Journal of Statistical Planning and Inference 19 (1988) 341-348

CONSTRUCTION OF STRONGLY BALANCED UNIFORM REPEATED MEASUREMENTS DESIGNS

B.K. ROY

Computer Science Unit, Indian Statistical Institute, Calcutta, India

Abstract: Strongly balanced uniform repeated measurements designs when the number of treatments is 0, 1 or 3 modulo 4 are constructed. The methods used are the methods of differences and Hamiltonian decomposition of the lexicographic product of two graphs.

AMS Subject Classification: 62K05.

Key words and phrases: Circular model; universal optimality; difference-set; cycles and circuits; graph decomposition; Hamiltonian circuits.

1. Introduction

In repeated measurements designs (RMD's) each experimental unit is exposed to a number of treatments applied sequentially over periods. If there are p periods $0, 1, \ldots, p-1$, t treatments and n experimental units, then an RMD(t, n, p) is an $n \times p$ array, say $D = ((d_{ij}))$ where d_{ij} denotes the treatment assigned to the i-th unit in the j-th period, $i = 1, 2, \ldots, n; j = 0, 1, \ldots, p-1$. RMD's have been studied quite extensively. For a general review of such designs, one may refer to Hedayat and Afsarinejad (1975) and for an excellent review of the literature on optimal RMD's reference is made to Hedayat (1981).

An RMD is called uniform if in each period the same number of units is assigned to each treatment and on each unit each treatment appears in the same number of periods.

The underlying statistical model is called circular if in each unit the residuals in the initial period are incurred from the last period. Under the circular model an RMD is called strongly balanced if the collection of ordered pairs $(d_{i,j}, d_{i,j+1})$, $0 \leq j \leq p-1$, $1 \leq i \leq n$ (operation on the second suffix is modulo p), contains each ordered pair of treatments, distinct or not, the same number of times. A strongly balanced uniform RMD(t, n, p) will be abbreviated by SBURMD(t, n, p).

Let $\Omega_{t,n,p}$ denote the class of all RMD(t, n, p)'s. Let D be an SBURMD(t, n, p). We assume the underlying model to be circular. Under an additive set-up Magda

(1980) proved the universal optimality of D over $\Omega_{t,n,p}$ for both direct and residual effects. Under a non-additive set-up, Sen and Mukherjee (1984) proved the universal optimality of D over $\Omega_{t,n,p}$ for the estimation of both direct and residual effects.

Sen and Mukherjee also give a method for constructing SBURMD(t,n,p) if $t \mid n$ and pt^{-1} is an even integer. This paper deals with the case when $t \mid n$ and pt^{-1} is an odd integer. It is shown that whenever $t \neq 0$, 1 or 3 (mod 4), the required SBURMD exists.

2. Method of differences

It is easy to see that a necessary condition for an SBURMD(t,n,p) to exist is $t \mid n$, $t \mid p$ and $p > t$. Let G be a group with operation $+$, B be a subset of G and $g \in G$, then $B+g$ is defined as follows:

$$B+g=\{b+g: b\in B\}.$$

The proof of the following theorem being trivial, is omitted.

Theorem 2.1. *Let G be a group with t elements. Consider the p-tuple B: $(a_0, a_1, \ldots, a_{p-1})$ where $a_i \in G$ $\forall i=0,1,\ldots,p-1$, each element of G occurs exactly s $(s=p/t)$ times in B and $\{a_i - a_{i+1}: i=0,1,\ldots,p-1\}$ (operation on the suffixes is modulo p) contains each element of G precisely s times. Then $\{B+g: g\in G\}$ arranged in t rows, forms* SBURMD(t,t,p).

Such a B will be called a *base-block*. B will also be referred to as a difference vector. It is easy to see that the following notes are true.

Note 1. SBURMD(t,n,p) may be constructed by repeating nt^{-1} SBURMD(t,t,p)'s vertically.

Note 2. If SBURMD$(t,t,2t)$ exists, then SBURMD(t,t,p^*t) where p^* is even, can be constructed repeating SBURMD$(t,t,2t)$'s horizontally.

Note 3. If SBURMD$(t,t,2t)$ and SBURMD$(t,t,3t)$ can be constructed from difference vectors, then SBURMD(t,t,p^*t) where p^* is odd and $p^*\geq 3$, can be constructed by taking SBURMD$(t,t,3t)$ followed by SBURMD$(t,t,2t)$'s.

The key behind Note 3 is that, if the constructions are from difference vectors, then we can always assume that the first and the last element in the base-block are 0's where 0 is the identity element in G.

Sen and Mukherjee (1984) have constructed a difference vector for SBURMD$(t,t,2t)$. They have taken G to be the set $\{0,1,\ldots,t-1\}$ with addition modulo t. The difference vector is

$$0, t-1, 1, t-2, \ldots, [\tfrac{1}{2}t], [\tfrac{1}{2}t], \ldots, t-2, 1, t-1, 0.$$

So one now attempts to get difference vectors for SBURMD$(t,t,3t)$.

For any ordered tuple $(a_1,a_2,\dots,a_k)$, the collection

$$\{a_1-a_2,a_2-a_3,\dots,a_{k-1}-a_k,a_k-a_1\}$$

will be called the *collection of linked differences* in the tuple.

Lemma 2.2. *Let $(G,+)$ be a group with identity element 0, $|G|=t$. Let $B'=(a_1,a_2,\dots,a_{t-1})$ be some ordering of the non-zero elements of G such that*

$$\{a_1-a_2,a_2-a_3,\dots,a_{t-2}-a_{t-1},a_{t-1}-a_1\}=G-\{0\}.$$

Let $B=(a_1,a_2,\dots,a_{t-2})$ and let A be the vector

$$(B,B+(a_{t-1}-a_1),B+2(a_{t-1}-a_1),3a_{t-1}-2a_1,2a_{t-1}-a_1,a_{t-1})$$

where kg means $g+g+\cdots+g$ (k times) and g_1-g_2 means $g_1+(-g_2)$. Then A along with some suitable *repetitions (to be explained at the end of the proof) of elements in it forms a difference vector for* SBURMD$(t,t,3t)$.

Proof. (i) $0\in A$: If the inverse of $a_{t-1}-a_1$ belongs to B then 0 belongs to $B+(a_{t-1}-a_1)$; otherwise $-(a_{t-1}-a_1)=a_{t-1}$ which implies $2a_{t-1}-a_1=0$.

(ii) $a_{t-1}\notin B$, $2a_{t-1}-a_1\notin B+(a_{t-1}-a_1)$ and $3a_{t-1}-2a_1\notin B+2(a_{t-1}-a_1)$: That $a_{t-1}\notin B$ is immediate. To show the second part, assume if possible, that there exists a_i $(1\le i\le t-2)$ such that $2a_{t-1}-a_1=a_i+a_{t-1}-a_1)$ and hence $a_i=a_{t-1}$ which is a contradiction.

The proof of third part is similar to this.

(iii) $3a_{t-1}-2a_1\neq 2a_{t-1}-a_1$ and $2a_{t-1}-a_1\neq a_{t-1}$.

It is easy to see the following:

(i) All non-zero elements in G occur 3 times as the linked differences in A.

(ii) No zero-difference occurs in A.

(iii) A contains each element of G at least once.

(iv) A contains each element of G at most thrice.

(v) A contains $3t-3$ elements.

Now, there will be some elements in A with frequency 1 or 2, and those are repeated so that the frequency becomes 3 and this will give rise to 3 zero-differences.

Thus A with some suitable repetitions gives the difference vector for SBURMD$(t,t,3t)$. □

Note that this construction does not work for $t=2$ since B becomes the null-set.

To illustrate the lemma, we show the case $t=3$.

$B'=(1\ 2)$, $B=(1)$, $A=(1\ 2\ 0\ 1\ 0\ 2)$,

Difference set $=(1\ 1\ 2\ 2\ 0\ 0\ 1\ 0\ 2)$.

Also observe that in the lemma, the condition "B' is some ordering of the non-zero elements of G" can be relaxed to "the ordering of any $t-1$ elements of G", because if B' contains 0, then B' does not contain some element, say g, then $B'+(-g)$ will have the required property and will not contain 0.

So, now the problem reduces to finding an ordering of any $t-1$ elements from a group of order t such that the linked-differences are all non-zero and distinct. Such an ordering will be called a *starting difference vector* for SBURMD$(t,t,3t)$.

Lemma 2.3. *If t is odd, then a starting difference vector for* SBURMD$(t,t,3t)$ *is given by the following:*

(i) *If $t=4k+1$, consider Z_{4k+1} and let the required vector be*

$$(1,4k,2,4k-1,\dots,k,3k+1,k+2,3k,k+3,3k-1,\dots,2k,2k+2,2k+1,0).$$

(ii) *If $t=4k+3$, consider Z_{4k+3} and let the required vector be*

$$(1,4k+2,2,4k+1,\dots,3k+3,k+1,k+2,3k+2,k+3,3k+1,\dots,2k+4,$$
$$2k+1,2k+3,2k+2).$$

The proof of this lemma and also the next lemma are trivial and hence omitted.

Lemma 2.4. *If t is prime or a prime power, consider* GF(t) *and let α be a primitive element; then $(1,\alpha,\alpha^2,\dots,\alpha^{t-2})$ is a starting difference vector for an* SBURMD$(t,t,3t)$.

In view of the previous lemmas and notes the following theorem is immediate.

Theorem 2.5. *Let $t\,|\,n$, $t\,|\,p$, $p>t$ and pt^{-1} be odd. If t is an odd integer or some power of 2, then* SBURMD(t,n,p) *always exists.*

3. Non-existence of SBURMD$(2,2,2p^*)$ where p^* is odd

Let us denote the treatments by 0 and 1.

If an SBURMD$(2,2,2p^*)$ exists then the second row must be the complement of the first row.

Assume that the frequencies of the pairs 00, 01, 10 and 11 in the first row are x, y, z and w respectively. Then the frequencies of 00, 01, 10 and 11 in the second row must be w, z, y and x respectively. Thus in the design, the frequency of both 00 and 11 is $x+w$ and that of 01 and 10 is $y+z$. So we have $x+w=y+z$ and these are equal to p^*. So what we have actually is that, if an SBURMD$(2,2,2p^*)$ exists, then the first row must be a difference vector, i.e. the collection of linked-differences in the first block must have p^* 0's and p^* 1's.

We may think of a graph with $2p^*$ vertices, p^* of which are labeled 0 and the remaining p^* vertices are labeled 1. It is easy to see that the first row of an SBURMD$(2,2,2p^*)$ is equivalent to a Hamiltonian walk in that graph with an odd (to be precise p^*) number of cross-overs between vertices labeled 0 and vertices labeled 1. But this is impossible. Hence we can state the following theorem:

Theorem 3.1. *If p^* is an odd integer, then* SBURMD$(2,2,2p^*)$ *does not exist.*

4. The case $t \equiv 0 \pmod 4$

Let $G_1=(X_1,E_1)$ and $G_2=(X_2,E_2)$ be two simple graphs. (X_i denotes the vertex set and E_i denotes the edge set of the graph G_i.) The lexicographic product $G_1 \otimes G_2$ has the vertex set $X_1 \times X_2$ (the Cartesian product) and the vertices (x_1,x_2) and (y_1,y_2) are joined iff $(x_1,y_1)\in E_1$ or $(x_1=y_1$ and $(x_2,y_2)\in E_2)$. We denote by S_n the graph consisting of n isolated vertices and by C_n the cycle of length n $(n\geq 3)$.

The fact that $C_r \otimes S_n$ is decomposable into Hamiltonian cycles, has been proved in different papers; one may refer to Baranyai and Szasz (1981) or Laskar (1978).

Our interest in this section is to get larger SBURMD's from the smaller ones. Let us think of a graph with p vertices; the vertices divided into different collections, each collection having pt^{-1} vertices. Let the vertices in the same collection be labeled by a treatment symbol and vertices in different collections be labeled by different treatment symbols. Then a row in an SBURMD is just a Hamiltonian circuit (i.e. directed cycle) in that graph. Let $C(i)$ denote such a circuit corresponding to the i-th row in D, an SBURMD(t,n,p). If $C(i)\otimes S_k$ can be decomposed into Hamiltonian circuits and these Hamiltonian circuits can be written as k rows of a matrix (call it D_i) such that each column in the matrix

$$D^* = \begin{bmatrix} D_1 \\ \hdashline D_2 \\ \hdashline \vdots \\ \hdashline D_n \end{bmatrix}$$

contains each vertex symbol the same number of times, then D^* is an SBURMD(tk,nk,pk). To see this first, one can easily observe that each treatment (now the treatments are labeled by (i,j), $i=1,2,\dots,t$; $j=1,2,\dots,k$) occurs pt^{-1} times in each row of D^*. Secondly, we have to ensure that each treatment pair (there are now t^2k^2 such pairs) appears λ $(\lambda=(np)/t^2)$) times in D^*. Consider a pair $((i_1,j_1),(i_2,j_2))$. The pair (i_1,i_2) appears λ times in D. Corresponding to each occurence of (i_1,i_2) in, say, row l of D, there is an occurence of the pair $((i_1,j_1),(i_2,j_2))$ in D_l. This ensures the occurence of $((i_1,j_1),(i_2,j_2))$ λ times in D.

If the graphs considered here were undirected and the occurence of each treatment in each column were not to be considered, then the Hamiltonian decomposition of $C_r \otimes S_n$, mentioned by Baranyai and Szasz (1981) and Laskar (1978), would have solved the problem of getting larger SBURMD's from the smaller ones. So, now we have to look for a further generalization of that decomposition theorem.

Lemma 4.1. *Let $C=(0,1,2,\dots,r-1)$ denote a circuit, r is even. Let k be a positive odd integer. Consider $C\otimes S_k$ whose vertices are labeled (i,j), $i=0,1,\dots,r-1$; $j=0,1,\dots,k-1$. Consider the following array and call it B:*

$$\begin{array}{ccccc}
(0,0) & (1,0) & \dots & (r-2,0) & (r-1,0)\\
(0,1) & (1,2\cdot 1) & \dots & (r-2,1) & (r-1,2\cdot 1)\\
(0,2) & (1,2\cdot 2) & \dots & (r-2,2) & (r-1,2\cdot 2)\\
\vdots & \vdots & & \vdots & \vdots\\
(0,k-1) & (1,2\cdot(k-1)) & \dots & (r-2,k-1) & (r-1,2\cdot(k-1))
\end{array}$$

Now consider the array

$$B^*=[B\,|\,B+1\,|\,B+2\,|\cdots|\,B+(k-1)]$$

where $B+i$ means B with each element, say (u,v), being replaced by $(u,v+i)$ (addition modulo k).

This array B^* forms a Hamiltonian decomposition of $C\otimes S_k$ with the additional property that each column has the elements (u,v), $v=0,1,\dots,k-1$, for some u.

Proof. (i) Each row is a Hamiltonian circuit.

(ii) Since k is odd, 2 has multiplicative inverse in Z_k, so each column has elements (u,v), $v=0,1,\dots,k-1$, for some u. In fact, for the e-th column, $0\le e<rk$, $u\equiv e \pmod r$.

(iii) Consider a pair of the form $((i,j),(i+1,j'))$ where i is even, $i+1$ is odd modulo r. If it belongs to the x-th row of $B+y$ then

$$j=x+y, \qquad j'=2x+y$$

which implies $x\equiv j'-j$, $y\equiv 2j-j' \pmod k$ and the solution is unique.

The pair $((i,j),(i+1,j'))$ where i is odd and $i\neq r-1$ may be found easily looking at $((i-1,j'),(i,j))$ ($i-1$ is even) which may be found uniquely in the array and so (i,j), $(i+1,j')$ appears just next to it.

Now let us consider the pairs of the form

$$(r-1,j),\ (0,j').$$

If it appears in such a way that $(0,j')$ appears in the x-th row of $B+y$, then

$$j=2x+y-1, \qquad j'=x+y,$$

which implies $x\equiv j-j'+1$, $y\equiv 2j'-j-1 \pmod k$ and the solution is unique. Hence the lemma. □

In view of the above lemma and the discussion in this section, we state the following theorem.

Theorem 4.2. *If* SBURMD(t,n,p) *exists, p is even and k is a positive odd integer, then* SBURMD(tk,nk,pk) *exists.*

Now let $t \equiv 0 \pmod 4$, $t>0$. Then

either $t=2^s$ for some $s>1$

or $t=2^s u$ for some $s>1$ and some odd integer u.

The existence of SBURMD(t,n,p) when $t=2^s$ for some $s>1$ is settled in view of Theorem 2.5. Let $t=2^s u$ and consider n and p such that

$$n=l_1 2^s u, \qquad p=l_2 2^s u,$$

where $l_1 \geq 1$, l_2 is odd and $l_2 \geq 3$. Let

$$n'=l_1 2^s, \qquad p'=l_2 2^s.$$

So by Theorem 2.5, SBURMD$(2^s,n',p')$ exists. Note that u is odd and p' is even.

Hence by Theorem 4.2, SBURMD$(2^s u, n'u, p'u)$, i.e. SBURMD(t,n,p), exists. So we can state the following theorem.

Theorem 4.3. *If* $t \equiv 0 \pmod 4$, $t \mid r$, $t \mid p$, $p>t$, *and* pt^{-1} *is odd then* SBURMD(t,n,p) *exists.*

5. Concluding remarks

The existence of SBURMD(t,n,p) when $t \equiv 2 \pmod 4$, and pt^{-1} is odd, is still an open problem. It has been shown in Section 3 that SBURMD$(2,2,p)$, where p is an odd multiple of 2, does not exist. The author has exhaustively checked that the difference technique mentioned in Section 2 fails to construct SBURMD(6, 6, 18). Now can anyone see any similarity (or even relationship) between SBURMD's and pairwise orthogonal latin squares?

6. Appendix

An SBURMD(5, 5, 25) constructed using

(i) Lemma 2.2 and 2.3;

(ii) Sen and Mukherjee's difference vector for SBURMD(5, 5, 10) (mentioned in Section 2); and

(iii) Note 3 in Section 2;

is as follows:

4	4	2	1	3	3	1	0	0	2	0	4	1	2	3	3	2	4	1	0	0	1	4	2	3
0	0	3	2	4	4	2	1	1	3	1	0	2	3	4	4	3	0	2	1	1	2	0	3	4
1	1	4	3	0	0	3	2	2	4	2	1	3	4	0	0	4	1	3	2	2	3	1	4	0
2	2	0	4	1	1	4	3	3	0	3	2	4	0	1	1	0	2	4	3	3	4	2	0	1
3	3	1	0	2	2	0	4	4	1	4	3	0	1	2	2	1	3	0	4	4	0	3	1	2

Acknowledgement

The author is thankful to Ms. Mousumi Sen for exposing the problem to him and to Dr. Rahul Mukherjee for his helpful suggestions.

References

Baranyai, Z. and Gy.R. Szasz (1981). Hamiltonian decomposition of lexicographic product. *J. Combinat. Theory Ser. B* 31, 253–261.

Hedayat, A. (1981). Repeated measurements designs, IV: Recent advances (with discussion). *Bull. Int. Statist. Inst.* 49 (1), 591–610.

Hedayat, A. and K. Afsarinejad (1975). Repeated measurements designs, I. In: J.N. Srivastava, Ed., *A Survey of Statistical Designs and Linear Models*. North-Holland, Amsterdam, 229–242.

Laskar, R. (1978). Decomposition of some composite graphs into Hamiltonian cycles. *Proceedings, Fifth Hungarian Colloquium*, 705–716.

Magda, C. (1980). Circular balanced repeated measurements designs, *Commun. Statist. Theory Methods A* 9, 1901–1918.

Sen, M. and R. Mukherjee (1984). Optimal Repeated Measurements Designs under Interaction. Tech. Report no. 27/84, Stat. Math. Division, Indian Statistical Institute, Calcutta, India (submitted to J. Statist. Plann. Inference).