Statistics & Probability Letters 43 (1999) 427–431

Bounds for rth order joint cumulant under rth order strong mixing

B.L.S. Prakasa Rao

Indian Statistical Institute, 7, S.J.S. Sansanwal Marg, New Delhi 110016, India

Received November 1997

Abstract

A new concept of rth order strong mixing for sub σ-algebras of a probability space is introduced and a bound for the rth order joint cumulant is obtained under a two-part dependence assumption, which includes the case of rth order strong mixing, generalizing the work of Bradley [(1996) Statist. Probab. Lett. 30, 287–293] and Rio [(1993) Ann. Inst. H. Poincare Probab. Statist. 29, 587–597].

MSC: primary 60E15

Keywords: Bounds for joint cumulant; rth order strong mixing; Two-part dependence assumption

1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathscr{F}_i$, $1 \leqslant i \leqslant r$ be sub σ-algebras on Ω contained in $\mathscr{F}$ for some positive integer $r \geqslant 2$. Define

$$\alpha(\mathscr{F}_1, \dots, \mathscr{F}_r) = \sup_{(A_1,\dots,A_r) \in \mathscr{F}_1 \times \cdots \times \mathscr{F}_r} |\operatorname{cum}(I_{A_1}, \dots, I_{A_r})|,$$

where, for any random vector $(X_1, \dots, X_r)$ with X_j $\mathscr{F}_j$-measurable for $1 \leqslant j \leqslant r$,

$$\operatorname{cum}(X_1, \dots, X_r) = \sum (-1)^{p-1}(p-1)! \left(E \prod_{j \in v_1} X_j\right) \dots \left(E \prod_{j \in v_p} X_j\right)$$

and $(v_1, \dots, v_p)$ denotes a partition of the set $\{1, \dots, r\}$ for $p = 1, \dots, r$ (cf. Block and Fang, 1988). Here I_A denotes the indicator function of the set A. The coefficient $\alpha(\mathscr{F}_1, \dots, \mathscr{F}_r)$ measures the association or dependence between the σ-algebras $\mathscr{F}_i$, $1 \leqslant i \leqslant r$. For the special case $r = 2$, the coefficient reduces to the usual strong-mixing coefficient between two sub σ-algebras.

Let $\{X_n, -\infty < n < \infty\}$ be a stochastic process defined on a probability space $(\Omega, \mathscr{F}, P)$ and $\mathscr{F}_J$ denote the σ-algebra generated by the random variables $\{X_k, k \in J\}$ for any subset J in the set $-\infty < n < \infty$. Consider any family of sets $\{J_{in},\ 1\leqslant i\leqslant r,\ n\geqslant 1\}$ having the property that $\bigcap_{i=1}^{r} J_{in}\to\phi$, the empty set as $n\to\infty$. The process $\{X_n,\ -\infty<n<\infty\}$ is said to be *r-strong mixing* if

$$\alpha(\mathscr{F}_{J_{1n}},\dots,\mathscr{F}_{J_{rn}})\to 0 \quad \text{as } n\to\infty$$

for every such collection of sets $\{J_{in},\ 1\leqslant i\leqslant r\}$.

If $r=2$, the definition reduces to the classical case of strong-mixing for a sequence of random variables. It is easy to see that if a process is r-strong mixing, then it is strong-mixing in the usual sense. Furthermore, every m-dependent process is r-strong mixing. However, there are processes which are strong-mixing but possibly not r-strong mixing. This can be checked from the observation that one can construct random variables X_1,X_2,X_3 such that they are pairwise independent but not mutually independent.

We now obtain a bound on the $\mathrm{cum}(X_1,\dots,X_r)$ for any random vector $(X_1,\dots,X_r)$ with X_j $\mathscr{F}_j$-measurable for $1\leqslant j\leqslant r$ based on the r-strong-mixing coefficient introduced above.

For any nonnegative random variable W defined on the probability space $(\Omega,\mathscr{F},P)$, define

$$Q_W(z)=\inf\{t\geqslant 0\colon P(W>t)\leqslant z\},\quad 0<z<1.$$

Theorem 1.1. *Suppose $(\Omega,\mathscr{F},P)$ is a probability space and $\mathscr{F}_i$, $1\leqslant i\leqslant r$ are sub σ-algebras of $\mathscr{F}$. Further suppose that there exists positive real numbers p_i, $1\leqslant i\leqslant r$, α, and λ such that $\sum_{i=1}^{r}\frac{1}{p_i}=1$, $0\leqslant\lambda,\alpha\leqslant 1$ and*

$$|\mathrm{cum}(I_{A_1},\dots,I_{A_r})|\leqslant\alpha+\lambda\prod_{i=1}^{r}[P(A_i)]^{1/p_i}$$

for all $A_i\in\mathscr{F}_i$, $1\leqslant i\leqslant r$. If X_i is $\mathscr{F}_i$-measurable for $1\leqslant i\leqslant r$, then

$$|\mathrm{cum}(X_1,\dots,X_r)|\leqslant C_1\int_0^{\alpha}\prod_{i=1}^{r}Q_{|X_i|}(z)\,\mathrm{d}z+C_2\prod_{i=1}^{r}\|X_i\|_{p_i}$$

where C_1 is a constant depending only on r and C_2 depends on r, λ, p_i, $1\leqslant i\leqslant r$. Here $\|X_i\|_{p_i}$ denotes $[E(|X_i|^{p_i})]^{1/p_i}$ for $1\leqslant i\leqslant r$.

Since

$$\mathrm{cum}(X_1+Y_1,X_2,\dots,X_r)=\mathrm{cum}(X_1,X_2,\dots,X_r)+\mathrm{cum}(Y_1,X_2,\dots,X_r),$$

it is sufficient to prove the theorem for nonnegative random variables X_i, $1\leqslant i\leqslant r$. Furthermore, $Q_{X^+}(z)\leqslant Q_{|X|}(z)$. Hence it is sufficient to prove the following theorem.

Theorem 1.2. *Suppose the conditions in the previous theorem hold and X_i, $1\leqslant i\leqslant r$ are nonnegative random variables such that X_i is $\mathscr{F}_i$-measurable for $1\leqslant i\leqslant r$. Then*

$$|\mathrm{cum}(X_1,\dots,X_r)|\leqslant C_1\int_0^{\alpha}\prod_{i=1}^{r}Q_{X_i}(z)\,\mathrm{d}z+C_2\prod_{i=1}^{r}\|X_i\|_{p_i}$$

where C_1 is a constant depending only on r and C_2 depends on r, λ, and p_i, $1\leqslant i\leqslant r$.

Proof. Following Theorem 1 of Block and Fang (1988), it follows that

$$\mathrm{cum}(X_1,\dots,X_r)=\int_0^{\infty}\dots\int_0^{\infty}\mathrm{cum}(I_{A_1}(x_1),\dots,I_{A_r}(x_r))\,\mathrm{d}x_1\dots\mathrm{d}x_r$$

where $A_i = [X_i > x_i]$, $1 \leqslant i \leqslant k$. Hence

$$|\text{cum}(X_1,\dots,X_r)| \leqslant \int_0^\infty \dots \int_0^\infty |\text{cum}(I_{A_1}(x_1),\dots,I_{A_r}(x_r))|\,\mathrm{d}x_1 \dots \mathrm{d}x_r.$$

It is easy to check that, for any $x_i > 0$, $1 \leqslant i \leqslant r$,

$$|\text{cum}(I_{A_1}(x_1),\dots,I_{A_r}(x_r))| \leqslant \min\left\{\beta_r P(X_i > x_i),\ 1 \leqslant i \leqslant r;\ \alpha + \lambda \prod_{i=1}^r [P(X_i > x_i)]^{1/p_i}\right\}$$

where

$$\beta_r = 1 + 2! + \dots + (r-2)! \quad \text{if } r \text{ is even,}$$

and

$$\beta_r = 1 + 2! + \dots + (r-1)! \quad \text{if } r \text{ is odd.}$$

Therefore

$$|\text{cum}(X_1,\dots,X_r)| \leqslant \int_0^\infty \dots \int_0^\infty \min\{\beta_r P(X_i > x_i),\ 1 \leqslant i \leqslant r;\ \alpha\}\,\mathrm{d}x_1 \dots \mathrm{d}x_r$$
$$+ \int_0^\infty \dots \int_0^\infty \min\left\{\beta_r P(X_i > x_i),\ 1 \leqslant i \leqslant r; \lambda \prod_{i=1}^r [P(X_i > x_i)]^{1/p_i}\right\} \mathrm{d}x_1 \dots \mathrm{d}x_r.$$

We will now get upper bounds on the two integrals given above following the techniques of Rio (1993) and Bradley (1996). Let

$$S = \{(x_1,\dots,x_r,z) \in R_+^{r+1} :\ z < \min\{\beta_r P(X_i > x_i),\ 1 \leqslant i \leqslant r; \alpha\}\}.$$

Then

$$S = \{(x_1,\dots,x_r,z) \in R_+^{r+1} :\ z < \alpha, x_i < Q_{X_i}(z/\beta_r), 1 \leqslant i \leqslant r\}.$$

Therefore

$$\begin{aligned}\int_0^\infty \dots \int_0^\infty \min\{\beta_r P(X_i > x_i),\ 1 \leqslant i \leqslant r;\ \alpha\}\,\mathrm{d}x_1 \dots \mathrm{d}x_r &= \int \dots \int_{(x_1,\dots,x_r,z)\in S} 1\,\mathrm{d}x_1 \dots \mathrm{d}x_r\,\mathrm{d}z \\ &= \int_{z=0}^\alpha \int_{x_1=0}^{Q_{X_1}(z/\beta_r)} \dots \int_{x_r=0}^{Q_{X_r}(z/\beta_r)} \mathrm{d}x_1 \dots \mathrm{d}x_r \\ &= \int_{z=0}^\alpha \prod_{i=1}^r Q_{X_i}\left(\frac{z}{\beta_r}\right) \mathrm{d}z. \qquad (1.1)\end{aligned}$$

Let

$$H(x_1,\dots,x_r) = \min\left\{\beta_r P(X_i > x_i),\ 1 \leqslant i \leqslant r; \lambda \prod_{i=1}^r [P(X_i > x_i)]^{1/p_i}\right\}.$$

We now obtain a bound on $H(x_1,\dots,x_r)$ following the techniques of Bradley (1996). If $\lambda = 0$, then Theorem 1.2 clearly is a consequence of Eq. (1.1). Suppose $0 < \lambda \leqslant 1$. Define $a_{k,n} = Q_{X_k}(2^{-n})$, $n \geqslant 1$ and $J_{k,n} = [a_{k,n}, a_{k,n+1}]$, $n \geqslant 0$. If $a_{k,n} = a_{k,n+1}$, then define $J_{k,n} = \phi$. Let $J_{k,\infty} = [a_k, \infty)$ if $\lim_{n\to\infty} a_{k,n} = a_k < \infty$. It is easy to see that

$$2^{-n-1} < P(X_k > x_k) \leqslant 2^{-n}, \quad n \geqslant 0,\ x_k \in J_{k,n},$$

and

$$P(X_k > x_k) = 0, \quad x_k \in J_{k,\infty}.$$

Define

$$G_{i_1,\dots,i_r} = \int \dots \int_{(x_1,\dots,x_r)\in J_{1,i_1}\times\cdots\times J_{r,i_r}} H(x_1,\dots,x_r)\,\mathrm{d}x_1\dots\mathrm{d}x_r.$$

It can be checked that

$$G_{i_1,\dots,i_r} = 0$$

if $i_k = \infty$ for some $1 \leqslant k \leqslant r$.

Let $j_{k,n}$ be the length of the interval $J_{k,n}$. Following the estimates derived in Bradley (1996), it can be shown that

$$H(x_1,\dots,x_r) \leqslant \min\{\beta_r 2^{-i_1},\dots,\beta_r 2^{-i_r},\lambda 2^{-\sum_{k=1}^r i_k/p_k}\}$$

and hence

$$G_{i_1,\dots,i_r} \leqslant \left\{\prod_{k=1}^r j_{k,i_k}\right\} \min\{\beta_r 2^{-i_1},\dots,\beta_r 2^{-i_r},\lambda 2^{-\sum_{k=1}^r i_k/p_k}\}.$$

For nonnegative integers $i_2,\dots,i_r$, we can obtain the following estimates:

$$\begin{aligned}\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r} &\leqslant \lambda \sum_{i_1=0}^{\infty}\left[j_{1,i_1}2^{-(i_1/p_1)}\left\{\prod_{k=2}^r j_{k,i_1+i_k}2^{-(i_1+i_k)/p_k}\right\}\right]\\ &\leqslant \lambda\left[\sum_{i_1=0}^{\infty} j_{1,i_1}^{p_1}2^{-(i_1)}\right]^{1/p_1}\prod_{k=2}^r\left[\sum_{i_1=0}^{\infty} j_{1,i_1+i_k}^{p_k}2^{-(i_1+i_k)}\right]^{1/p_k}\\ &\leqslant 2\lambda\prod_{k=1}^r \|X_k\|_{p_k}.\end{aligned}$$

Similarly

$$\sum_{i_1=0}^{\infty} G_{i_1+i_2,i_1,\dots,i_1+i_r} \leqslant 2\lambda\prod_{k=1}^r \|X_k\|_{p_k}.$$

Another bound for

$$\sum_{i_1=0}^{\infty} G_{i_1,i_1+i_2,\dots,i_1+i_r}$$

is

$$\beta_r^r.2.2^{-\sum_{k=2}^r i_k(1-\frac{1}{p_k})}\prod_{k=1}^r \|X_k\|_{p_k}.$$

by Holder's inequality. Similarly, another bound for

$$\sum_{i_1=0}^{\infty} G_{i_1+i_2,i_1,\dots,i_1+i_r}$$

is

$$\beta_r^r.2.2^{-\sum_{k=2}^r i_k(1-\frac{1}{p_k})}\prod_{k=1}^r \|X_k\|_{p_k}.$$

The above bounds involve sums of $G_{i_1,\dots,i_r}$ with i_1 varying over 0 to ∞. Similar bounds hold for other indices i_j, $1\leqslant j\leqslant r$. Note that

$$\int_0^\infty \dots \int_0^\infty H(x_1,\dots,x_r)\,\mathrm{d}x_1\dots\,\mathrm{d}x_r = \sum_{i_1=0}^{\infty}\dots\sum_{i_r=0}^{\infty} G_{i_1,\dots,i_r}.$$

By the arguments similar to those given in Bradley (1996) and using the bounds given above, it follows that there exists a constant C_2 depending on λ, r, p_i, $1\leqslant i\leqslant r$ such that

$$\int_0^\infty \dots \int_0^\infty H(x_1,\dots,x_r)\,\mathrm{d}x_1\dots\mathrm{d}x_r \leqslant C_2\prod_{k=1}^r \|X_k\|_{p_k}.$$

Combining this bound with the earlier bound obtained in Eq. (1.1), we have

$$|\mathrm{cum}(X_1,\dots,X_r)|\leqslant C_1\int_0^\alpha \prod_{i=1}^r Q_{X_i}(z)\,\mathrm{d}z + C_2\prod_{k=1}^r \|X_k\|_{p_k}.$$

Remarks. More explicit bounds involving λ, r, p_i, $1\leqslant i\leqslant r$ for C_1 and C_2 can be obtained as in the case when $r=2$ (cf. Bradley, 1996). It is however clear from the proof of the result that C_2 can be chosen to be zero in case $\lambda=0$.

References

Block, H.W., Fang, Z., 1988. A multivariate extension of Hoeffding's lemma. Ann. Probab. 16, 1803–1820.
Bradley, R.C., 1996. A covariance inequality under a two-part dependence assumption. Statist. Probab. Lett. 30, 287–293.
Rio, E., 1993. Covariance inequalities for strongly mixing processes. Ann. Inst. H. Poincare Probab. Statist. 29, 587–597.