INTEGRATION, the VLSI journal 30 (2001) 157–168

Enumerating catastrophic fault patterns in VLSI arrays with both uni- and bidirectional links ☆

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Received 30 January 2001

Abstract

Characterization of catastrophic fault patterns (CFPs) and their enumeration have been studied by several authors. Given a linear array with a set of bypass links, an important problem is how to count the number of CFPs. Enumeration of CFPs for two link redundancy $G = \{1, g\}$ has been solved for both unidirectional and bidirectional link cases. In this paper, we consider the more general case of link redundancy $G = \{1, 2, \ldots, k, g\}$, $2 \leqslant k < g$. Using random walk as a tool, we enumerate CFPs for both unidirectional and bidirectional cases.

Keywords: Catastrophic fault patterns; Combinatorial problems; Random walk

1. Introduction

A standard technique to lower the production costs of VLSI circuits is the provision of on-chip redundancy, and accompanying mechanisms to reconfigure the chip component at the occurrence of fabrication faults. Without the presence of reconfiguration capabilities, the yield of very large VLSI chips would be so poor to will make their production unacceptable. In the case of linear arrays of identical processing elements (PEs), redundant ones (called spares) are often placed on the chip to replace faulty PEs and, therefore, preserve the network connectivity. Besides the

☆This work was supported in part by Natural Sciences and Engineering Research Council of Canada under Operating Grant 9167.

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regular links between neighboring PEs, bypass links are also included to recreate the array topology in the reconfiguration phase. However, regardless of any amount of redundancy and configuration capabilities, there are always sets of faults occurring at strategic position which affect the chip in an unrepairable way. Such sets of faults, called catastrophic fault patterns (CFPs) has been extensively studied in [1–3] for linear arrays with different varieties of link redundancy.

Let $A = \{p_0, p_1, \dots, p_N\}$ denote a one-dimensional array of processing elements. There exists a direct link (regular link) between p_i and p_{i+1}, $0 \leqslant i < N$. Any link connecting p_i and p_j, where $j > i+1$ is said to be a bypass link of length $j-i$. The bypass links are used strictly for reconfiguration purposes when a fault is detected. The links can be either unidirectional or bidirectional.

Given an integer $g \in [1, N]$, A is said to have link redundancy g, if for every $p_i \in A$ with $i \leqslant N-g$, there exists a link between p_i and p_{i+g}. Let $G = \{g_1, g_2, \dots, g_k\}$, where $g_j < g_{j+1}$ and $g_j \in [1, N]$. The array A is said to have link redundancy G if A has link redundancy $g_1, g_2, \dots, g_k$. Notice that $g_1 = 1$ is the regular link, while all other g_i's correspond to bypass links. A fault pattern for A is a set of integers $F = \{f_0, f_1, \dots, f_m\}$, where $m \leqslant N$, $f_j < f_{j+1}$ and $f_i \in [0, N]$. An assignment of a fault pattern F to A means that for every $f \in F$, p_f is faulty. The width W_F of a fault pattern $F = \{f_0, f_1, \dots, f_{g-1}\}$ is defined to be the number of PEs between and including the first and the last fault in F; that is, $W_F = f_{g-1} - f_0 + 1$. At the two ends of the array, two special PEs called I (for input) and O (for output) are responsible for I/O function of the system. It is assumed that I is connected to $p_0, p_1, \dots, p_{g_k-1}$ while I is connected to $p_{N-g_k}, p_{N-g_k-1}, \dots, p_{N-1}$ so that all PEs in the system have same degree and reliability bottlenecks at the borders of the array are avoided. Given a link-redundant linear array A, a fault pattern $F = \{f_0, f_1, \dots, f_m\}$ is catastrophic for A if and only if no path exists between I and O, once the faulty p_i, $i \in F$, and their incident links are removed.

Characterization of catastrophic fault patterns (CFPs) and their enumeration have been studied by several authors, e.g. in [2–5]. Given a linear array with a set of bypass links, an important problem is to count the number of CFPs. The knowledge of the number of CFPs enables us to estimate the probability that the system operates correctly. Enumeration of CFPs for $G = \{1, g\}$ has been done in [1] for bidirectional case and in [6] for unidirectional case. A method of enumeration of CFPs in the more general context is given in [7], but no closed form solution has been obtained. Using random walk as a tool, we derive the number of CFPs for both bidirectional and unidirectional linear arrays with link redundancy $G = \{1, 2, \dots, k, g\}$, $2 \leqslant k < g$.

2. Preliminaries

For $G = \{g_1, g_2, \dots, g_k\}$ with $g_1 = 1$, CFPs with exactly g_k faults are considered because of their minimality [5]. A fault pattern $F = \{f_0, f_1, \dots, f_{g_k-1}\}$ is represented by a Boolean matrix [3] W of size (W_F^+, g_k) where $W_F^+ = \lceil W_F / g_k \rceil$ and

$$W[i,j] = \begin{cases} 1 & \text{if } (ig_k + j) \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $W[0,0] = 1$ which indicates the location of the first fault. Each column of W contains exactly one 1. Let $W[h_i, i] = 1$ for all i and define $m_i = h_{i-1} - h_i$. We call $\{m_1, m_2, \ldots, m_{g_k}\}$ the catastrophic sequence of F.

Definition 2.1 (Feller [8]). A random walk is a sequence $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\}$, where each $\varepsilon_i = +1$ or -1. The sequence is normally represented by a polynomial line on a X–Y plane where the kth side has slope ε_k and the kth vertex has ordinate $S_k = \sum_{i=1}^{k} \varepsilon_i$; such lines are called paths. For example, the row $\{1, -1, -1, 1, -1, -1\}$ is represented by a path from $(0,0)$ to $(6,-2)$, with intermediate points $(1,1), (2,0), (3,-1), (4,0), (5,-1)$ in the given order.

Definition 2.2. A subsequence $\{\varepsilon_{s+1}, \varepsilon_{s+2}, \ldots, \varepsilon_{s+r}\}$ of $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, $r \geqslant 1$ is called a run of length r if $\varepsilon_s \neq \varepsilon_{s+1} = \varepsilon_{s+2} = \cdots = \varepsilon_{s+r} \neq \varepsilon_{s+r+1}$. Let ρ_1 and ρ_{-1} denote the numbers of runs whose elements are 1 and -1, respectively, and $R = \rho_1 + \rho_{-1}$. Note that R is the number of runs.

Notations. We assume $m \geqslant 0$.

$E_{n,m}$: a path from $(0,0)$ to (n,m),

$E_{n,m}^{R}$: an $E_{n,m}$ path with R runs,

$E_{n,m}^{R+}$: an $E_{n,m}^{R}$ path starting with a positive step,

$E_{n,m}^{R-}$: an $E_{n,m}^{R}$ path starting with a negative step,

$E_{n,m}^{R+,t}$: an $E_{n,m}^{R+}$ path crossing the line $y = t$, $t > 0$ at least once,

$E_{n,m}^{R-,t}$: an $E_{n,m}^{R-}$ path crossing the line $y = t$, $t > 0$ at least once,

$N(A)$: the number of all A paths, e.g.

$$N(E_{n,m}) = \binom{n}{\frac{n-m}{2}}.$$

Theorem 2.1 (Feller [8]). *Among the* $\binom{2n}{n}$ *paths joining the origin to the point* $(2n, 0)$ *there are exactly* $[1/(n+1)]\binom{2n}{n}$ *paths such that* $S_1 \leqslant 0, S_2 \leqslant 0, \ldots, S_{2n-1} \leqslant 0, S_{2n} = 0$.

Theorem 2.2 (Vellore [9]). *For* $m \leqslant t < (n+m)/2$ *and* $m \geqslant 0$,

$$N(E_{n,m}^{(2r-1)+,t}) = \binom{\frac{n-m}{2} + t - 1}{r-2}\binom{\frac{n+m}{2} - t - 1}{r-1},$$

$$N(E_{n,m}^{2r-,t}) = \binom{\frac{n-m}{2} + t - 1}{r-2}\binom{\frac{n+m}{2} - t - 1}{r}.$$

Theorem 2.3. *The number of paths from the origin to the point* $(2n, 0)$ *such that* $S_1 \leqslant 0, S_2 \leqslant 0, \ldots, S_{2n-1} \leqslant 0, S_{2n} = 0$ *and there are exactly* $2r$ *runs, is*

$$\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r}.$$

Proof. The required number of paths equals

$$N(E^{2r-}_{2n,0}) - N(E^{2r-,0*}_{2n,0}), \tag{1}$$

where $E^{2r-,0*}_{2n,0}$ is an $E^{2r-}_{2n,0}$ path crossing the line $y = 0$ at least once. It is known that

$$N(E^{2r-}_{2n,0}) = \binom{n-1}{r-1}^2 \tag{2}$$

(see Wald and Wolfowitz [10]). To find $N(E^{2r-,0*}_{2n,0})$, let P be an $E^{2r-,0*}_{2n,0}$ path, i.e., a path from $(0, 0)$ to $(2n, 0)$ with the first step negative, with $2r$ runs and crossing $y = 0$ at least once. Then dropping the first step and taking $(1, -1)$ as the new origin, we have an $E_{2n-1,1}$ path Q. Note that Q crosses the line $y = 1$ (w.r.t. the new origin) and has $2r - 1$ runs if it starts with a positive step and $2r$ runs if it starts with a negative step. Moreover, any path Q with these properties arises from an unique $E^{2r-,0*}_{2n,0}$ path P as above. Thus,

$$\begin{aligned} N(E^{2r-,0*}_{2n,0}) &= N(E^{2r-,1}_{2n-1,1}) + N(E^{(2r-1)+,1}_{2n-1,1}) \\ &= \binom{n-1}{r-2}\binom{n-2}{r} - \binom{n-1}{r-2}\binom{n-2}{r-1} \\ &= \binom{n-1}{r-2}\binom{n-1}{r}. \end{aligned} \tag{3}$$

The theorem follows from (1)–(3). $\square$

3. The case of bidirectional links

Proposition 3.1 (Pagli et al. [11]). *Necessary and sufficient conditions for* $\{m_1, m_2, \ldots, m_{g-1}\}$ *to be a catastrophic sequence of a minimal CFP for a bidirectional linear array with link redundancy* $G = \{1, g\}$ *are*

(1) $m_i = -1,\ 0$ *or* 1, *for* $1 \leqslant i \leqslant g - 1$,

(2) $S_k = \sum_{i=1}^{k} m_i \leqslant 0$ *for any* $1 \leqslant k \leqslant g - 2$,

(3) $S_{g-1} = \sum_{i=1}^{g-1} m_i = 0$.

Theorem 3.1 (Nayak [1]). *For $G=\{1,g\}$, the number of CFPs for bidirectional links is given by*

$$\sum_{n=0}^{\lfloor (g-1)/2 \rfloor} \frac{1}{n+1}\binom{2n}{n}\binom{g-1}{2n}.$$

Proposition 3.2. *Necessary and sufficient conditions for $\{m_1, m_2, \ldots, m_{g-1}\}$ to be the catastrophic sequence of a minimal CFP for a bidirectional linear array with link redundancy $G=\{1,2,g\}$ are*

(1) $m_{g-1}=0$,
(2) $m_j=-1,0,+1$ *for* $j=1,2,3,\ldots,g-2$,
(3) $\sum_{j=1}^{k} m_j \leqslant 0$ *for* $k=1,2,3,\ldots,g-3$,
(4) $\sum_{j=1}^{g-2} m_j=0$,
(5) $m_i+m_{i+1}=-1,0,+1$ *for* $i=1,2,3,\ldots,g-3$.

That is, two or more consecutive $+1$'s or -1's are not allowed.

In general, we have the following characterization.

Proposition 3.3. *Necessary and sufficient conditions for $\{m_1, m_2, \ldots, m_{g-1}\}$ to be the catastrophic sequence of a minimal CFP for a bidirectional linear array with link redundancy $G=\{1,2,\ldots,k,g\}$, $k<g$, are*

(1) $m_{g-1}=m_{g-2}=\cdots=m_{g-k+1}=0$,
(2) $m_j=-1,0,+1$ *for* $j=1,2,3,\ldots,g-k$,
(3) $\sum_{j=1}^{k} m_j \leqslant 0$ *for* $k=1,2,3,\ldots,g-k-1$,
(4) $\sum_{j=1}^{g-k} m_j=0$,
(5) $m_i+m_{i+1}+\cdots+m_{i+s}=-1,0,+1$ *for* $s=1,2,3,\ldots,k-1$ *for* $i=1,2,\ldots,g-k-s$.

Theorem 3.2. *The number of CFPs for a linear array with bidirectional bypass links of lengths 2 and g (i.e., with link redundancy $G=\{1,2,g\}$) is*

$$F^{\mathrm{B}}(1,2,g)=1+\sum_{n=1}^{\lfloor (g-2)/2 \rfloor}\sum_{r=1}^{n}\left[\binom{n-1}{r-1}^2-\binom{n-1}{r-2}\binom{n-1}{r}\right]\binom{g-2(n-r)-2}{2n}.$$

Proof. The number of catastrophic fault patterns is equal to the number of catastrophic sequences $\{m_1, m_2, \ldots, m_{g-2}\}$ satisfying the conditions of Proposition 3.2. Let the number of -1's (and so the number of $+1$'s) in the sequence be n. Clearly then the number of zeroes is $g-2-2n$. We start with a path from $(0,0)$ to $(2n,0)$ such that $S_1 \leqslant 0, S_2 \leqslant 0, \ldots, S_{2n-1} \leqslant 0$, $(S_{2n}=0)$ and having $2r$ runs. $R(\text{run})=1+$ number of changes of the type $(-1,+1)$ or $(+1,-1)$.

So, the number of paths from $(0,0)$ to $(2n,0)$ having $(2r-1)$ changes of the type $(-1,+1)$ or $(+1,-1)$ and satisfying $S_1 \leqslant 0, S_2 \leqslant 0, \ldots, S_{2n-1} \leqslant 0$, $(S_{2n}=0)$ is

$$\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r}.$$

All these paths have $2n-1-2r+1=2(n-r)$ identical pairs of the type $(+1,+1)$ or $(-1,-1)$. So, to satisfy condition (5) of Proposition 3.2, we have to plug in a 0 between every two consecutive $+1$'s and every two consecutive -1's. So the number of zeroes plugged in are $2(n-r)$. The remaining $g-2-2n-2(n-r)=g-4n+2r-2$ positions are also to be filled with 0's. There are $(2n+1)$ distinguishable positions in which $(g-4n+2r-2)$ 0's can be distributed in

$$\binom{g-2(n-r)-2}{2n}$$

ways. Since n can vary from 1 to $\lfloor (g-2)/2 \rfloor$, the total number of such paths is

$$\sum_{n=1}^{\lfloor (g-2)/2 \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r}\right]\binom{g-2(n-r)-2}{2n}.$$

Note that these paths do not include the trivial path corresponding to the sequence $(0,0,\ldots,0)$. Hence the theorem follows. □

The next theorem is a straight forward generalization of Theorem 3.2.

Theorem 3.3. *The number of CFPs for a linear array with bidirectional bypass links of length* $2,3,\ldots,k$ *and* g (*i.e., with link redundancy* $G=\{1,2,\ldots,k,g\}$, $k<g$) *is*

$$F^{\mathrm{B}}(1,2,3,\ldots,k,g) = 1 + \sum_{n=1}^{\lfloor (g-k)/2 \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r}\right] \times \binom{g-k-2(n-r)(k-1)}{2n}.$$

4. The case of unidirectional links

Proposition 4.1 (Pagli et al. [11]). *Necessary and sufficient conditions for* $(m_1, m_2, \ldots, m_{g-1})$ *to be the catastrophic sequence of a minimal CFP for a unidirectional linear array with link redundancy* $G=\{1,g\}$ *are*

(1) $m_j \leqslant 1$ *for* $j=1,2,\ldots,g-1$,
(2) $\sum_{j=1}^{k} m_j \leqslant 0$ *for* $k=1,2,\ldots,g-2$,
(3) $\sum_{j=1}^{g-1} m_j = 0$.

In general, we have the following characterization for $k > 1$.

Proposition 4.2. *Necessary and sufficient conditions for* $(m_1, m_2, \ldots, m_{g-1})$ *to be the catastrophic sequence of a minimal CFP for a unidirectional linear array with link redundancy* $G = \{1, 2, 3, \ldots, k, g\}$, $k < g$ *are*

(1) $m_{g-1} = m_{g-2} = \cdots = m_{g-k+1} = 0$,
(2) $m_j \leqslant 1$ *for* $j = 1, 2, \ldots, g-k$,
(3) $\sum_{j=1}^{p} m_j \leqslant 0$ *for* $p = 1, 2, 3, \ldots, g-k-1$,
(4) $\sum_{j=1}^{g-k} m_j = 0$,
(5) $m_i + m_{i+1} + \cdots + m_j \leqslant 1$ *if* $1 \leqslant j - i \leqslant k-1$.

Theorem 4.1 (De Prisco [6]). *The number of CFPs for a linear array with unidirectional bypass links of length g (i.e., with link redundancy* $G = \{1, g\}$*) is*

$$F^{\mathrm{U}}(1, g) = \frac{1}{g}\binom{2g-2}{g-1}.$$

Proof. The proof given here is little different from that of De Prisco [6] and is relevant to the proof of theorems to follow. The theorem is proved by establishing a bijective mapping between the set of all sequences $(m_1, m_2, \ldots, m_{g-1})$ satisfying the conditions of Proposition 4.1 and the set of all paths from $(0, 0)$ to $(2g-2, 0)$ such that $S_1 \leqslant 0, S_2 \leqslant 0, \ldots, S_{2g-3} \leqslant 0, S_{2g-2} = 0$ since the number of such paths is $(1/g)\binom{2g-2}{g-1}$. Let F be a CFP and $(m_1, m_2, \ldots, m_{g-1})$ be its catastrophic sequence. Let $s(m_i)$ be the string $-1, -1, \ldots, -1, +1$ of length $2 - m_i$. To the CFP F associate string $s(F)$ obtained by concatenating $s(m_1), s(m_2), \ldots, s(m_{g-1})$. From Proposition 4.1 (with $k = 1$), it is clear that $s(F)$ corresponds to a path from $(0, 0)$ to $(2g-2, 0)$ with the properties stated above. □

Theorem 4.2. *The number of CFPs for a linear array with unidirectional bypass links of lengths* 2 *and g (i.e., with link redundancy* $G = \{1, 2, g\}$*) is*

$$F^{\mathrm{U}}(1, 2, g) = \sum_{n=0}^{\lfloor (g-2)/2 \rfloor} \frac{1}{n+1}\binom{2n}{n}\binom{g-2}{2n}.$$

Proof. The number of CFPs is equal to the number of catastrophic sequences $(m_1, m_2, \ldots, m_{g-2})$ satisfying conditions (2)–(5) of Proposition 4.2 with $k = 2$. Given such a catastrophic sequence by using the above mapping we get $s(F) = (x_1, x_2, \ldots, x_{2(g-2)})$ with the following properties:

(1) $x_i = -1$ or $+1$ for $i = 1, 2, \ldots, 2(g-2)$,

(2) $\displaystyle\sum_{i=1}^{k} x_i \leqslant 0$ for $k = 1, 2, \ldots, 2(g-2) - 1$,

(3) $\sum_{i=1}^{2(g-2)} x_i = 0,$

(4) $x_i + x_{i+1} + x_{i+2} \leqslant 1 \quad \text{for } i = 1, 2, \ldots, 2(g-2) - 2.$ (A)

Given any sequence $(x_1, x_2, \ldots, x_{2(g-2)})$ satisfying (A), let y_i be the number of $+1$'s between the ith -1 and $(i+1)$th -1 and $z_i = y_i - 1$. Then (4), (2) and (3) of (A) give

(a) $z_i = -1, \ 0 \text{ or } +1 \quad \text{for } i = 1, 2, \ldots, g-3,$

(b) $z_1 + z_2 + \cdots + z_k \leqslant 0 \quad \text{for } k = 1, 2, \ldots, g-4,$

(c) $z_1 + z_2 + \cdots + z_{g-3} = 0 \quad \text{or} \quad -1,$ (B)

respectively. It is also clear that any sequence $(z_1, z_2, \ldots, z_{g-3})$ satisfying (B) arises from a unique sequence $(x_1, x_2, \ldots, x_{2(g-2)})$ satisfying (A). So the number of CFPs is equal to the number of sequences $(z_1, z_2, \ldots, z_{g-3})$ satisfying (B). Now the number of such sequences is equal to the number of sequences $(z_1, z_2, \ldots, z_{g-2})$ satisfying the following conditions (where z_{g-2} is taken to be 0 or 1 according as $z_1 + z_2 + \cdots + z_{g-3}$ is 0 or -1).

(a) $z_i = -1, \ 0 \text{ or } -1 \quad \text{for } i = 1, 2, \ldots, g-2,$

(b) $z_1 + z_2 + \cdots + z_k \leqslant 0 \quad \text{for } k = 1, 2, \ldots, g-3,$

(c) $\sum_{i=1}^{g-2} z_i = 0.$

Any such sequence with n $-$ 1's has n + 1's and $g - 2 - 2n$ 0's and can be obtained by starting with a path from $(0,0)$ to $(2n, 0)$ such that $S_1 \leqslant 0, S_2 \leqslant 0, \ldots, S_{2n-1} \leqslant 0, S_{2n} = 0$ and plugging in $g - 2 - 2n$ 0's in the $(2n+1)$ distinct places $(2n-1$ intermediate places and two more places before and after the sequence). Clearly the number of such paths is $[1/(n+1)] \binom{2n}{n}$ and for each path, the number of ways of plugging in the 0's is

$$\binom{(g-2-2n) + (2n+1) - 1}{2n} = \binom{g-2}{2n}.$$

Hence the theorem follows. □

Theorem 4.3. *The number of CFPs for a linear array with unidirectional bypass links of lengths* 2, 3 *and* g (*i.e., with link redundancy* $G = \{1, 2, 3, g\}$) *is*

$$F^{\mathrm{U}}(1,2,3,g) = 1 + \sum_{n=1}^{\lfloor (g-3)/2 \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r}\binom{n-1}{r-2} \right] \binom{g-3-(n-r)}{2n}.$$

Proof. The number of CFPs is equal to the number of catastrophic sequences $(m_1, m_2, \ldots, m_{g-3})$ satisfying conditions (2)–(5) of Proposition 4.2 with $k = 3$. Given a catastrophic sequence by using the mapping given in the proof of Theorem 4.1, we get $s(F) = (x_1, x_2, \ldots, x_{2(g-3)})$ with the following properties:

(1) $x_i = -1$ or $+1$ for $i = 1, 2, 3, \ldots, 2(g-3)$,

(2) $\sum_{i=1}^{k} x_i \leqslant 0$ for $k = 1, 2, \ldots, 2(g-3) - 1$,

(3) $\sum_{i=1}^{2(g-3)} x_i = 0$,

(4.1) $x_i + x_{i+1} + x_{i+2} \leqslant 1$ for $i = 1, 2, \ldots, 2(g-3) - 2$ and

(4.2) $x_i + x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4} \leqslant 1$ for $i = 1, 2, \ldots, 2(g-3) - 4$. (C)

Given any sequence $(x_1, x_2, \ldots, x_{2(g-3)})$ satisfying (C), let y_i be the number of $+1$'s between ith -1 and $(i+1)$th -1 and $z_i = y_i - 1$. Then (4.1), (4.2), (2) and (3) of (C) give

(a.1) $z_i = -1$, 0 or $+1$ for $i = 1, 2, \ldots, g-4$,

(a.2) $-2 \leqslant z_i + z_{i+1} \leqslant 1$ for $i = 1, 2, \ldots, g-5$,

(b) $z_1 + z_2 + \cdots + z_k \leqslant 0$ for $k = 1, 2, \ldots, g-5$,

(c) $z_1 + z_2 + \cdots + z_{g-4} = -1$ or 0. (D)

respectively. It is also clear that any sequence $(z_1, z_2, \ldots, z_{g-4})$ satisfying (D) arises from a unique sequence $(x_1, x_2, \ldots, x_{2(g-3)})$ satisfying (C). So the number of CFPs is equal to the number of sequences $(z_1, z_2, \ldots, z_{g-4})$ satisfying (D). Now the number of such sequences is equal to the number of sequences $(z_1, z_2, \ldots, z_{g-3})$ satisfying the following conditions:

(a.1) $z_i = -1$, 0 or $+1$ for $i = 1, 2, \ldots, g-3$,

(a.2) $-2 \leqslant z_i + z_{i+1} \leqslant 1$ for $i = 1, 2, \ldots, g-4$,

(b) $z_1 + z_2 + \cdots + z_k \leqslant 0$ for $k = 1, 2, \ldots, g-4$,

(c) $\sum_{i=1}^{g-3} z_i = 0$. (E)

Any such sequence with n -1's has n $+1$'s and $g-3-2n$ 0's and can be obtained by starting with a path from $(0,0)$ to $(2n,0)$ such that $S_1 \leqslant 0, S_2 \leqslant 0, \ldots, S_{2n-1} \leqslant 0, S_{2n} = 0$ and having exactly $2r$ runs. The number of such paths is

$$\binom{n-1}{r-1}^2 - \binom{n-1}{r}\binom{n-1}{r-2}$$

by Theorem 2.3 $R(\text{run}) = 1 +$ number of changes of the type $(-1,+1)$ or $(+1,-1)$. All these paths have $(n-r)$ identical pairs of the type $(+1,+1)$ and $(n-r)$ identical pairs of the type $(-1,-1)$. Now to satisfy condition (a.2) of (E) we have to plug in a 0 between every two consecutive $+1$'s. So the number of 0's plugged in is $(n-r)$. The remaining places are also to be filled up with 0's. There are $(2n+1)$ distinct places for each such path in which $g-3-2n-(n-r)$ 0's can be plugged in

$$\binom{g-3-(n-r)}{2n}$$

ways. Since n can vary from 1 to $\lfloor (g-3)/2 \rfloor$, the total number of such paths is

$$\sum_{n=1}^{\lfloor (g-3)/2 \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r}\binom{n-1}{r-2} \right] \binom{g-3-(n-r)}{2n}.$$

Note that these paths do not include the trivial path corresponding the sequence $(0,0,\ldots,0)$. Hence the theorem follows. $\square$

Theorem 4.4. *The number of CFPs for a linear array with unidirectional bypass links of length* $2,3,\ldots,k$ *and* g (*i.e., with link redundancy* $G=\{1,2,\ldots,k,g\}$) *is*

$$F^{\mathrm{U}}(1,2,3,\ldots,k,g) = 1 + \sum_{n=1}^{\lfloor (g-k)/2 \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r}\binom{n-1}{r-2} \right] \times \binom{g-k-(n-r)(k-2)}{2n}.$$

Proof. The number of CFPs is equal to the number of catastrophic sequences $(m_1, m_2, \ldots, m_{g-k})$ satisfying conditions (2)–(5) of Proposition 4.2. By using the same argument as in the proof of Theorem 4.3, the number of such sequences is equal to the number of the sequences $(z_1, z_2, \ldots, z_{g-k})$ satisfying the following conditions:

(a.1) $z_i = -1,\ 0$ or $+1$ for $i = 1,2,\ldots,g-k$,

(a.2) $-2 \leqslant z_i + z_{i+1} \leqslant 1$ for $i = 1,2,\ldots,g-k-1$

$\vdots$

(a.$k-1$) $-(k-1) \leqslant z_i + z_{i+1} + \cdots + z_{i+k-2} \leqslant 1$ for $i = 1,2,\ldots,g-2k+2$

(b) $z_1 + z_2 + \cdots + z_p \leqslant 0$ for $p = 1,2,\ldots,g-k-1$

(c) $\sum_{i=1}^{g-k} z_i = 0.$ (F)

Counting the number of such sequences is done as in Theorem 4.3 except that instead of plugging in one 0 we have to plug in $(k-2)$ 0's between any two consecutive $+1$'s to satisfy conditions (a.2)–(a.$k-1$) of (F). Hence the theorem follows. $\square$

Acknowledgements

We are grateful to Professor A.R. Rao for his suggestions.

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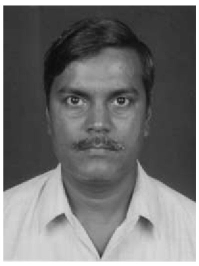

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