

# Characterization of catastrophic faults in two-dimensional reconfigurable systolic arrays with unidirectional links

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## Abstract

The catastrophic fault pattern is a pattern of faults occurring at strategic locations that may render a system unusable regardless of its component redundancy and of its reconfiguration capabilities. In this paper, we extend the characterization of catastrophic fault patterns known for linear arrays to two-dimensional VLSI arrays in which all links are unidirectional. We determine the minimum number of faults required for a fault pattern to be catastrophic and give algorithm for the construction of catastrophic fault patterns with minimum number of faults.

*Keywords:* Catastrophic fault patterns; Fault tolerance; Algorithms

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## 1. Introduction

Systolic systems have been widely used as parallel models of computation [5,7]. These systems consist of a large number of identical and elementary processing elements locally connected in a regular fashion. Each element receives data from its neighbors, computes and sends the results again to its neighbors. Few par-

ticular elements located at the extremes of the systems (these extremes depend on the particular system) are allowed to communicate with the external world. In this paper, we will focus on systolic two-dimensional array with unidirectional links.

Fault tolerant techniques are very important to systolic systems. Here we assume that only processors can fail. Indeed, since the number of processing elements is very large, the probability that a set of processing elements becomes faulty is fairly high. Thus, fault-tolerant mechanisms must be provided in order to avoid that faulty processing elements take

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part in the computation. A widely used technique to achieve reconfigurability consists of providing redundancy to the desired architecture [1,2]. In systolic arrays the redundancy consists of additional processing elements, called spares, and additional connections, called bypass links. The redundant processing elements are used to replace any faulty processing element; the redundant links are used to bypass the faulty processing elements and reach others. The effectiveness of using redundancy to increase fault tolerance clearly depends on both the amount of redundancy and the reconfiguration capability of the system. It does however depend also on the distribution of faults in the system. There are sets of faulty processing elements for which no reconfiguration strategy is possible. Such sets are called catastrophic fault patterns (CFPs).

If we have to reconfigure a system when a fault set occurs, it is necessary to know if the set is catastrophic or not. Therefore it is important to study the properties of catastrophic sets. Till today, the characterization of CFPs is known only for linear arrays with the following results. The characterization has been used to obtain efficient testing algorithms both for unidirectional and bidirectional cases [9] with order of magnitude improvement over [3,4]. Efficient techniques have been obtained for constructing CFPs [8]. Recently, using random walk as a tool, a closed form solution for the number of CFPs for bidirectional links has been provided in [6], an improvement over [11,4].

The main contribution of this paper is complete characterization of catastrophic fault patterns for two-dimensional arrays with unidirectional links. The requirement on the minimum number of faults in a fault pattern for it to be catastrophic is shown to be a function of the length of the longest horizontal bypass link and the number of rows in the two-dimensional array. From a practical viewpoint, above result allows to prove some answers to the question about the guaranteed level of fault tolerance of a design. Guaranteed fault tolerance indicates positive answer to the question as: will the system withstand up to  $k$  faults always regardless of how and where they occur? We analyze catastrophic faults having the minimal number of defective processors. Throughout this paper, by catastrophic fault pattern we mean a catastrophic fault pattern having minimum number of faulty processing elements. We describe algorithm for constructing a catastrophic fault pattern with maximum width

(i.e., the distance between the first and the last fault in the fault pattern). The width of a fault pattern must fall within precise bounds for the pattern to be catastrophic. This algorithm gives us the framework for achieving the upper bounds on the width of a catastrophic fault pattern for different link configurations.

## 2. Preliminaries

In this paper, we will focus on *two-dimensional networks*. The basic components of such a network are the processing elements (PEs) indicated by circles in Fig. 1. The links are unidirectional. There are two kinds of links: *regular* and *bypass*. Regular links connect neighboring (either horizontal or vertical) PEs while bypass links connect non-neighbors. The bypass links are used strictly for reconfiguration purposes when a fault is detected, otherwise they are considered to be the redundant links. We now introduce the following definitions:

**Definition 2.1.** A two-dimensional network  $\mathcal{N}$  consists of a set  $V$  of PEs and a set  $E$  of links (where a link joins a pair of distinct PEs) satisfying the conditions listed below.

$V$  is the union of three disjoint sets: the set  $ICUL = \{ICUL_1, ICUL_2, \dots, ICUL_{N_1}\}$  of left interface control units, the set  $ICUR = \{ICUR_1, ICUR_2, \dots, ICUR_{N_1}\}$  of right interface control units and a two-dimensional array  $A = \{p_{ij} : 1 \leq i \leq N_1, 1 \leq j \leq N_2\}$  of PEs. We sometimes refer to the processing element  $p_{ij}$  as  $(i, j)$ .

$E$  consists of the links obtained as follows. Fix integers  $1 = g_1 < g_2 < \dots < g_k \leq N_2 - 1$  and  $1 = v_1 < v_2 < \dots < v_l \leq N_1 - 1$ . Join  $p_{ij}$  to  $p_{i'j'}$  by a link if and only if

- (i)  $i = i'$  and  $(j - j')$  is one of  $g_1, g_2, \dots, g_k$  or
- (ii)  $j = j'$  and  $(i - i')$  is one of  $v_1, v_2, \dots, v_l$ .

Also join  $ICUL_i$  to  $p_{i2}, \dots, p_{ig_k}$  and join  $p_{i, N_2 - g_k + 1}, p_{i, N_2 - g_k + 2}, \dots, p_{i, N_2 - 1}$  to  $ICUR_i$  by links, for  $i = 1, 2, \dots, N_1$ .

We assume that  $N_1 > v_l$  and  $N_2 > g_k(N_1 - 1) + (g_k - 1)^2 + 1$ . The idea behind this specific choice of lower bound for  $N_2$  is given in Result 3.1.

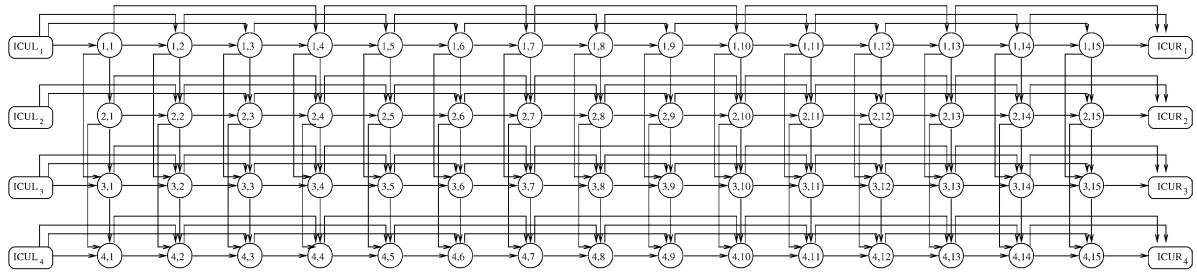


Fig. 1. Two-dimensional network of PEs.

**Definition 2.2.** We call  $g_1, g_2, \dots, g_k$  the horizontal link redundancies of  $\mathcal{N}$  and  $v_1, v_2, \dots, v_l$  the vertical link redundancies of  $\mathcal{N}$ . We refer to  $G = (g_1, g_2, \dots, g_k | v_1, v_2, \dots, v_l)$  as the link redundancy of  $\mathcal{N}$ .

Fig. 1 shows a two-dimensional network with  $N_1 = 4, N_2 = 15$  and  $G = (1, 3 | 1, 2)$ . A link joining two PEs of the type  $p_{ij}$  and  $p_{i,j+1}$  is called a horizontal direct link and a link joining two PEs of the type  $p_{ij}$  and  $p_{i+1,j}$  is called a vertical direct link. Direct links are also called regular links. Links joining  $p_{ij}$  and  $p_{i,j+g}$  with  $g > 1$  are called horizontal bypass links and links joining  $p_{ij}$  and  $p_{i+v,j}$  with  $v > 1$  are called vertical bypass links.

The length of the horizontal bypass link joining  $p_{ij}$  to  $p_{i,j+g}$  is  $g$  and the length of the vertical bypass link joining  $p_{ij}$  to  $p_{i+v,j}$  is  $v$ .

We assume that the direction of information flow through horizontal links is from left to right and the direction of information flow through vertical links is from top to bottom of the array. Note that, no links exist in the network  $\mathcal{N}$  except the ones specified by  $G$  as in Definition 2.1. It is assumed that  $ICUL$  and  $ICUR$  always operate correctly and we are considering information flow from  $ICUL$  to  $ICUR$ .

**Definition 2.3.** Given a two-dimensional array  $A$ , a fault pattern (FP) for  $A$  is simply a non-empty subset  $F$  of the set of processing elements in  $A$ . An assignment of a fault pattern  $F$  to  $A$  means that every processing element belonging to  $F$  is faulty (and the others operate correctly).

Given a fault pattern  $F$ , define  $m = \min\{j : (i, j) \in F\}$  and  $M = \max\{j : (i, j) \in F\}$ .

**Definition 2.4.** The window  $W_F$  of a fault pattern  $F$  is the sub-array of  $A$  consisting of  $\{p_{ij} : 1 \leq i \leq N_1, m \leq j \leq M\}$ . By the width  $\|W_F\|$  of  $F$  we mean  $M - m + 1$ .

**Definition 2.5.** A fault pattern is catastrophic for the network  $\mathcal{N}$  if  $ICUL$  and  $ICUR$  are not connected (i.e., there is no path connecting any  $ICUL_i$  to any  $ICUR_j$  which does not involve a faulty PE) when the fault pattern  $F$  is assigned to  $A$ .

**Example 2.1.** Consider the fault pattern  $F = \{(1, 5), (1, 6), (1, 8), (1, 11), (2, 5), (2, 8), (2, 10), (2, 11), (3, 6), (3, 8), (3, 9), (3, 11), (4, 7), (4, 8), (4, 10), (4, 13)\}$  with unidirectional link redundancy  $G = (1, 4 | 1)$  in a  $4 \times 24$  array as shown in Fig. 2. Links and some processors are not drawn in the figure.

We see from Fig. 3, that the removal of the processing elements belonging to  $F$  along with their inci-

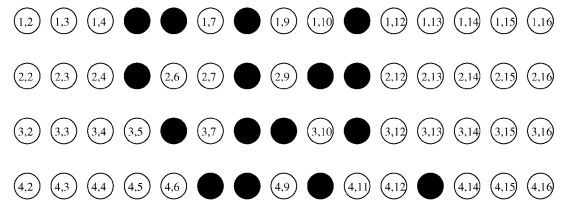


Fig. 2. Fault pattern  $F$ .

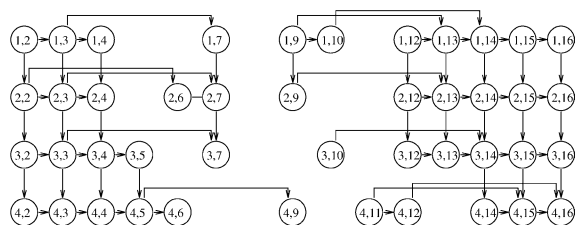


Fig. 3. Network  $\mathcal{N}$  after the removal of  $F$  and their incident links.

dent links disconnects *ICUL* and *ICUR*. Hence *F* is catastrophic. It is easy to check that *F* remains catastrophic with respect to unidirectional link redundancy  $G = (1, 4 | 1, 2)$ .

### 3. Characterization of catastrophic fault patterns

In this section, we will characterize the catastrophic fault patterns for two-dimensional networks and prove that the minimum number of faults in a catastrophic fault pattern is a function of  $N_1$  and the length of the longest horizontal bypass link.

**Theorem 3.1.** *F* is catastrophic with respect to  $\mathcal{N}$  implies that the cardinality of *F*,  $|F| \geq N_1 g_k$ .

**Proof.** Suppose to the contrary that  $|F| < N_1 g_k$ . Then partition the two-dimensional array *A* of PEs into blocks of  $g_k$  columns as  $A = (A_1 \vdots A_2 \vdots \dots \vdots A_c)$  where  $c = \lceil N_2/g_k \rceil$  and place the blocks as consecutive floors to form a cuboid as shown in Fig. 4. Observe that, in this cuboid representation, each horizontal regular link joins two consecutive elements in the same row of a floor or the last element of a row of a floor with the first element of the same row of the floor just above it whereas each vertical regular link joins two consecutive elements in the same column of a floor. On the other hand, each horizontal bypass link of the maximum length joins two consecutive elements in the same pillar. So, in this cuboid, going up along a pillar corresponds to using the longest horizontal bypass links. Since the number of faulty elements  $|F|$  is less than the number of pillars, there must be a pillar with no faulty element, regardless of the distribution of the fault pattern. Since the bottom and top of each pillar are linked to *ICUL* and *ICUR*, respectively, *F* cannot be catastrophic since we can use the bypass links of length  $g_k$  to avoid the faulty PEs, a contradiction which proves the theorem.  $\square$

This theorem gives us a necessary condition on the minimum number of faults required for blocking a two-dimensional array. This also tells us that fewer than  $N_1 g_k$  faults occurring in *A* will not be catastrophic. In the following we will restrict ourselves to the case where there are at least  $N_1 g_k$  faults, and we

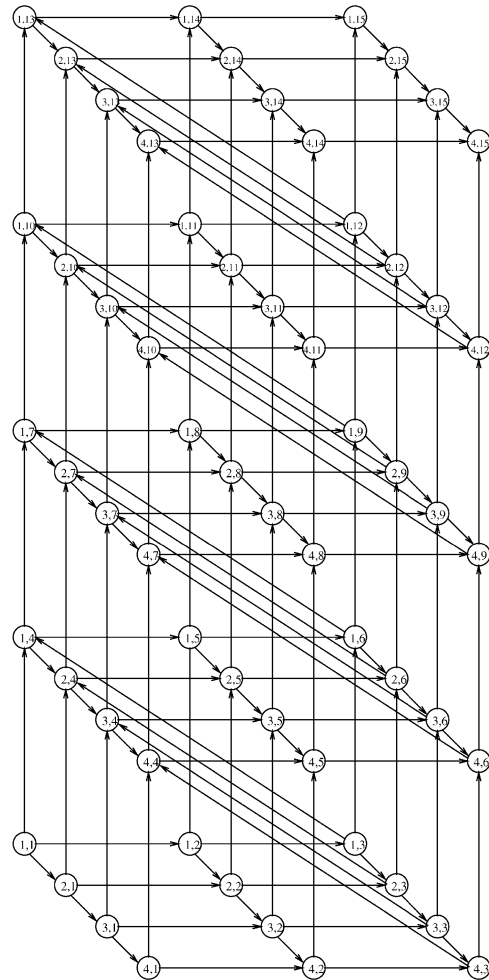


Fig. 4. Cuboid representation of a  $4 \times 15$  array with link redundancy  $G = (1, 3 | 1)$ .

will characterize the blocking fault patterns containing exactly  $N_1 g_k$  faults.

Not all fault patterns consisting of  $N_1 g_k$  faults are catastrophic. Some additional properties must be satisfied. Before we describe further characteristics of a catastrophic fault pattern, we outline an algorithm for the construction of a CFP with the maximum width for a given link redundancy *G* when links are unidirectional. Recall that,  $N_2$  is very large compare to  $g_k(N_1 - 1) + (g_k - 1)^2 + 1$  and  $N_1 > v_l$ . For simplicity, we also assume that  $N_2$  is divisible by  $g_k$ .

**Algorithm UCFP.** Construction of a catastrophic fault pattern for unidirectional horizontal and unidirectional vertical links.

*Input:*  $G$ .

*Output:* A catastrophic fault pattern  $F$  with the maximum width.

*Step 1.* Partition the two-dimensional array of PEs into blocks of  $g_k$  columns and list the blocks as the floors of a cuboid. Mark the first element of the  $N_1$ th row in floor 0 by an  $X$  and set  $f = 1$ .

*Step 2.* If there exists an unmarked element  $u = (i, j)$  in floor  $f$  such that the element  $v = (i, j - g_k)$  below  $u$  in floor  $f - 1$  is marked, choose one such  $u$  and go to Step 4. Otherwise go to Step 3.

*Step 3.* If there is an unmarked element in floor  $f$ , then increase  $f$  by 1 and go to Step 2. Otherwise, go to Step 5.

*Step 4.* If  $v$  is marked  $Y$ , then mark  $u$  by  $Y$  and go to Step 2. If  $v$  is marked  $X$ , then mark  $u$  by  $Y$  and mark every unmarked element  $w$  which is of the form  $(i, j - g)$  where  $g \in \{g_1, g_2, \dots, g_{k-1}\}$  or  $(i - v, j)$  where  $v \in \{v_1, v_2, \dots, v_l\}$ . Mark  $w$  by  $Y$  if the pillar of  $w$  contains another marked element; otherwise mark  $w$  by  $X$ . Go to Step 2.

*Step 5.* Stop. Note that all elements in floor  $f$  are marked. The elements marked  $X$  form a catastrophic fault pattern  $F$  with maximum width for link redundancy  $G$ .

**Observations.** (1) Note that the algorithm assigns exactly  $N_1 g_k$  number of  $X$ 's.

(2) In Step 3, if there is an unmarked element in floor  $f$ , then there always exists a floor above floor  $f$ . The reason is this. Since  $N_2 > g_k(N_1 - 1) + (g_k - 1)^2 + 1$ , there are at least  $N_1 + g_k - 3$  floors in the cuboid representation. Note that, the elements in floor  $N_1 + g_k - 3$  are all marked. Therefore, a floor  $f$  with some unmarked elements must be below the floor  $N_1 + g_k - 3$ , and hence there exists a floor above that floor.

**Example 3.1.** Fig. 5 shows a CFP obtained by the above algorithm consisting of 16 faults corresponding to  $G = (1, 4 | 1, 3)$  in a  $4 \times 24$  array  $A$  when the links are unidirectional.

**Theorem 3.2.** Algorithm UCFP generates a catastrophic fault pattern.

**Proof.** We make the following simple observations on the algorithm:

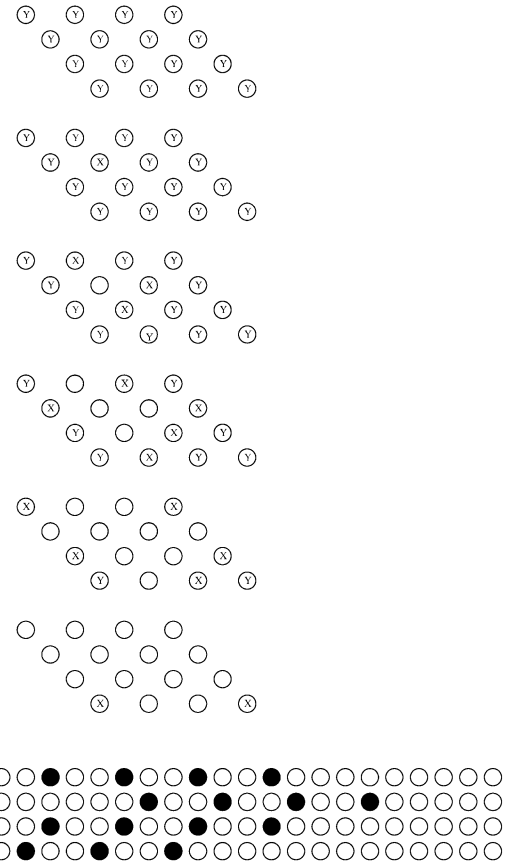


Fig. 5. Fault pattern for  $G = (1, 4 | 1, 3)$ .

- Marking takes place only in Step 4. When some PE is marked there are two cases: (a) if the pillar has no marked PE then the current PE is marked by  $X$  and (b) if the pillar has at least one marked PE then all marked PEs are below the current PE and the current PE is marked by  $Y$ . When the algorithm terminates, we also have the following:
- There is exactly one  $X$  in each pillar, hence  $|F| = N_1 g_k$ .
- For each pillar the  $X$  occurs below the  $Y$ 's.
- Let the final value of  $f$  be  $f_0$ . If a PE  $p$  is marked  $Y$  and is adjacent to a PE  $q$  then  $q$  is marked (with  $X$  or  $Y$ ) unless  $p$  is in floor  $f_0$  and  $q$  is in a floor  $f_0 + 1$ .

We next prove that any PE  $\omega_1$  marked with a  $Y$  is inaccessible, i.e., there is no way to reach this PE from ICUL without using any faulty PEs (those marked

with  $X$ ). Suppose  $\omega_1$  is accessible, i.e.,  $\omega_1$  is connected to  $ICUL$  by a path  $\mu = [ICUL, \dots, \omega_2, \omega_1]$  not containing any faulty PEs (those marked  $X$ ). We consider two cases:

*Case 1.*  $\omega_1, \omega_2, \dots$  are all marked  $Y$ . Now, the PEs adjacent to  $ICUL$  all lie in the 0th floor. But no PE in the 0th floor can be marked  $Y$  by the algorithm, a contradiction which proves that  $\omega_1$  is not accessible from  $ICUL$ .

*Case 2.*  $\omega_1, \omega_2, \dots, \omega_i$  are marked  $Y$  and  $\omega_{i+1}, \omega_{i+2}, \dots$  are unmarked. But there is no direct link from an unmarked PE to a PE marked with a  $Y$ , a contradiction which proves that  $\omega_1$  is not accessible from  $ICUL$ .

Clearly when the algorithm terminates, all PEs in floor  $f_0$  are either faulty or inaccessible from  $ICUL$ , so no PE in any floor  $\geq f_0 + 1$  is accessible from  $ICUL$ ; in particular,  $ICUR$  is not accessible from  $ICUL$  and  $F$  is a CFP.

We prove that the algorithm terminates by showing that the number of marked elements in a floor increases strictly until it reaches  $N_{1g_k}$ . Note that all elements of the floor  $f + 1$ , which are above marked elements in floor  $f$ , are marked. If there is an unmarked element in the  $f$ th floor, note that some marked element  $v$  is adjacent to an unmarked element  $z$  in the  $f$ th floor. Then, by the algorithm, the element  $w$  in floor  $f + 1$  above  $z$  will be marked.  $\square$

**Conjecture 3.1.** *The catastrophic fault pattern  $F$ , generated by Algorithm UCFP, has the maximum width of the window  $W_F$ .*

We now indicate our basis for the above conjecture. Let  $F^*$  be any catastrophic fault pattern with  $N_{1g_k}$  faulty PEs. We assume that PE  $(N_1, 1)$  belongs to  $F^*$ . We now consider the cuboid representation of  $A$  used in the proof of Theorem 3.1. Note that, since there are only  $N_{1g_k}$  faulty PEs, each pillar can contain only one faulty PE. Now mark the faulty PEs by  $X$ 's and in each pillar mark the PEs above the faulty PE by  $Y$ 's. A PE  $(i, j)$  marked by  $X$  and  $\neq (N_1, 1)$  is called *fair PE* if no PE among the PEs  $(i, j - g)$  where  $g \in \{g_1, g_2, \dots, g_{k-1}\}$  or  $(i - v, j)$  where  $v \in \{v_1, v_2, \dots, v_l\}$  is marked by  $Y$ . In other words, a PE (marked by  $X$ ) is called *fair PE* if it is not accessible from any PE marked by  $Y$  using the links in  $G$ . In the given fault pattern  $F^*$ , if there exists a fair PE  $(i, j)$ , we obtain a new fault pattern called *derived fault pat-*

*tern* by making the PE  $(i, j + g_k)$  faulty instead of PE  $(i, j)$ . Then PE  $(i, j + g_k)$  becomes faulty whereas PE  $(i, j)$  becomes accessible from  $ICUL$ . Note that, the derived fault pattern is still catastrophic and its width is greater than or equal to the width of  $F^*$ . If there is any fair PE  $(i', j')$  in the derived fault pattern then make PE  $(i', j' + g_k)$  faulty instead of PE  $(i', j')$ . Continuing this process we get a derived fault pattern  $F^0$  with no fair PE. It is easy to check that this process terminates in finite number of steps. Clearly  $\|W_{F^0}\| \geq \|W_{F^*}\|$ . Let  $F$  be the fault pattern obtained from Algorithm UCFP. Then we think  $F$  will be the only catastrophic fault pattern which does not have any fair PE, so  $F^0 = F$  and  $\|W_F\| \geq \|W_{F^*}\|$ . Since  $F^*$  is arbitrary,  $F$  has the maximum width.

This conjecture gives us the framework for achieving specific upper bounds and exact bounds on the size of the largest window for a given link configuration. Given a link configuration  $G$ , we can obtain, by applying the algorithm, a catastrophic fault pattern  $F$  which is contained in the largest window; that is,  $W_F$  is the maximum value possible.

We study the effect of  $G$  on the maximum width of window of a CFP. We start by showing that the window size decreases as the size of  $G$  increases.

**Theorem 3.3.** *Let  $G = (G_1 | G_2)$  and  $G' = (G'_1 | G'_2)$  be two link redundancies with the same largest horizontal link redundancy. If  $G_1 \subseteq G'_1$  and  $G_2 \subseteq G'_2$  and  $W_F$  and  $W'_F$  are the corresponding widest fault windows, then  $\|W_F\| \geq \|W'_F\|$ .*

**Proof.** In Algorithm UCFP, we note that there will be more  $X$ 's for  $G'$  than for  $G$  in each floor, so the algorithm terminates sooner. Hence the final value of  $f$  for  $G'$  will be less than or equal to the final value of  $f$  for  $G$ . Note that the width of window of a CFP increases as the final value of  $f$  increases. Hence the theorem follows.  $\square$

We now present some results which give the maximum width of a window of a CFP when there are at most two horizontal and at most two vertical link redundancies and an upper bound for the width of a window of a CFP in the general case. These results follow directly from Conjecture 3.1 and application of Algorithm UCFP.

**Result 3.1.** Let  $G = (1, g | 1)$ . Then, the maximum width of the window of a CFP with  $N_1 g$  faults is  $g(N_1 - 1) + (g - 1)^2 + 1$ .

**Proof.** Let  $F$  be the catastrophic fault pattern with respect to link redundancy  $G = (1, g | 1)$  generated by Algorithm UCFP. Let  $F_i \subseteq F$  be the set of faulty PEs occur only in the  $i$ th row of  $A$ . Note that all  $F_i$ 's are identical and of width  $\lceil 10 \rceil (g - 1)^2 + 1$ . We observe that, if  $F_i$  begins at PE  $(i, j)$  then  $F_{i-1}$  will begin at PE  $(i - 1, j + g)$  for  $2 \leq i \leq N_1$  and Algorithm UCFP starts at PE  $(N_1, 1)$ . Therefore  $F$  begins at PE  $(N_1, 1)$  and ends at PE  $(1, g(N_1 - 1) + (g - 1)^2 + 1)$ .  $\square$

**Result 3.2.** Let  $G = (1, g | 1, v)$  and  $v$  divides  $(N_1 - 1)$ . Then, the maximum width of the window of a CFP with  $N_1 g$  faults is given by  $g((N_1 - 1)/v + v - 2) + (g - 1)^2 + 1$ .

**Proof.** The proof of the present result is similar to that of previous result. Let  $F$  be the catastrophic fault pattern with respect to link redundancy  $G = (1, g | 1, v)$  generated by Algorithm UCFP. Here we observe that if  $F_{rv}$ ,  $r = 1, 2, \dots, (N_1 - 1)/v$ , begins at PE  $(rv, j)$  then  $F_i$ ,  $rv - v + 2 \leq i \leq rv - 1$ , will begin at PE  $(i, j + g(rv - l))$  and  $F_{rv-v+1}$  will begin at PE  $(rv - v + 1, j)$ . Note that Algorithm UCFP starts at PE  $(N_1, 1)$ . Now it is easy to check that  $F$  begins at PE  $(N_1, 1)$  and ends at PE  $(2, g((N_1 - 1)/v + v - 2) + (g - 1)^2 + 1)$ .  $\square$

In view of Theorem 3.3, when  $G = (1, g_2, \dots, g_k | 1, v_2, \dots, v_l)$ , we get an upper bound for the width of the window of a CFP with  $N_1 g_k$  faults by replacing  $g$  and  $v$  by  $g_k$  and  $v_l$ , respectively in the expression given in the preceding result. A similar statement holds for Result 3.1.

#### 4. Cuboid representation for fault pattern

Suppose we are given a fault pattern  $F$  with  $N_1 g_k$  faults in a two-dimensional array with link redundancy  $G = (g_1, g_2, \dots, g_k | v_1, v_2, \dots, v_l)$ . Without loss generality we will assume that the first column of  $A$  contains a fault. We now consider the cuboid representation of  $A$  used in the proof of Theorem 3.1. However, we label the  $N_1$  rows in any floor of the cuboid

with  $0, 1, \dots, N_1 - 1$  instead of  $1, 2, \dots, N_1$  and  $g_k$  columns in any floor with  $0, 1, \dots, g_k - 1$  instead of  $1, 2, \dots, g_k$ . The floors are labelled using  $0, 1, 2, \dots$  as before. With every PE  $(i, j)$  we can uniquely associate the triple  $(x, y, z)$  where  $x, y$  and  $z$  are the row label, column label and floor label of the position  $(i, j)$  occupies in the cuboid. (Note that  $x = i - 1$ ,  $y$  is the remainder obtained when  $j - 1$  is divided by  $g_k$  and  $z$  is  $\lfloor (j - 1)/g_k \rfloor$ .) We will write

$$W(x, y, z) = \begin{cases} 1 & \text{if } (i, j) \in F, \\ 0 & \text{otherwise.} \end{cases}$$

We will some time refer to  $(x, y, z)$  as the *location of the PE*  $(i, j)$ .

Suppose now  $F$  is a fault pattern such that for any  $(x, y)$ , there is exactly one  $z$  for which  $W(x, y, z) = 1$  (i.e., there is exactly one faulty PE in each pillar). We then denote this  $z$  by  $h_{xy}$  and call the matrix

$$H = \begin{pmatrix} h_{00} & h_{01} & \cdots & h_{0, g_k - 1} \\ h_{10} & h_{11} & \cdots & h_{1, g_k - 1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N_1 - 1, 0} & h_{N_1 - 1, 1} & \cdots & h_{N_1 - 1, g_k - 1} \end{pmatrix}$$

the height matrix of  $F$ .

**Example 4.1.** Consider the CFP  $F = \{(1, 5), (1, 8), (1, 11), (1, 14), (2, 9), (2, 12), (2, 15), (2, 18), (3, 5), (3, 8), (3, 11), (3, 14), (4, 1), (4, 4), (4, 7), (4, 10)\}$  with 16 faults for a two-dimensional array  $A$  with link redundancy  $G = (1, 4 | 1, 3)$  which has  $\|W_F\| = 18$  as shown in Fig. 5.

The height matrix for this CFP is

$$H = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 4 & 3 & 2 \\ 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}.$$

Note that every minimal CFP (i.e., CFP with  $N_1 g_k$  faulty PEs) satisfies the conditions stated at the beginning of the preceding paragraph. We now define the interior, exterior and border elements in the cuboid representation of a minimal CFP.

**Definition 4.1.** Let  $F$  be a minimal CFP. Then the PE of  $A$  corresponding to the location  $(x, y, z)$  is said to be *interior*, *border* or *exterior* with respect to  $F$  according as  $z < h_{xy}$ ,  $z = h_{xy}$  or  $z > h_{xy}$ . The interior  $I(F)$  of  $F$ , the border  $B(F)$  of  $F$  and the exterior

$E(F)$  of  $F$  are defined to be the set of all interior elements, the set of all border elements and the set of all exterior elements of  $F$ , respectively.

**Lemma 4.1.** *A fault pattern  $F$  is catastrophic for a two-dimensional network  $\mathcal{N}$  with unidirectional link redundancy  $G$  iff it is not possible to reach any exterior element from any interior element using the links in  $\mathcal{N}$ .*

**Proof.** It is easy to see that all interior elements are reachable from  $ICUL$  and  $ICUR$  is reachable from all exterior elements. The lemma follows from Definition 2.5.  $\square$

## 5. Necessary and sufficient conditions for catastrophic fault patterns

We give the necessary and sufficient conditions for the existence of minimal CFPs in terms of the height matrix.

**Proposition 5.1.** *An  $N_1 \times g_k$  matrix  $H = ((h_{xy}))$  with non-negative integer entries is the height matrix of a minimal CFP for  $\mathcal{N}$  with unidirectional link redundancy  $G = (1, g | 1)$  iff the following conditions are satisfied:*

- (i)  $h_{x0} - h_{x,g-1} = 0$  or  $+1$  for all  $x$  such that  $0 \leq x \leq N_1 - 1$ ;  $h_{x0} = h_{x,g-1} = 0$  for at least one  $x$ .
- (ii)  $h_{xy} - h_{x,y+1} \leq 1$  whenever  $0 \leq x \leq N_1 - 1$  and  $0 \leq y \leq g - 2$  and
- (iii)  $h_{xy} - h_{x+1,y} \leq 1$  whenever  $0 \leq x \leq N_1 - 2$  and  $0 \leq y \leq g - 1$ .

Clearly the number of minimal CFPs for  $\mathcal{N}$  with unidirectional link redundancy  $G = (1, g | 1)$  is equal to the number of height matrices  $H$  which satisfy the conditions of Proposition 5.1. We shall illustrate Proposition 5.1 by an example.

**Example 5.1.** Consider the fault pattern  $F = \{(1, 1), (1, 4), (1, 7), (1, 14), (2, 1), (2, 7), (2, 8), (2, 10), (3, 5), (3, 8), (3, 11), (3, 14)\}$  in a  $3 \times 20$  array  $A$  with link redundancy  $G = (1, 4 | 1)$ . Note that in the cuboid representation for  $F$  there is exactly one faulty PE in each pillar.

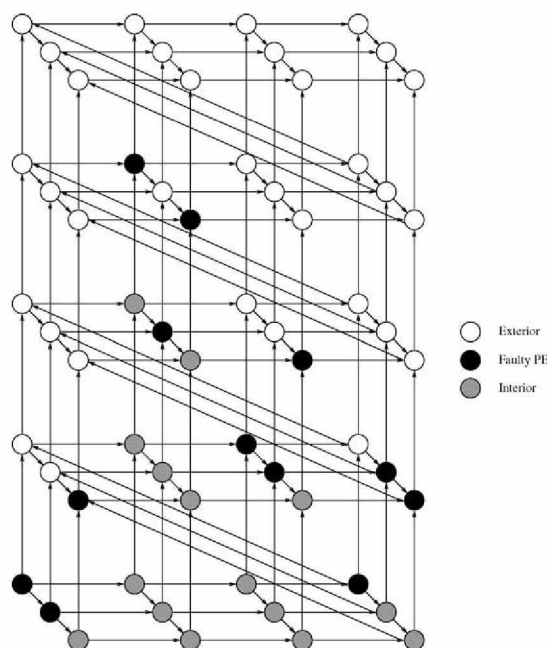


Fig. 6. Cuboid representation of  $F$ .

The height matrix for this fault pattern is

$$H = \begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & h_{22} & h_{23} \end{pmatrix} = \begin{pmatrix} 0 & 3 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{pmatrix}.$$

Note that,  $h_{10} - h_{13} = -1$  which violates condition (i) of Proposition 5.1. We see from Fig. 6, that the exterior processor at location  $(1, 0, 1)$  and the interior processor at location  $(1, 3, 0)$  are connected by a horizontal regular link. Hence  $F$  is not a catastrophic fault pattern by Lemma 4.1. Similarly condition (ii) of Proposition 5.1 is violated since  $h_{01} - h_{02} = 2$ . Note that, locations  $(0, 1, 2)$  and  $(0, 2, 2)$  contain an interior processor and an exterior processor, respectively, which are connected by a horizontal regular link. However it can easily be verified that  $F$  satisfies conditions (iii) of Proposition 5.1 even though  $F$  is not catastrophic.

In the general case, we have the following proposition:

**Proposition 5.2.** *An  $N_1 \times g_k$  matrix  $H = ((h_{xy}))$  with non-negative integer entries is the height matrix of a minimal CFP for  $\mathcal{N}$  with unidirectional link redundancy  $G = (g_1, g_2, \dots, g_k | v_1, v_2, \dots, v_l)$  iff the following conditions are satisfied:*



- (i)  $h_{x0} - h_{x, g_k - 1} = 0$  or  $+1$  for all  $x$  such that  $0 \leq x \leq N_1 - 1$ ;  $h_{x0} = h_{x, g_k - 1} = 0$  for at least one  $x$ .
- (ii)  $h_{xy} - h_{x, y + g_i} \leq 1$  for all  $g_i$ ,  $1 \leq i \leq k - 1$  whenever  $0 \leq x \leq N_1 - 1$  and  $0 \leq y \leq g_k - g_i - 1$  and
- (iii)  $h_{xy} - h_{x + v_i, y} \leq 1$  for all  $v_i$ ,  $1 \leq i \leq l$  whenever  $0 \leq x \leq N_1 - v_i - 1$  and  $0 \leq y \leq g_k - 1$ .

As before, the number of minimal CFPs for  $\mathcal{N}$  with unidirectional link redundancy  $G = (g_1, g_2, \dots, g_k \mid v_1, v_2, \dots, v_l)$  is equal to the number of height matrices  $H$  which satisfy the conditions of Proposition 5.2.

## 6. Conclusions

In this paper, we extended the characterization of CFPs known for linear arrays to two-dimensional VLSI arrays. We determined the minimum number of faults required for a fault pattern to be catastrophic. We gave an algorithm for the construction of minimal CFPs with largest fault window when all the links are unidirectional, and studied the effect of different link configurations on the size of the fault window. We gave the necessary and sufficient conditions for the existence of CFPs. However, the number of catastrophic fault patterns is not known even for the link redundancy  $G = (1, g \mid 1, v)$ .

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