| &= \|x - c\| - \frac{\lambda}{2} \quad \text{if} \quad \lambda/2 \leqslant \|x - c\| \leqslant \lambda \\ &= \frac{\lambda}{2} - \|x - c\| \quad \text{if} \quad 0 \leqslant \|x - c\| \leqslant \lambda/2 \end{aligned} \tag{15}$$

We can write

$$\begin{aligned} \hat{\pi}(x; c, \lambda) &= 2(1 - \|x - c\|/\lambda)^2, \quad \frac{\lambda}{2} \leqslant \|x - c\| \leqslant \lambda \\ &= 1 - 2\frac{\|x - c\|^2}{\lambda^2}, \quad 0 \leqslant \|x - c\| \leqslant \lambda/2 \end{aligned} \tag{16}$$

$\hat{\pi}(x; c, \lambda)$ represents the compatibility function corresponding to a 'cluster' when $x \in \mathbb{R}^n$.

3. Fuzzy relations and composition

Let $X = \{x\}$ and $Y = \{y\}$ be the two universes of discourse. The cartesian product $X \times Y$ is the collection of ordered pairs (x, y); $x \in X$, $y \in Y$. A binary fuzzy relation R from X to Y is a fuzzy subset on $X \times Y$ and is characterized by a bivariate membership function $\pi_R(x, y) \in [0, 1]$.

An n-ary fuzzy relation is thus a fuzzy subset on $X_1 \times X_2 \times \cdots \times X_n$ and is characterized by a multivariate (n-parameter) membership function

$$\mu_R(x_1, x_2, \ldots, x_n) \in [0, 1]$$
$$x_i \in X_i, \quad i = 1, 2, \ldots, n$$

Typical forms of μ_R are given by equations (12)–(16).

Example 1

Let $X = \{\text{Ram, Shyam}\}$ and $Y = \{\text{Jadu, Madhu}\}$. Then the binary ordinary relation is a subset of the cartesian product of X and Y. The binary fuzzy relation 'similarity' or 'resemblance', say, may be expressed as a matrix

	Jadu	Madhu
Ram	0·9	0·5
Shyam	0·2	0·7

where the (i, j)th element is the value of $\mu_R(x_i, y_j)$, $x_i \in X$, $y_j \in Y$, $i, j = 1, 2$.

It is to be mentioned here that each entry of the similarity matrix may again be determined from the attributes characterizing a human being. Let $p_1, p_2, \ldots, p_N$ be such attributes, each denoting the properties, say, behaviour, intelligence, looking, opinion, etc. Then we may write

$$\mu_R(x_i, y_j) = 1 - \max_n |\mu_n(x_i) - \mu_n(y_j)| \tag{17}$$

or

$$\mu_R(x_i, y_j) = 1 - \frac{1}{N} \sum_n |\mu_n(x_i) - \mu_n(y_j)| \tag{18}$$
$$n = 1, 2, \ldots, N$$

where $\mu_n(\cdot)$ denotes the degree of possessing the nth property.

Similarly, if $p_1, p_2, \ldots, p_N$ denote some properties that characterize the template of an object pattern, then μ_R may be used to represent the degree of equality or consistency between two such templates.

Let R_1 and R_2 be two fuzzy relations from X to Y and from Y to Z. The composed fuzzy relation C from X to Z then written as

$$C = R_1 \circ R_2 \tag{19}$$

and is characterized by the membership function

$$\mu_C(x, z) = \max_y \min \{\mu_{R_1}(x, y), \mu_{R_2}(y, z)\}$$
$$x \in X, \quad y \in Y, \quad z \in Z \tag{20}$$

Considering Example 1 of a relation matrix from X to Y, if we have another fuzzy 'similarity' relation matrix from Y to Z, {Lalu, Bhulu}, as

	Lalu	Bhulu
Jadu	0·3	0·8
Madhu	0·6	0·7

then the composed fuzzy relation denoting the 'similarity' between {Ram, Shyam} and {Lalu, Bhulu} is

$$C = \begin{bmatrix} 0{\cdot}9 & 0{\cdot}5 \\ 0{\cdot}2 & 0{\cdot}7 \end{bmatrix} \circ \begin{bmatrix} 0{\cdot}3 & 0{\cdot}8 \\ 0{\cdot}6 & 0{\cdot}7 \end{bmatrix} = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}8 \\ 0{\cdot}6 & 0{\cdot}7 \end{bmatrix}$$

Example 2

Let us consider the speech recognition problem (Pal and Dutta Majumder 1977) where, say

$$X = \{300, 700: \text{first formant frequencies } F_1\}$$
$$Y = \{1000, 2000: \text{second formant frequencies } F_2\}$$
$$Z = \{2500, 3000: \text{third formant frequencies } F_3\}$$

and let the fuzzy relation matrices for vowel sound $/u/$ from X to Y and Y to Z be

$$R_1 = \begin{array}{c} \\ 300 \\ 700 \end{array} \begin{array}{c} \begin{array}{cc} 1000 & 2000 \end{array} \\ \begin{bmatrix} 0{\cdot}8 & 0{\cdot}05 \\ 0{\cdot}1 & 0{\cdot}05 \end{bmatrix} \end{array}, \quad R_2 = \begin{array}{c} \\ 1000 \\ 2000 \end{array} \begin{array}{c} \begin{array}{cc} 2500 & 3000 \end{array} \\ \begin{bmatrix} 0{\cdot}7 & 0{\cdot}25 \\ 0 & 0 \end{bmatrix} \end{array}$$

Then the composed relation matrix from X to Z is

$$\begin{array}{c} \\ 300 \\ 700 \end{array} \begin{array}{c} \begin{array}{cc} 2500 & 3000 \end{array} \\ \begin{bmatrix} 0{\cdot}7 & 0{\cdot}25 \\ 0{\cdot}1 & 0{\cdot}1 \end{bmatrix} \end{array}$$

In the above example, each entry of the matrix denotes the degree to which the corresponding combination (F_i, F_j), $i, j = 1, 2, \ldots, 3$, $i \neq j$, represents the vowel sound $/u/$.

Some other examples of fuzzy relations are 'much larger/taller/younger than'.

4. Measures of fuzziness

The index of fuzziness reflects the average amount of ambiguity (fuzziness) present in A by measuring its distance (linear and quadratic corresponding to the linear index of fuzziness and the quadratic index of fuzziness) from its nearest ordinary set $\mathscr{A}$. The term 'entropy', on the other hand, uses Shannon's function, but its meaning is quite different from classical entropy because no probabilistic concept is needed to define it. The 'index of non-fuzziness', as its name implies, measures the non-fuzziness (crispness) in A by computing its distance from its complement set. These quantities are defined below.

4.1. *Linear index of fuzziness* (Kaufmann 1975)

$$\begin{aligned} \nu_l(A) &= \frac{2}{n} \sum_i |\mu_A(x_i) - \mu_{\mathscr{A}}(x_i)| \\ &= \frac{2}{n} \sum_i \mu_{A \cap \bar{A}}(x_i) \\ &= \frac{2}{n} \sum_i \min(\mu_A(x_i), \quad 1 - \mu_A(x_i)), \quad i = 1, 2, \ldots, n \end{aligned} \tag{21}$$

where $\mu_{\mathscr{A}}(x_i)$ is defined as

$$\mu_{\mathscr{A}}(x_i) = 0, \quad \text{if } \mu_A(x_i) \leqslant 0{\cdot}5 \\ = 1, \quad \text{otherwise} \tag{22}$$

4.2. *Quadratic index of fuzziness* (Kaufmann 1975)

$$v_q(A) = \frac{2}{\sqrt{n}}\left[\sum_i (\mu_A(x_i) - \mu_{\mathscr{A}}(x_i))^2\right]^{0{\cdot}5}, \quad i = 1,2,\ldots,n \tag{23}$$

4.3. *Entropy* (De Luca and Termini 1972)

$$H(A) = \frac{1}{n\ln 2}\sum_i S_n(\mu_A(x_i)) \tag{24}$$

with

$$S_n(\mu_A(x_i)) = -\mu_A(x_i)\ln\mu_A(x_i) - (1 - \mu_A(x_i))\ln(1 - \mu_A(x_i)), \quad i = 1,2,\ldots,n \tag{25}$$

4.4. *Index of non-fuzziness* (*crispness*) (Pal 1986)

$$\eta(A) = \frac{1}{n}\sum_i |\mu_A(x_i) - \mu_{\bar{A}}(x_i)|, \quad i = 1,2,\ldots,n \tag{26}$$

All these measures lie in [0, 1] and have the following properties:

$$I(A) = 0 \text{ (minimum) for } \mu_A(x) = 0 \text{ or } 1, \quad \forall x \tag{27 a}$$

$$I(A) = 1 \text{ (maximum) for } \mu_A(x) = 0{\cdot}5, \quad \forall x \tag{27 b}$$

$$I(A) \geqslant I(A^*) \tag{27 c}$$

$$I(A) = I(\bar{A}) \tag{27 d}$$

where I stands for $v(A)$, $H(A)$ and $1 - \eta(A)$. A^* is the sharpened or intensified version of A such that

$$\mu_{A^*}(x_i)\begin{cases}\geqslant \mu_A(x_i), & \text{if} \\ \leqslant \mu_A(x_i), & \text{if}\end{cases} \quad \mu_A(x_i)\begin{cases}\geqslant 0{\cdot}5 \\ \leqslant 0{\cdot}5\end{cases} \tag{28}$$

4.5. *π-ness* (Pal 1982 b)

The π-ness of A is defined as

$$\pi(A) = \frac{1}{n}\sum_i G_\pi(x_i), \quad i = 1,2,\ldots,n \tag{29}$$

where G_π is any π function such that $0 \leqslant G_\pi \leqslant 1$ and it increases monotonically in [$x_i = 0$ to $x_i = x_{max}/2$, say] and then decreases monotonically in $[x_{max}/2, x_{max}]$ with a maximum of unity at $x_{max}/2$, where x_{max} denotes the maximum value of x_i.

5. Measures of dispersion (De Luca 1985)

An energy measure E of a fuzzy set A satisfies the following axioms:

$$E(A) = 0 \quad \text{iff} \quad \mu_A(x) = 0, \quad \forall x \in X \tag{30 a}$$

$$\text{if } \mu_A \leqslant \mu_B \quad \text{then} \quad E(A) \leqslant E(B) \tag{30 b}$$

$$E(A) \text{ reaches it maximum} \quad \text{iff} \quad \mu_A(x) = 1, \quad \forall x \in X \tag{30 c}$$

The power (or cardinality) which represents an energy measure is defined as

$$P(A) = \sum_{x \in X} \mu_A(x) = \sum_i \mu_A(x_i), \quad i = 1, 2, \ldots, n \tag{31}$$

where X is a finite set of cardinality n.

The dispersion measure of a fuzzy set gives a measure of the size (cardinality, in cases of finite supports) in which almost all the energy of A is concentrated. Let $\hat{A}$ denote the fuzzy set obtained from A by rearranging its membership values $\mu_A(x_i)$, $i = 1, 2, \ldots, n$, in a non-increasing way. In other words

$$\mu_{\hat{A}}(x) \geqslant \mu_{\hat{A}}(x + 1), \quad 1 \leqslant x \leqslant n - 1 \tag{32}$$

One would obviously have

$$P(\hat{A}) = P(A) = P \tag{33}$$

$$d(\hat{A}) = d(A) \tag{34}$$

where d stands for the measure of fuzziness as described in §4.

The dispersion of A may then be defined as

$$\delta_e(A) = \min\left\{k \in [n] \,\middle|\, \sum_{x \leqslant k} \mu_{\hat{A}}(x) > P - e\right\}$$

$$e > 0, \quad [n] = \{1, 2, \ldots, n\} \tag{35}$$

This definition implies that

$$\sum_{x > \delta_e} \mu_{\hat{A}}(x) < e \tag{36a}$$

$$\sum_{x \geqslant \delta_e} \mu_{\hat{A}}(x) \geqslant e \tag{36b}$$

$$\delta_e(A) = \delta_e(\hat{A}) \tag{36c}$$

If $e = P/n$ so that $0 < e \leqslant 1$, then $\delta(A)$ gives a measure of the minimal cardinality of a subset of the universe X in which an amount of power greater than $P - P/n$ is concentrated.

6. Fuzzy geometry

Rosenfeld (1984) and Rosenfeld and Haber (1985) extended the concept of digital picture geometry to fuzzy subsets and generalized some of the standard geometric properties of the relationships among regions to fuzzy subsets. Among the extensions of the various properties, we discuss only the area, perimeter and compactness of a fuzzy image subset, which may be used for pattern recognition and image processing problems. In defining the above mentioned parameters we replace $\mu_A(x)$ by μ for simplicity.

The area of μ is defined as

$$a(\mu) \triangleq \int \mu \tag{37}$$

where the integral is taken over any region outside which $\mu = 0$.

If μ is piecewise constant (for example, in a digital image) $a(\mu)$ is the weighted sum of the areas of the regions on which μ has constant values, weighted by those values.

For the piecewise constant case, the perimeter of μ is defined as

$$p(\mu) \triangleq \sum_{i,j}\sum_{k} |\mu_i - \mu_j||A_{ijk}|, \quad i,j = 1,2,\ldots,r,\ i<j,\ k = 1,2,\ldots,r_{ij} \tag{38}$$

This is just the weighted sum of the length of the arcs A_{ijk} along which the ith and jth regions with constant μ values μ_i and μ_j, respectively, meet, weighted by the absolute difference of these values.

The compactness of μ is defined as

$$\text{comp}(\mu) \triangleq a(\mu)/p^2(\mu) \tag{39}$$

For the crisp sets, this is largest for a disc, where it is equal to $1/4\pi$. For a fuzzy disc where μ depends only on the distance from the origin (centre), it can be shown that

$$a(\mu)/p^2(\mu) \geqslant \frac{1}{4\pi} \tag{40}$$

In other words, of all possible fuzzy discs, the compactness is smallest for its crisp version.

7. Correlation between membership functions (Murthy *et al.* 1985)

In real-life phenomena we come across many fuzzy subsets, e.g. tall, very tall, short, medium, etc., where if one membership function increases the other also increases or decreases, or vice versa; or else when one membership function takes low values the other also takes low values or high values, and vice versa. A similar phenomenon is studied in statistics and is called correlation. A measure of such relationship was studied by Murthy *et al.* (1985), and is explained here in brief.

Let μ_1 and μ_2 be two membership functions defined on the same domain Ω. Let C_{μ_1,μ_2} represent the correlation between them. The properties which C_{μ_1,μ_2} may possess are given below.

(i) If for high values of $\mu_1(x)$, $\mu_2(x)$ takes high values, and if the converse is also true, then C_{μ_1,μ_2} must be very high.

(ii) If Q is a subset of $\mathbb{R}$ and

(*a*) $x\uparrow \Rightarrow \mu_1(x)\uparrow$ and $\mu_2(x)\uparrow$ then $C_{\mu_1,\mu_2} > 0$

(*b*) $x\uparrow \Rightarrow \mu_1(x)\uparrow$ and $\mu_2(x)\downarrow$ then $C_{\mu_1,\mu_2} < 0$

(iii) $|C_{\mu_1,\mu_2}| \leqslant 1, \quad \forall \mu_1, \mu_2$

(iv) $C_{\mu,\mu} = 1, \quad C_{\mu,1-\mu} = -1, \quad \forall \mu$

(v) $C_{\mu_1,\mu_2} = -C_{\mu_1,1-\mu_2}, \quad \forall \mu_1, \mu_2$

(vi) $C_{\mu_1,\mu_2} = C_{\mu_2,\mu_1}, \quad \forall \mu_1, \mu_2$

(vii) $C_{\mu_1,\mu_2} = C_{1-\mu_1,1-\mu_2}, \quad \forall \mu_1, \mu_2$

Let $\Omega \subseteq \mathbb{R}$ and let the domain Ω be the minimal set. Let Ω be a closed interval of $\mathbb{R}$, and let $\mu_1 : \Omega \to [0, 1]$ and $\mu_2 : \Omega \to [0, 1]$ be continuous. $\mu_1(\Omega) = \mu_2(\Omega) = [0, 1]$ and for all $x \in \Omega^c$, $\mu_i(x) = 0$ or 1 or is undefined for all $i = 1, 2$.

Then define

$$C_{\mu_1,\mu_2} = 1 - \frac{4}{X_1 + X_2}\int_{\Omega} (\mu_1 + \mu_2)^2\, dx \tag{41}$$

where

$$X_1 = \int_\Omega (2\mu_1 - 1)^2\,dx \quad \text{and} \quad X_2 = \int_\Omega (2\mu_2 - 1)^2\,dx$$

This definition satisfies all the seven properties mentioned above. When Ω is finite (Ω need not be a subset of $\mathbb{R}$)

$$C_{\mu_1,\mu_2} = 1 - \frac{4}{X_1 + X_2} \sum_{x\in\Omega} (\mu_1 - \mu_2)^2 \tag{42a}$$

$$= 1, \quad \text{if } X_1 + X_2 = 0 \tag{42b}$$

where

$$X_1 = \sum_{x\in\Omega} (2\mu_1 - 1)^2 \quad \text{and} \quad X_2 = \sum_{x\in\Omega} (2\mu_2 - 1)^2$$

Example 3

Let $\Omega = [0, 1]$, $\mu_1 = x$, denoting the 'tall' men membership function, and $\mu_2 = 2\min(x, 1 - x)$, denoting the 'medium' men membership function. Then

$$X_{\mu_1} = \int_\Omega (2x - 1)^2\,dx = \tfrac{1}{3}, \quad X_{\mu_2} = \tfrac{1}{3}$$

$$\int_\Omega (\mu_1 - \mu_2)^2\,dx = \tfrac{1}{6}$$

$$C_{\mu_1,\mu_2} = 1 - \frac{4}{\frac{1}{3} + \frac{1}{3}} \cdot \tfrac{1}{6} = 0$$

Similarly, if we consider $\mu_2 = x^2$, $1 - x$ and $(1 - x)^2$ to represent the membership functions for 'very tall', 'not tall' and 'very not tall' men, then the corresponding C_{μ_1,μ_2} will be $\frac{5}{6}$, -1 and $-\frac{5}{6}$.

Details of the correlation were given by Murthy *et al.* (1985).

8. Fuzzy expected value and interval

Kandel and Byatt (1978) defined the fuzzy expected value (FEV) of a membership function μ over a fuzzy set A with respect to a fuzzy measure χ as follows.

Let μ_A be a B-measurable function such that $\mu_A \in [0, 1]$. The fuzzy expected value of μ_A over A, with respect to the fuzzy measure $\chi(\cdot)$ is defined as

$$\mathrm{FEV}(\mu_A) = \sup_{T\in[0,1]} \{\min[T, \chi(\xi_T)]\} \tag{43}$$

where

$$\xi_T = \{x \mid \mu_A(x) \geqslant T\} \tag{44}$$

Now, $\chi\{x \mid \mu_A(x) \geqslant T\} = f_A(T)$ is a function of the threshold T and the function χ maps ξ into $[0, 1]$. In other words, the method of evaluating $\mathrm{FEV}(\mu_A)$ consists of finding the point of intersection of the curves $g(T) = T$ and $f_A(T)$. These curves will therefore intersect at $T = H$ so that $\mathrm{FEV}(\mu_A) = H \in [0, 1]$.

FEV can thus be regarded as an indicative measure of some sort of central tendency.

Example 4

For a given population and a given membership function for the set 'Old', let us consider the following data (Schneider and Kandel 1991):

10 people are 20 years old, i.e. $\mu = 0{\cdot}20$
15 people are 30 years old, i.e. $\mu = 0{\cdot}30$
25 people are 45 years old, i.e. $\mu = 0{\cdot}45$
30 people are 55 years old, i.e. $\mu = 0{\cdot}55$
20 people are 60 years old, i.e. $\mu = 0{\cdot}60$

Here we have $T = 0{\cdot}20, 0{\cdot}30, 0{\cdot}45, 0{\cdot}55$ and $0{\cdot}60$. For a given threshold we can now determine the number of people (as a percentage) who are above the threshold. For example, the numbers are 100, 90, 75, 50 and 20, corresponding to the thresholds 0·20, 0·30, 0·45, 0·55 and 0·60. Thus we have $\chi = 1{\cdot}0, 0{\cdot}90, 0{\cdot}75, 0{\cdot}5$ and $0{\cdot}20$.

Now the minimum values of each (T, χ) pairs are 0·20, 0·30, 0·45, 0·50 and 0·20. The FEV(μ_A), which is the maximum of all these minima, is thus 0·50.

The fuzzy expected age of the population is 50.

Suppose we have the following data for a population: more or less 20 people are between the ages of 20 and 30; 20 to 25 people are 15 years old; 25 people are almost 40 years old.

The FEV denoting the typical age of the group of people is not applicable here, because the data do not have complete information about the distribution of the population and their grades of membership. In order to tackle this kind of problem the concept of the fuzzy expected interval (FEI) is introduced (Schneider and Kandel 1991).

The upper and lower bounds of any χ_j are defined as

$$UB_j = \frac{\sum_{i=j}^{n} \max(p_{i1}, p_{i2})}{\sum_{i=j}^{n} \max(p_{i1}, p_{i2}) + \sum_{i=1}^{j-1} \min(p_{i1}, p_{i2})} \tag{45}$$

$$LB_j = \frac{\sum_{i=j}^{n} \min(p_{i1}, p_{i2})}{\sum_{i=j}^{n} \min(p_{i1}, p_{i2}) + \sum_{i=1}^{j-1} \max(p_{i1}, p_{i2})} \tag{46}$$

where p_{i1} and p_{i2} are the lower bound and upper bound, respectively, of group i.

Therefore, arranging the data in order of increasing age, we may write: 20 to 25 people are 15 years old; more or less 20 people are between the ages of 20 and 30; 25 people are almost 40 years old.

Let us assume that the adjectives 'almost' and 'more or less' for the variable x have lower and upper bounds $x - 10\%$ and $x - 1$, and $x - 10\%$ and $x + 10\%$, respectively. Therefore, we have: 20 to 25 people are of μ_1 in $0{\cdot}15 - 0{\cdot}15$; more or less 20 people are of μ_2 in $0{\cdot}2 - 0{\cdot}3$; 25 people are of μ_3 in $0{\cdot}36 - 0{\cdot}39$.

The corresponding upper and lower bounds of the χ_j values are (using equations (45) and (46))

$$\chi_1 : \frac{20 + 18 + 25}{20 + 18 + 25 + 0} - \frac{25 + 22 + 25}{25 + 22 + 25 + 0} = 1 - 1$$

L_∞-norm and the L_1-norm are used). Since the poles of the obtained square-magnitude function are prevented from occurring on the $j\Omega$ axis or unit circle by constraints, the reduced model determined by the factorization technique is always stable. Therefore, this approach both ensures the stability of the reduced model and preserves the advantages of using the linear programming technique. This method applies to both continuous and discrete time systems. Examples are given to illustrate its applicability.

We proceed as follows. The reduction of continuous-time systems via the proposed approach is formulated in § 2. An illustrative example is given in § 3. Section 4 discusses the formulations for reduction of discrete-time systems. In § 5, an example is introduced to illustrate its usefulness. Finally, the discussion and conclusion are presented in § 6.

2. Model reduction for continuous-time systems

Let $G(s)$ be the transfer function of a given system and its reduced model be

$$H(s) = \frac{C(s)}{D(s)} = \frac{\sum_{m=0}^{M} c_m s^m}{\sum_{n=0}^{N} d_n s^n} \tag{1}$$

where the coefficients $d_n, c_m, 0 \leqslant n \leqslant N, 0 \leqslant m \leqslant M$, are real and N, M are the desired degrees of the denominator and numerator polynomials, respectively. Without loss of generality, $d_0 = 1$.

2.1. *Squared-magnitude function*

Consider $P(s), Q(s)$ to be the squared-magnitude functions of $G(s)$ and $H(s)$, respectively. Then

$$P(s) = G(s)G(-s) = \frac{B(s)}{A(s)} \tag{2}$$

$$Q(s) = H(s)H(-s) = \frac{B_r(s)}{A_r(s)} = \frac{\sum_{m=0}^{M} b_m s^{2m}}{\sum_{n=0}^{N} a_n s^{2n}} \tag{3}$$

where $a_0 = d_0^2 = 1$. Replacing s by $j\Omega$, the frequency responses of $P(s)$ and $Q(s)$ can be obtained as follows

$$P(j\Omega) = G(j\Omega)G(-j\Omega) = G(j\Omega)G^*(j\Omega) = \frac{B(j\Omega)}{A(j\Omega)} \tag{4}$$

$$Q(j\Omega) = \frac{B_r(j\Omega)}{A_r(j\Omega)} = \frac{\sum_{m=0}^{M} (-1)^m b_m \Omega^{2m}}{\sum_{n=0}^{N} (-1)^n a_n \Omega^{2n}} \tag{5}$$

where $G^*(j\Omega)$ is the complex conjugate of $G(j\Omega)$. Since $A_r(s)$ and $B_r(s)$ are squared-magnitude functions of $D(s)$ and $C(s)$, respectively

$$A_r(j\Omega) = D(j\Omega)D^*(j\Omega) = \sum_{n=0}^{N} (-1)^n a_n \Omega^{2n} > 0 \tag{6}$$

$$B_r(j\Omega) = C(j\Omega)C^*(j\Omega) = \sum_{m=0}^{M} (-1)^m b_m \Omega^{2m} \geqslant 0 \tag{7}$$

$A_r(j\Omega)$ should not be equal to zero because no pole exists on the $j\Omega$ axis for a stable reduced model.

2.2. *Approximation problem*

In order to find $Q(j\Omega)$ that approximates $P(j\Omega)$, the error function is defined as follows:

$$E(j\Omega) = |P(j\Omega) - Q(j\Omega)| = \left|\frac{B(j\Omega)}{A(j\Omega)} - \frac{B_r(j\Omega)}{A_r(j\Omega)}\right|$$

$$= \left|\frac{B(j\Omega)}{A(j\Omega)} - \frac{\sum_{m=0}^{M} (-1)^m b_m \Omega^{2m}}{\sum_{n=0}^{N} (-1)^n a_n \Omega^{2n}}\right| \tag{8}$$

Then, the approximation problem is to find the coefficients $\{a_n, b_m, 1 \leqslant n \leqslant N, 0 \leqslant m \leqslant M\}$ such that the error function $E(j\Omega)$ is minimum in some design sense.

2.3. *Linear programming formulation*

In order to be adapted to the linear programming environment, the error function is modified in the following form:

$$E(j\Omega) = \left|\frac{A_r(j\Omega)}{A(j\Omega)} - \frac{B_r(j\Omega)}{B(j\Omega)}\right| \tag{9}$$

Such a definition is adequate because when $A_r(j\Omega)/A(j\Omega)$ approximate $B_r(j\Omega)/\mathrm{B}(j\Omega)$, $B(j\Omega)/A(j\Omega)$ will also approximate $B_r(j\Omega)/A_r(j\Omega)$ and vice versa. Next, consider a set of frequency points $\{\Omega_i, i = 1, \ldots, L\}$ for matching. Two formulations based on the respective criteria of the L_1-norm and L_∞-norm of the error function for linear programming to solve the approximation problem can be derived as follows. as follows.

Formulation 1: *L_1-norm approach*

Define a set of auxiliary non-negative variables $\{\varepsilon_i, i = 1, \ldots, L\}$ such that

$$\left|\frac{A_r(j\Omega_i)}{A(j\Omega_i)} - \frac{B_r(j\Omega_i)}{B(j\Omega_i)}\right| = E(j\Omega_i) = \varepsilon_i \tag{10}$$

Multiplying (10) by $A(j\Omega_i)B(j\Omega_i)$, we get

$$|B(j\Omega_i)A_r(j\Omega_i) - A(j\Omega_i)B_r(j\Omega_i)| = \varepsilon_i A(j\Omega_i)B(j\Omega_i) \tag{11}$$

Since ε_i is non-negative, the following two inequality constraints can be applied to obtain the same minimum ε_i as (11):

$$B(j\Omega_i)A_r(j\Omega_i) - A(j\Omega_i)B_r(j\Omega_i) \leqslant \varepsilon_i A(j\Omega_i)B(j\Omega_i) \tag{12}$$

$$-B(j\Omega_i)A_r(j\Omega_i) + A(j\Omega_i)B_r(j\Omega_i) \leqslant \varepsilon_i A(j\Omega_i)B(j\Omega_i) \tag{13}$$

In addition, the constraints (6), (7) are included in order to obtain the factorizable $A_r(s)$ and $B_r(s)$ and to ensure the stability of the reduced model. Therefore, the formulation of the linear programming technique to find optimum coefficients $\{a_n, b_m, 1 \leqslant n \leqslant N, 0 \leqslant m \leqslant M\}$ based on the L_1-norm as criterion can be described as

$$\text{Minimize} \sum_{i=1}^{L} W(j\Omega_i)\varepsilon_i$$

Subject to

$$B(j\Omega_i)\sum_{n=0}^{N}(-1)^n a_n \Omega_i^{2n} - A(j\Omega_i)\sum_{m=0}^{M}(-1)^m b_m \Omega_i^{2m} \leqslant \varepsilon_i A(j\Omega_i)B(j\Omega_i) \quad (14)$$

$$-B(j\Omega_i)\sum_{n=0}^{N}(-1)^n a_n \Omega_i^{2n} + A(j\Omega_i)\sum_{m=0}^{M}(-1)^m b_m \Omega_i^{2m} \leqslant \varepsilon_i A(j\Omega_i)B(j\Omega_i) \quad (15)$$

$$\sum_{n=0}^{N}(-1)^n a_n \Omega_i^{2n} > 0 \quad (16)$$

$$\sum_{m=0}^{M}(-1)^m b_m \Omega_i^{2m} \geqslant 0 \quad (17)$$

where $W(j\Omega)$ is the weighting function and constraints (14), (15) are obtained by stating $A_r(j\Omega)$ and $B_r(j\Omega)$ of (12), (13) in terms of coefficients $\{a_n, b_m\}$.

Formulation 2: L_∞-norm (or mini-max) approach

Define an auxiliary non-negative variable ε which has the following property:

$$\left|\frac{A_r(j\Omega_i)}{A(j\Omega_i)} - \frac{B_r(j\Omega_i}{B(j\Omega_i)}\right| \leqslant \frac{\varepsilon}{W(j\Omega_i)} \quad (18)$$

where $W(j\Omega)$ is the weighting function and $i = 1, \ldots, L$. Multiplying (18) by $A(j\Omega_i)B(j\Omega_i)$, two equivalent constraints are

$$B(j\Omega_i)A_r(j\Omega_i) - A(j\Omega_i)B_r(j\Omega_i) \leqslant \frac{\varepsilon A(j\Omega_i)B(j\Omega_i)}{W(j\Omega_i)} \quad (19)$$

$$-B(j\Omega_i)A_r(j\Omega_i) + A(j\Omega_i)B_r(j\Omega_i) \leqslant \frac{\varepsilon A(j\Omega_i)B(j\Omega_i)}{W(j\Omega_i)} \quad (20)$$

As in Formulation 1, the constraints (6), (7) are also necessary. Thus, the formulation using the L_∞-norm as the criterion is

$$\text{Minimize } \varepsilon$$

Subject to

$$B(j\Omega_i)\sum_{n=0}^{N}(-1)^n a_n \Omega_i^{2n} - A(j\Omega_i)\sum_{m=0}^{M}(-1)^m b_m \Omega_i^{2m} \leqslant \frac{\varepsilon A(j\Omega_i)B(j\Omega_i)}{W(j\Omega_i)} \quad (21)$$

$$-B(j\Omega_i)\sum_{n=0}^{N}(-1)^n a_n \Omega_i^{2n} + A(j\Omega_i)\sum_{m=0}^{M}(-1)^m b_m \Omega_i^{2m} \leqslant \frac{\varepsilon A(j\Omega_i)B(j\Omega_i)}{W(j\Omega_i)} \quad (22)$$

$$\sum_{n=0}^{N}(-1)^n a_n \Omega_i^{2n} > 0 \quad (23)$$

$$\sum_{m=0}^{M}(-1)^m b_m \Omega_i^{2m} \geqslant 0 \quad (24)$$

where the subscript $i = 1, \ldots, L$.

In practice, the error between the responses of a given system and its reduced model is usually required to be small in one band (such as the passband or the transition band) and allowed to be a little larger in another band (such as the stopband). Thus the weighting function $W(j\Omega)$ of the mini–max approach is often not uniform.

2.4. *Steady-state value preservation*

In order for the reduced model to preserve the same steady-state value of step responses as that of the original system, it is necessary that $G(j0) = H(j0), P(j0) = Q(j0)$. Therefore, the following constraint is employed:

$$P(j0)a_0 = b_0 \tag{25}$$

2.5. *Factorization*

After obtaining the coefficients a_n, b_m, the squared-magnitude function $Q(s)$ of the reduced model can be factorized as below

$$Q(s) = K\frac{(1 - s/z_1)(1 + s/z_1)\dots(1 - s/z_M)(1 + s/z_M)}{(1 - s/p_1)(1 + s/p_1)\dots(1 - s/z_N)(1 + s/z_N)} \tag{26}$$

where $\operatorname{Re}(p_n) < 0$ and $\operatorname{Re}(z_m) < 0$. Taking the roots in the left-hand side of the s-plane, the reduced model is

$$H(s) = K^* \frac{\sum_{m=1}^{M} (1 - s/z_m)}{\sum_{n=1}^{N} (1 - s/p_n)} \tag{27}$$

where $K^* = \sqrt{K}$ since $Q(j0) = H(j0)H(-j0)$.

3. Illustrative example for continuous-time systems

Consider the following system:

$$\begin{aligned} G(s) = {} & (1441{\cdot}53s^3 + 78319s^2 + 525286{\cdot}125s + 607693{\cdot}25)/(s^7 + 112{\cdot}04s^6 \\ & + 3755{\cdot}92s^5 + 39756{\cdot}73s^4 + 363650{\cdot}56s^3 + 759894{\cdot}195s^2 \\ & + 683656{\cdot}25s + 617497{\cdot}375) \end{aligned} \tag{28}$$

The reduced model of this system by the squared-magnitude CFE method (2) is

$$F_2(s) = \frac{1{\cdot}2157 + 0{\cdot}6646s}{1{\cdot}2354 + 0{\cdot}5427s + s^2} \tag{29}$$

For the proposed methods, we choose 100 frequency matching points $\{\Omega_i, i = 1, \dots, 100\}$, logarithmically spaced within frequency range $[0{\cdot}05, 7{\cdot}5]\,\mathrm{rad\,s^{-1}}$, which include from passband to stopband of the squared-magnitude responses of the original system. The weighting function is

$$W(j\Omega) = \begin{cases} 100 & \Omega \leqslant 1{\cdot}7\,\mathrm{rad\,s^{-1}} \\ 1 & \text{otherwise} \end{cases} \tag{30}$$

in order to emphasize the matching in the passband and transition band of the squared-magnitude responses of the original system and its reduced model. Assume $a_0 = 1$. Next, the squared-magnitude functions $Q_2^1(s)$ and $Q_2^\infty(s)$ of the reduced models, corresponding to L_1-norm and L_∞-norm approaches, respectively, are

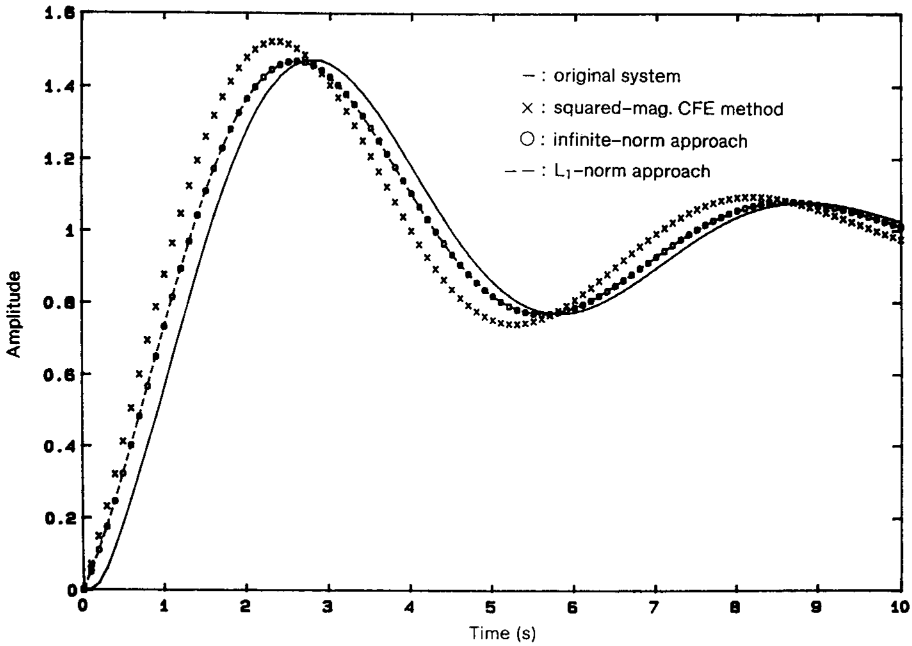

Figure 1. Step responses (continuous-time).

obtained by the linear programming technique:

$$Q_2^1(s) = \frac{-0{\cdot}165770s^2 + 0{\cdot}968554}{0{\cdot}748883s^4 + 1{\cdot}513820s^2 + 1} \tag{31}$$

$$Q_2^\infty(s) = \frac{-0{\cdot}163653s^2 + 0{\cdot}968554}{0{\cdot}749614s^4 + 1{\cdot}515786s^2 + 1} \tag{32}$$

Finally, the reduced models $H_2^1(s)$ and $H_2^\infty(s)$ can be computed by the factorization technique (26), (27):

$$H_2^1(s) = \frac{0{\cdot}407149s + 0{\cdot}984151}{0{\cdot}865380s^2 + 0{\cdot}465769s + 1} \tag{33}$$

$$H_2^\infty(s) = \frac{0{\cdot}40450s + 0{\cdot}984151}{0{\cdot}865803s^2 + 0{\cdot}464563s + 1} \tag{34}$$

The step responses and squared-magnitude responses of the original system $G(s)$ and its reduced models $F_2(s)$, $H_2^1(s)$, $H_2^\infty(s)$, are shown in Figs 1 and 2, respectively. These results show that the proposed methods are satisfactory and comparable to those of the squared-magnitude CFE method.

4. Model reduction for discrete-time systems

The formulation procedure for model reduction of discrete-time systems is similar to that of continuous-time systems. Consider the transfer function of a given system

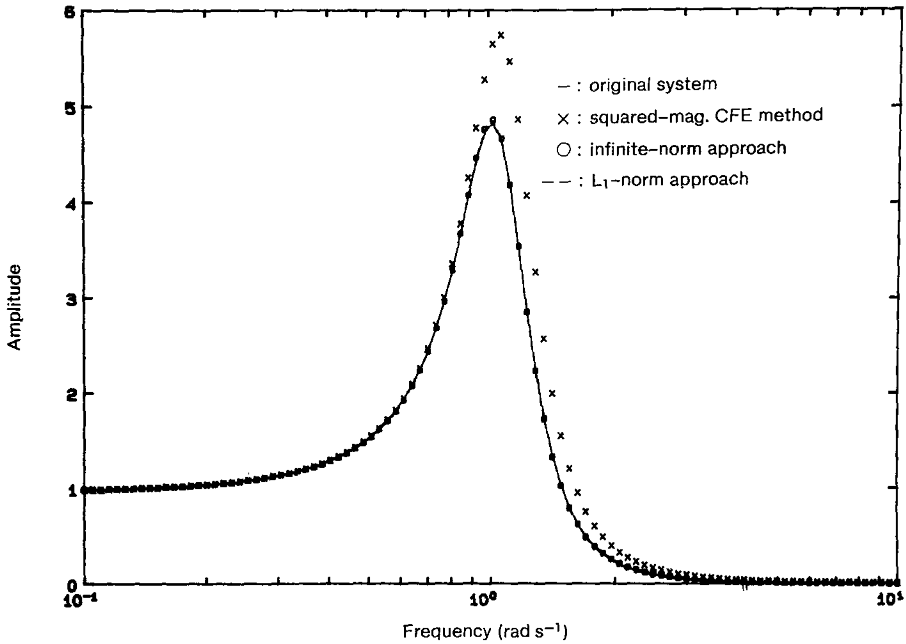

Figure 2. Squared-magnitude responses (continuous-time).

$G(z)$ and its reduced model $H(z)$

$$H(z) = \frac{C(z)}{D(z)} = \frac{\sum_{m=0}^{M} c_m z^{-m}}{\sum_{n=0}^{N} d_n z^{-n}} \tag{35}$$

where the coefficients $d_n, c_m, 0 \leqslant n \leqslant N, 0 \leqslant m \leqslant M$, are real. Without loss of generality, $d_0 = 1$.

4.1. *Squared-magnitude function*

Let $P(z), Q(z)$, be the squared-magnitude functions of $G(z)$ and $H(z)$, respectively. Then

$$P(z) = G(z)G(z^{-1}) = \frac{B(z)}{A(z)} \tag{36}$$

$$Q(z) = H(z)H(z^{-1}) = \frac{B_r(z)}{A_r(z)} = \frac{\sum_{m=-M}^{M} b_m z^{-m}}{\sum_{n=-N}^{N} a_n z^{-n}} \tag{37}$$

where $a_n = a_{-n}, b_m = b_{-m}$ for $1 \leqslant n \leqslant N, 1 \leqslant m \leqslant M$. a_0 can be assumed to be one without loss of generality. Replacing z by $\exp(jw)$, the frequency responses of $P(z)$ and $Q(z)$ are

$$P(\exp jw) = \frac{B(\exp jw)}{A(\exp jw)} = G(\exp jw)G^*(\exp jw) \tag{38}$$

$$Q(\exp jw) = \frac{B_r(\exp jw)}{A_r(\exp jw)} = \frac{b_0 + 2\sum_{m=1}^{M} b_m \cos(m\omega)}{a_0 + 2\sum_{m=1}^{N} a_n \cos(nw)} \tag{39}$$

Similar to continuous-time systems

$$A_r(\exp jw) = D(\exp jw)D^*(\exp jw) = a_0 + 2\sum_{n=1}^{N} a_n \cos(nw) > 0 \tag{40}$$

$$B_r(\exp jw) = C(\exp jw)C^*(\exp jw) = b_0 + 2\sum_{m=1}^{M} b_m \cos(mw) \geqslant 0 \tag{41}$$

$A_r(\exp jw)$ will not be equal to zero since no poles of stable $H(z)$ exist on the unit circle.

4.2. *Approximation problem*

Similarly, the error function is defined as follows

$$\begin{aligned} E(\exp jw) &= |P(\exp jw) - Q(\exp jw)| \\ &= \left| \frac{B(\exp jw)}{A(\exp jw)} - \frac{B_r(\exp jw)}{A_r(\exp jw)} \right| \\ &= \left| \frac{B(\exp jw)}{A(\exp jw)} - \frac{b_0 + 2\sum_{m=1}^{M} b_m \cos(mw)}{a_0 + 2\sum_{n=1}^{N} a_n \cos(nw)} \right| \end{aligned} \tag{42}$$

Next, the approximation problem is to find the coefficients $\{a_n, b_m, 1 \leqslant n \leqslant N, 0 \leqslant m \leqslant M\}$ so that the object function $E(\exp jw)$ is minimum in some design sense.

4.3. *Linear programming formulation*

The error function is modified as follows:

$$E(\exp jw) = \left| \frac{A_r(\exp jw)}{A(\exp jw)} - \frac{B_r(\exp jw)}{B(\exp jw)} \right| \tag{43}$$

in order to adapt to the linear programming technique. Next, consider a set of frequency points $\{w_i, i = 1, \ldots, L\}$ for matching. Two formulations based on the L_1-norm and the L_∞-norm as criteria are also presented.

Formulation 1: *L_1-norm approach*

Define a set of auxiliary non-negative variables $\{\varepsilon_i, i = 1, \ldots, L\}$ such that

$$\left| \frac{A_r(\exp jw_i)}{A(\exp jw_i)} - \frac{B_r(\exp jw_i)}{B(\exp jw_i)} \right| = E(\exp jw_i) = \varepsilon_i \tag{44}$$

Multiplying (44) by $A(\exp jw_i)B(\exp jw_i)$, two equivalent constraints to obtain the same minimum ε_i are

$$B(\exp jw_i)A_r(\exp jw_i) - A(\exp jw_i)B_r(\exp jw_i) \leqslant \varepsilon_i A(\exp jw_i)B(\exp jw_i) \tag{45}$$

$$-B(\exp jw_i)A_r(\exp jw_i) + A(\exp jw_i)B_r(\exp jw_i) \leqslant \varepsilon_i A(\exp jw_i)B(\exp jw_i) \tag{46}$$

In addition, the constraints (40), (41) are added to obtain the factorizable $A_r(z)$ and $B_r(z)$ and to ensure the stability of the reduced model. Therefore, the formulation based on the L_1-norm as criterion is

$$\text{Minimize} \sum_{i=1}^{L} W(\exp jw_i)\varepsilon_i$$

Subject to

$$B(\exp jw_i)\left[a_0 + 2\sum_{n=1}^{N} a_n \cos(nw_i)\right] - A(\exp jw_i)\left[b_0 + 2\sum_{m=1}^{M} b_m \cos(mw_i)\right] \leqslant \varepsilon_i A(\exp jw_i)B(\exp jw_i) \quad (47)$$

$$-B(\exp jw_i)\left[a_0 + 2\sum_{n=1}^{N} a_n \cos(nw_i)\right] + A(\exp jw_i)\left[b_0 + 2\sum_{m=1}^{M} b_m \cos(mw_i)\right] \leqslant \varepsilon_i A(\exp jw_i)B(\exp jw_i) \quad (48)$$

$$a_0 + 2\sum_{n=1}^{N} a_n \cos(nw_i) > 0 \quad (49)$$

$$b_0 + 2\sum_{m=1}^{M} b_m \cos(mw_i) \geqslant 0 \quad (50)$$

where $W(\exp jw)$ is a weighting function and constraints (47), (48) are obtained by stating $A_r(\exp jw)$ and $B_r(\exp jw)$ of (45), (46) in terms of coefficients $\{a_n, b_m\}$.

Formulation 2: L_∞-norm (or mini–max) approach

Define an auxiliary non-negative variable ε which has the following property

$$\left|\frac{A_r(\exp jw_i)}{A(\exp jw_i)} - \frac{B_r(\exp jw_i)}{B(\exp jw_i)}\right| \leqslant \frac{\varepsilon}{W(\exp jw_i)} \quad (51)$$

where $W(\exp jw)$ is the weighting function. Multiplying (51) by $A(\exp jw_i)/B(\exp jw_i)$, two equivalent constraints are

$$B(\exp jw_i)A_r(\exp jw_i) - A(\exp jw_i)B_r(\exp jw_i) \leqslant \frac{\varepsilon A(\exp jw_i)B(\exp jw_i)}{W(\exp jw_i)} \quad (52)$$

$$-B(\exp jw_i)A_r(\exp jw_i)) + A(\exp jw_i)B_r(\exp jw_i) \leqslant \frac{\varepsilon A(\exp jw_i)B(\exp jw_i)}{W(\exp jw_i)} \quad (53)$$

As with Formulation 1, the constraints (40), (41) are necessary. Thus, the formulation based on the criterion of the L_∞-norm is

$$\text{Minimize}\ \varepsilon$$

Subject to

$$B(\exp jw_i)\left[a_0 + 2\sum_{n=1}^{N} a_n \cos(nw_i)\right] - A(\exp jw_i)\left[b_0 + 2\sum_{m=1}^{M} b_m \cos(mw_i)\right] \leqslant \frac{\varepsilon A(\exp jw_i)B(\exp jw_i)}{W(\exp jw_i)} \quad (54)$$

$$-B(\exp jw_i)\left[a_0 + 2\sum_{n=1}^{N} a_n \cos(nw_i)\right] + A(\exp jw_i)\left[b_0 + 2\sum_{m=1}^{M} b_m \cos(mw_i)\right] \leqslant \frac{\varepsilon A(\exp jw_i)B(\exp jw_i)}{W(\exp jw_i)} \quad (55)$$

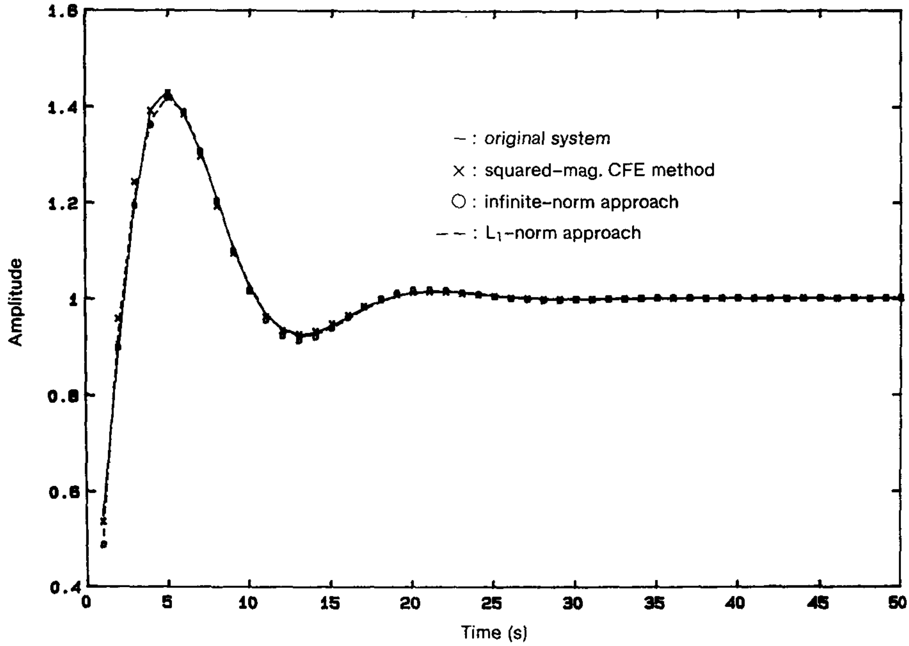

Figure 3. Step responses (discrete-time).

$$a_0 + 2 \sum_{n=1}^{N} a_n \cos(nw_i) > 0 \tag{56}$$

$$b_0 + 2 \sum_{m=1}^{M} b_m \cos(mw_i) \geq 0 \tag{57}$$

where the subscript $i = 1, \ldots, L$.

4.4. *Steady-state value preservation*

In order for the reduced model to preserve the same steady-state value of step responses as that of the original system, it is necessary that $G(1) = H(1), P(1) = Q(1)$. Therefore, the following constraint is applied

$$P(1)\left(a_0 + 2 \sum_{n=1}^{N} a_n\right) = b_0 + 2 \sum_{m=1}^{M} b_m \tag{58}$$

4.5. *Factorization*

After obtaining the coefficients a_n, b_m, the squared-magnitude function $Q(z)$ can be factorized as

$$Q(z) = K \frac{z^{-M}(z - z_1)(z - 1/z_1) \ldots (z - z_M)(z - 1/z_M)}{z^{-N}(z - p_1)(z - 1/p_1) \ldots (z - z_N)(z - 1/z_N)} \tag{59}$$

where $|z_m| < 1$ and $|p_n| < 1$. Taking the roots within the unit circle of the z-plane, the reduced model is

$$H(z) = K^* \frac{z^{-M} \sum_{m=1}^{M} (z - z_m)}{z^{-N} \sum_{n=1}^{N} (z - p_n)} \tag{60}$$

where K^* can be obtained from the relation $H(1) = \sqrt{Q(1)}$.

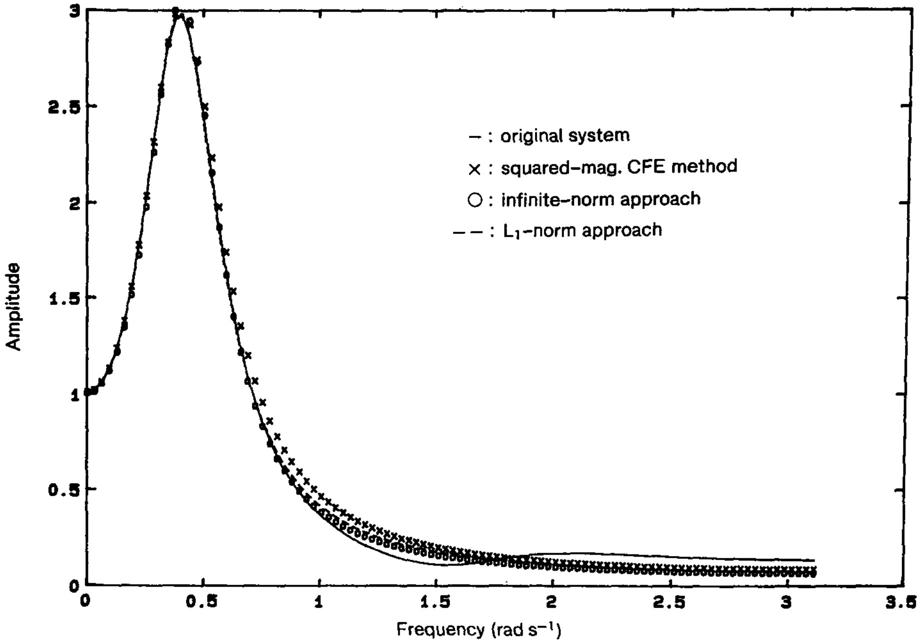

Figure 4. Squared-magnitude responses (discrete-time).

5. Illustrative example for discrete-time systems

Consider a fourth-order system

$$G(z) = \frac{0{\cdot}547377 - 0{\cdot}404730z^{-1} + 0{\cdot}319216z^{-2} - 0{\cdot}216608z^{-3}}{1 - 1{\cdot}361780z^{-1} + 0{\cdot}875599z^{-2} - 0{\cdot}551205z^{-3} + 0{\cdot}282145z^{-4}} \tag{61}$$

The reduced model of this system by the squared-magnitude CFE method in the z-domain (3) is

$$F_2(z) = \frac{0{\cdot}535150 - 0{\cdot}377930z^{-1}}{1 - 1{\cdot}497814z^{-1} + 0{\cdot}654717z^{-2}} \tag{62}$$

For the present methods, 100 frequency points $\{w_i, i = 1, \ldots, 100\}$, logarithmically spaced within frequency range $[0{\cdot}1, \Pi]$ rad s^{-1}, which include from passband to stopband of the squared-magnitude response of the original system, are chosen for matching. The weighting function is

$$W(\exp jw) = \begin{cases} 100 & w \leqslant 0{\cdot}8 \text{ rad s}^{-1} \\ 1 & \text{otherwise} \end{cases} \tag{63}$$

in order to emphasize the matching in passband and transition band of the squared-magnitude responses of the original system and its reduced model. Assume $a_0 = 1$. Then, the obtained squared-magnitude functions $Q_2^1(z)$ and $Q_2^\infty(z)$ of reduced models corresponding to L_1-norm and L_∞-norm approaches, respectively, are

$$Q_2^1(z) = \frac{0{\cdot}101807 - 0{\cdot}047722(z + z^{-1})}{1 - 0{\cdot}675408(z + z^{-1}) + 0{\cdot}178577(z^2 + z^{-2})} \tag{64}$$

$$Q_2^\infty(z) = \frac{0{\cdot}091766 - 0{\cdot}042441(z + z^{-1})}{1 - 0{\cdot}676627(z + z^{-1}) + 0{\cdot}180055(z^2 + z^{-2})} \tag{65}$$

Finally, the reduced models $H_2^1(z)$ and $H_2^\infty(z)$ can be computed by the factorization technique (59), (60):

$$H_2^1(z) = \frac{0\cdot506137 - 0\cdot352008z^{-1}}{1 - 1\cdot512805z^{-1} + 0\cdot666621z^{-2}} \tag{66}$$

$$H_2^\infty(z) = \frac{0\cdot487448 - 0\cdot326732z^{-1}}{1 - 1\cdot515292z^{-1} + 0\cdot675683z^{-2}} \tag{67}$$

The step responses and squared-magnitude responses of the original system $G(z)$ and its reduced models $F_2(z), H_2^1(z), H_2^\infty(z)$, are shown in Figs 3 and 4, respectively. The results obtained by the present methods are clearly comparable to those of the squared-magnitude CFE method.

6. Discussion and conclusion

The linear programming technique is used for model reduction of continuous- and discrete-time systems by matching the squared-magnitude responses of the original system and its reduced model. A novel error function is prevented. Two formulations based on the L_1-norm and the L_∞-norm of the error function as respective criteria are introduced. Since the poles of the obtained squared-magnitude function occurring on the $j\Omega$ axis (continuous-time case) or unit circle (discrete-time case) are prevented by constraints (6) or (40), the application of the factorization technique will ensure the stability of the reduced model. A useful property of this approach that needs further elaboration is its flexibility. If necessary, constraints may be added to the formulation. For example, constraints may easily be added to the reduced model in order to preserve the initial value of step responses of the original system.

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