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ON MIXING FOR FLOWS OF σ-ALGEBRAS

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SUMMARY. The notion of mixing is extended to flows of σ-algebras. Suppose a stochastic process is mixing in some sense. Conditions under which this process, observed at random times, inherits mixing property are discussed. Moment inequalities for mixing flows of σ-algebras are obtained. Applications to random fields are studied.

1. Introduction

The concept of strong mixing for sequences of random variables was introduced by Rosenblatt (1956) to study long range dependence or independence. This concept was generalized and several applications are discussed in the literature. Our aim here is not to give a survey of these results but to study a more general concept of mixing for σ-algebras. For a nice survey of mixing sequences and their properties, see Roussas and Ioannides (1987). In order to motivate the reason for developing the noting of mixing for σ-algebras (not be be confused with mixing transformations on measure spaces), let us consider the following problem.

Suppose $\{X(t), t \geqslant 0\}$ is a stochastic process defined on a probability space $(\Omega, \mathscr{F}, P)$ and the finite dimensional distributions of the process are determined by a parameter θ. If the process X is continuously observable over $[0, T]$, asymptotic properties of maximum likelihood estimator and other types of estimators of θ are studied for certain classes of processes by several authors. For instance, see Basawa and Prakasa Rao (1980), Kutoyants (1984), Grenander (1981) and Karr (1986). Nonparametric inference for stochastic processes, based on continuous realization of X over $[O, T]$, is discussed in Prakasa Rao (1983). In practice, the entire sample path is not available and suppose the process is observed only at random time points $\{\tau_n\}$. The problem is to infer about the characteristics of X based on $\{X(\tau_i), 1 \leqslant i \leqslant n\}$. In general, $\{X(\tau_i), i \geqslant 1\}$ does not possess all the information about X. For instance, if $\tau_{n+1} > \tau_n + s$ for all n and some $s > 0$, then

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it might not be possible to get information about $(X(0), X(s))$ unless some additional information on the process X is available. This problem has been considered earlier by several people. We will discuss nonparametric and parametric inference aspects of this problem in a separate publication.

The problem of interest in this paper is to find out whether a mixing condition on a process X is inherited by the sequence $\{X(\tau_n), n \geqslant 1\}$. In general, it need not hold. We extend the notion of mixing to flows of σ-algebras and obtain some consequences. We restrict our attention to extending the notion of ϕ-mixing (or some times referred to as uniform mixing). Other concepts of mixing can be developed and studied in this larger frame work of σ-algebras.

2. Mixing for flows

Let $(\Omega\ \mathscr{F}, P)$ be a probability space. Let $\{\mathscr{F}_t, t \geqslant 0\}$ be an increasing flow of σ-algebras contained in $\mathscr{F}$ and $\{\zeta_t, t \geqslant 0\}$ be a decreasing flow of σ-algebras contained in $\mathscr{F}$, that is,

$$\mathscr{F}_t \subset \mathscr{F}_s \text{ if } 0 \leqslant t \leqslant s < \infty,$$

and

$$\zeta_t \supset \zeta_s \text{ if } 0 \leqslant t \leqslant s < \infty.$$

Definition 2.1 : The increasing flow $\{\mathscr{F}_t\}$ is said to be *ϕ-mixing weakly* with the decreasing flow $\{\zeta_s\}$, if for every $A \in \mathscr{F}_t, t \geqslant 0$,

$$|P(A \cap B)-P(A)P(B)| \leqslant \phi(|s-t|)P(A) \quad \ldots \quad (2.1)$$

for every $B \in \zeta_s$, $s \geqslant 0$ where $\phi(u) \downarrow 0$ as $u \to \infty$.

Definition 2.2. For any real-valued non-negative random variable τ, defined $\mathscr{F}_\tau$ to be the σ-algebra generated by sets $A \in \mathscr{F}$ such that $A \cap [\tau \leqslant t] \in \mathscr{F}_t, t \geqslant 0$ when $\{\mathscr{F}_t\}$ is an increasing flow of σ-algebras and ζ_τ to be the σ-algebra generated by sets $B \in \mathscr{F}$ such that $B \cap [\tau \geqslant s] \in \zeta_s$, $s \geqslant 0$ when $\{\zeta_s\}$ is a decreasing flow of σ-algebras.

Definition 2.3 : Let $\{\tau_n, n \geqslant 1\}$ and $\{S_n, n \geqslant 1\}$ be increasing sequences of non-negative random variables. The increasing flow $\{\mathscr{F}_t\}$ is said to be *ϕ-mixing strongly* with the decreasing flow $\{\zeta_s\}$ with respect to $\{\tau_n\}$ and $\{S_n\}$ if, for every $A \in \mathscr{F}_{\tau_n},$ $n \geqslant 1$ and $B \in \zeta_{S_m}$,

$$|P(A \cap B)-P(A)P(B)| \leqslant E\{\phi(|\tau_n-S_m|)\}P(A) \quad \ldots \quad (2.2)$$

and $E\{\phi(|\tau_n-S_m|)\} \to 0$ whenever $|\tau_n-S_m| \xrightarrow{p} \infty$ as $m \to \infty$.

Definition 2.4 : If the increasing flow $\{\mathscr{F}_t\}$ is *ϕ-mixing strongly* with the decreasing flow $\{\zeta_s\}$ with respect to every pair $\{\tau_n\}$ and $\{S_n\}$ of increasing sequences of non-negative random variables, then the increasing flow $\{\mathcal{F}_t\}$ is said to be *ϕ-mixing strongly* with the decreasing flow $\{\zeta_s\}$.

Definition 2.5 : Let $\{X_t, t \geqslant 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose $\{X_t, t \geqslant 0\}$ is progressively measurable and $\{\tau_n, n \geqslant 1\}$ is an increasing sequence of non-negative random variables. Define $\mathcal{F}_t^X$ and ζ_s^X as in Example 2.1 given below. If $\{\mathcal{F}_{\tau_n}^X\}$ is ϕ-mixing strongly with the flow $\{\zeta_{\tau_n}^X\}$ with respect to $\{\tau_n\}$, then $\{X_t\}$ is said to be *ϕ-mixing strongly* with respect to $\{\tau_n\}$.

Example 2.1 : Let $\{X_t, t \geqslant 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. Define

$$\mathcal{F}_t^X = \sigma\text{-algebra generated by } X_u,\ 0 \leqslant u \leqslant t$$

and
$$\zeta_s^X = \sigma\text{-algebra generated by } X_v,\ v \geqslant s.$$

Clearly $\{\mathcal{F}_t^X\}$ is an increasing flow and $\{\zeta_s^X\}$ is a decreasing flow of σ-algebras. If $\{X_t, t \geqslant 0\}$ is ϕ-mixing in the classical sense, then $\{\mathcal{F}_t^X\}$ is ϕ-mixing weakly with $\{\zeta_s^X\}$ in the sense of Definition 2.1.

Example 2.2 : Suppose $\{X_t, t \geqslant 0\}$ is a stationary ϕ-mixing stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\{\tau_n, n \geqslant 1\}$ be an increasing sequence of non-negative random variables defined on $(\Omega, \mathcal{F}, P)$ independent of $\{X_t, t \geqslant 0\}$. We assume that $\{X_{\tau_n}, n \geqslant 1\}$ is well-defined and $|\tau_n - \tau_{n+m}| \xrightarrow{p} \infty$ as $m \to \infty$ for every $n \geqslant 1$. Further assume that the conditional distributions indicated in the following exist. For any $n \geqslant 1$,

$$P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y)$$

$$= \int_{R_+^2} P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y \mid \tau_k = t, \tau_{k+n} = s) d\mu_{\tau_k, \tau_{k+n}}(t, s)$$

where $\mu_{\tau_k, \tau_{k+n}}$ is the joint probability measure of (τ_k, τ_{k+n}).

Hence

$$P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y)$$

$$= \int_{R_+^2} (PX_t \leqslant x, X_s \leqslant y \mid \tau_k = t, \tau_{k+n} = s)\, d\mu_{\tau_k, \tau_{k+n}}(t, s)$$

$$= \int_{R_+^2} P(X_t \leqslant x, X_s \leqslant y)\, d\mu_{\tau_k, \tau_{k+n}}(t, s)$$

(by independence of $\{X_t\}$ and $\{\tau_n\}$)

$$= \int_{R_+^2} [P(X_t \leqslant x)\, P(X_s \leqslant y) + \psi(s, t\,; x, y)]\, d\mu_{\tau_k, \tau_{k+n}}(t, s)$$

where

$$|\psi(s, t; x, y)| \leqslant \phi(|s-t|)P(X_t \leqslant x) \quad \text{(since } X \text{ is } \phi\text{-mixing)}$$

$$= \phi(|s-t|)P(X_0 \leqslant x) \quad \text{(by stationarity of } X).$$

Therefore

$$|P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y) - P(X_0 \leqslant x)\, P(X_0 \leqslant y)|$$

$$\leqslant \left\{ \int_{R_+^2} \phi(|s-t|)\, d\mu_{\tau_k, \tau_{k+n}}(t, s) \right\} P(X_0 \leqslant x)$$

$$= E[\phi(|\tau_k - \tau_{k+n}|)]\, P(X_0 \leqslant x). \qquad \ldots \quad (2.3)$$

Note that, for any $k \geqslant 1$,

$$P(X_{\tau_k} \leqslant x) = \int_{R_+} P(X_{\tau_k} \leqslant x \mid \tau_k = t)\, d\mu_{\tau_k}(t)$$

$$= \int_{R_+} P(X_t \leqslant x \mid \tau_k = t)\, d\mu_{\tau_k}(t)$$

$$= \int_{R_+} P(X_t \leqslant x)\, d\mu_{\tau_k}(t)$$

$$= \int_{R_+} P(X_0 \leqslant x)\, d\mu_{\tau_k}(t)$$

$$= P(X_0 \leqslant x). \qquad \ldots \quad (2.4)$$

(2.3) and (2.4) imply that

$$|P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y) - P(X_{\tau_k} \leqslant x) P(X_{\tau_{k+n}} \leqslant y)|$$

$$\leqslant P(X_{\tau_k} \leqslant x)\, E[\phi(|\tau_k - \tau_{k+n}|)]. \qquad \ldots \quad (2.5)$$

Observe that

$$E[\phi(|\tau_k - \tau_{k+n}|)] \to 0 \text{ as } n \to \infty \text{ for fixed } k \geqslant 1$$

by monotone convergence theorem since $\phi(\cdot) \downarrow 0$ and $|\tau_k - \tau_{k+n}| \xrightarrow{p} \infty$ as $n \to \infty$. It can now be shown that, for any $A \in \mathscr{F}^X_{\tau_k}$ and $B \in \zeta^X_{\tau_{k+n}}$

$$|P(A \cap B) - P(A)P(B)| \leqslant P(A) E[\phi(|\tau_k - \tau_{k+n}|)]$$

where $E[\phi(|\tau_k - \tau_{k+n}|)] \to 0$ as $n \to \infty$. Hence $\{X_t\}$ is ϕ-mixing strongly with respect to $\{\tau_n\}$.

Example 2.3 : Suppose $\{X_t, t \geqslant 0\}$ is a stationary ϕ-mixing process defined on a probability space $(\Omega, \mathscr{F}, P)$. Let $\{\tau_n, n \geqslant 1\}$ be an increasing sequence of non-negative random variables defined on $(\Omega, \mathscr{F}, P)$. Let $\mathscr{F}^X_t$ be the σ-algebra generated by X_u, $0 \leqslant u \leqslant t$ and ζ^τ_s be the σ-algebra generated by sets of the form $[\tau_k \geqslant s]$, $k \geqslant 1$. Suppose the flows $\{\mathscr{F}^X_t, t \geqslant 0\}$ and $\{\zeta^\tau_s, s \geqslant 0\}$ are ψ-mixing in the sense that

$$|P(A \mid B) - P(A)| \leqslant \psi(|t-s|) P(A)$$

for all $A \in \mathscr{F}_t^X$ and $B \in \varsigma_s^\tau$ and $\psi(s) \to 0$ as $s \to \infty$. With the same notation as in Example 2.2, let us compute, for $n \geqslant 1$,

$$\begin{aligned} &P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y) \\ &= \int_{R_+^2} P(X_t \leqslant x, X_s \leqslant y \mid \tau_k = t, \tau_{k+n} = s)\, d\mu_{\tau_k, \tau_{k+n}}(t, s) \\ &= \int_{R_+^2} [P(X_t \leqslant x, X_s \leqslant y) + H_1(t, s; x, y)] d\mu_{\tau_k, \tau_{k+n}}(t, s) \qquad \ldots (2.6) \end{aligned}$$

where $$|H_1(t, s; x, y)| \leqslant \psi(|t-s|)\, P(X_t \leqslant x, X_s \leqslant y).$$

Similarly

$$\begin{aligned} P(X_{\tau_k} \leqslant x) &= \int_{R_+} P(X_t \leqslant x \mid \tau_k = t)\, d\mu_{\tau_k}(t) \\ &= \int_{R_+} [P(X_t \leqslant x) + H_2(t; x)]\, d\mu_{\tau_k}(t) \qquad \ldots (2.7) \end{aligned}$$

where $$|H_2(t; x)| \leqslant \psi(0)\, P(X_t \leqslant x). \qquad \ldots (2.8)$$

By stationarity and ϕ-mixing properties of stochastic process X, it follows that

$$\begin{aligned} |P(X_t \leqslant x, X_s \leqslant y) - P(X_t \leqslant x)\, P(X_s \leqslant y)| \\ \leqslant \phi(|t-s|)\, P(X_t \leqslant x) \\ = \phi(|t-s|)\, P(X_0 \leqslant x), \qquad \ldots (2.9) \end{aligned}$$

$$\begin{aligned} &|H_1(t, s; x, y)| \\ &\leqslant \psi(|t-s|)\, [P(X_t \leqslant x)\, P(X_s \leqslant y) + \phi(|t-s|)\, P(X_0 \leqslant x)] \\ &= \psi(|t-s|) P(X_0 \leqslant x) P(X_0 \leqslant y) + \psi(|t-s|) \phi(|t-s| P(X_0 \leqslant x). \ldots (2.10) \end{aligned}$$

Relations (2.7) and (2.8) prove that, for any $k \geqslant 1$,

$$P(X_{\tau_k} \leqslant x) = P(X_0 \leqslant x) + H_3(x), \qquad \ldots (2.11)$$

where $$|H_3(x)| \leqslant \psi(0)\, P(X_0 \leqslant x).$$

Relations (2.6), (2.9) and (2.10) show that

$$\begin{aligned} &P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y) \\ &= \int_{R_+^2} [P(X_0 \leqslant x)\, P(X_0 \leqslant y) + H_4(t, s; x, y) + H_1(t, s; x, y)]\, d\mu_{\tau_k, \tau_{k+n}}(t, s) \\ &\qquad \ldots (2.12) \end{aligned}$$

where $$|H_4(t, s; x, y)| \leqslant \phi(|t-s|)\, P(X_0 \leqslant x). \qquad \ldots (2.13)$$

Hence

$$P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y) = P(X_0 \leqslant x)\, P(X_0 \leqslant y) + H_5(x, y) \qquad \ldots (2.14)$$

where

$$\begin{aligned} |H_5(x, y)| &\leqslant \int_{R_+^2} \{\phi(|t-s|)\, P(X_0 \leqslant x) + \psi(|t-s|)\, P(X_0 \leqslant x)\, P(X_0 \leqslant y) \\ &\quad + \psi(|t-s|)\, \phi(|t-s|)\, P(X_0 \leqslant x)\}\, d\mu_{\tau_k, \tau_{k+n}}(t, s) \\ &\leqslant P(X_0 \leqslant x)\, E[\eta(|\tau_k - \tau_{k+n}|)] \qquad \ldots (2.15) \end{aligned}$$

where $$\eta = \phi + \psi + \psi\phi.$$

Note that, from (2.11),

$$P(X_0 \leqslant x)\, P(X_0 \leqslant y) = [P(X_{\tau_k} \leqslant x) - H_3(x)]\,[P(X_{\tau_k} \leqslant y) - H_3(y)]$$

$$= P(X_{\tau_k} \leqslant x)\, P(X_{\tau_k} \leqslant y) + H_6(x, y) \quad \ldots \quad (2.16)$$

where

$$|H_6(x, y)| \leqslant |H_3(x)| + |H_3(y)| + |H_3(x)\, H_3(y)|$$

$$\leqslant \psi(0)\, [P(X_0 \leqslant x) + P(X_0 \leqslant y) + \psi(0)\, P(X_0 \leqslant x)\, P(X_0 \leqslant y)]. \quad \ldots \quad (2.17)$$

Relations (2.14)—(2.17) show that

$$P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y) - P(X_{\tau_k} \leqslant x) P(X_{\tau_{k+n}} \leqslant y)$$

$$= H_7(x, y),$$

where

$$|H_7(x, y)| \leqslant |H_5(x, y)| + |H_6(x, y)|$$

$$\leqslant P(X_0 \leqslant x)\, E[\eta(\tau_k - \tau_{k+n})]$$

$$+\psi(0)\, [P(X_0 \leqslant x) + P(X_0 \leqslant y) + \psi(0) P(X_0 \leqslant x) P(X_0 \leqslant y] \quad \ldots \quad (2.18)$$

Hence

$$|P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y) - P(X_{\tau_k} \leqslant x) P(X_{\tau_{k+n}} \leqslant y)|$$

$$\leqslant \{P(X_{\tau_k} \leqslant x) - H_3(x)\}\, E[\eta(|\tau_k - \tau_{k+n}|)]$$

$$+\psi(0)[P(X_0 \leqslant x) + P(X_0 \leqslant y)$$

$$+\psi(0) P(X_0 \leqslant y) P(X_0 \leqslant y)]. \quad \ldots \quad (2.19)$$

If, in addition $\psi(0) = 0$, then $H_3(x) = 0$ and

$$|P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y) - P(X_{\tau_k} \leqslant x) P(X_{\tau_{k+n}} \leqslant y)|$$

$$\leqslant P(X_{\tau_k} \leqslant x)\, E[\eta(|\tau_k - \tau_{k+n}|)] \quad \ldots \quad (2.20)$$

where $\qquad \eta = \phi + \psi + \phi\psi.$

This proves that X is η-mixing strongly with respect to $\{\tau_n\}$ provided $\psi(0) = 0$.

Example 2.4 : Let $\{X_t, t \geqslant 0\}$ be a stationary ϕ-mixing process and $\tau_n = \sum_{i=l}^{n} Y_i$ where Y_i are i.i.d. non-negative random variables independent of $\{X_t, t \geqslant 0\}$ with $E(Y_i) > 0$. Then $\{X_{\tau_i}, \text{i} \geqslant 1\}$ is ϕ-mixing strongly with respect to $\{\tau_n\}$. Assume that the conditional distributions in the following exist.

Note that

$$
\begin{aligned}
&P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y) \\
&= \int_{R_+^2} P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y \mid \tau_k = t, \tau_{k+n} = s)\, d\mu_{\tau_k, \tau_{k+n}}(t, s) \\
&= \int_{R_+^2} P(X_t \leqslant x, X_s \leqslant y \mid \tau_k = t, \tau_{k+n} = s)\, d\mu_{\tau_k, \tau_{k+n}}(t, s) \\
&= \int_{R_+^2} P(X_t \leqslant x, X_s \leqslant y)\, d\mu_{\tau_k, \tau_{k+n}}(t, s) \\
&\quad \text{(By independence of } \{X_t, t \geqslant 0\} \text{ and } \{\tau_n\}) \\
&= \int_{R_+^2} P(X_t \leqslant x, X_s \leqslant y)\, d\mu_{\tau_k, \tau_{k+n}-\tau_k}(t, s-t) \\
&= \int_{R_+^2} P(X_t \leqslant x, X_s \leqslant y)\, d\mu_{\tau_k}(t)\, d\mu_{\tau_{k+n}-\tau_k}(s-t) \\
&\quad \text{(By independence of } \{Y_i\}) \\
&= \int_{R_+^2} P(X_0 \leqslant x, X_{s-t} \leqslant y)\, d\mu_{\tau_k}(t)\, d\mu_{\tau_{k+n}-\tau_k}(s-t) \\
&\quad \text{(By stationarity of } X) \\
&= \int_{R_+^2} \{P(X_0 \leqslant x,)\, P(X_{s-t} \leqslant y) + O(\phi(|s-t|)P(X_0 \leqslant x)\} \\
&\qquad d\mu_{\tau_k}(t) d\mu_{\tau_{k+n}-\tau_k}(s-t) \\
&\quad \text{(By } \phi\text{-mixing property of } X) \\
&= P(X_0 \leqslant x)P(X_0 \leqslant y) + P(X_0 \leqslant x) O\{E\, \phi[(|\tau_{k+n}-\tau_k|)]\} \\
&= P(X_{\tau_k} \leqslant x)P(X_{\tau_{k+n}} \leqslant y) + P(X_{\tau_k} \leqslant x) O\{E[\phi(|\tau_{k+n}-\tau_k|)]\}
\end{aligned}
$$

since

$$
\begin{aligned}
P(X_{\tau_k} \leqslant x) &= \int_{R_+} P(X_{\tau_k} \leqslant x \mid \tau_k = t)\, d\mu_{\tau_k}(t) \\
&= \int_{R_+} P(X_t \leqslant x \mid \tau_k = t)\, d\mu_{\tau_k}(t) \\
&= \int_{R_+} P(X_t \leqslant x)\, d\mu_{\tau_k}(t) \\
&= \int_{R_+} P(X_0 \leqslant x)\, d\mu_{\tau_k}(t) \\
&= P(X_0 \leqslant x)
\end{aligned}
$$

for all $k \geqslant 1$. Hence

$$
\begin{aligned}
&|P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y) - P(X_{\tau_k} \leqslant x)P(X_{\tau_{k+n}} \leqslant y)| \\
&\leqslant P(X_{\tau_k} \leqslant x)\, E[\phi(|\tau_{k+n}-\tau_k|)].
\end{aligned}
$$

Since $\tau_{k+n} - \tau_k \xrightarrow{p} \infty$ as $n \to \infty$ for any fixed k and $\phi(s) \downarrow 0$ as $s \to \infty$, an application of monotone convergence theorem implies that

$$\{X_{\tau_i},\ i \geqslant 1\} \text{ is } \phi\text{-mixing strongly with respect to } \{\tau_n\}.$$

Definition 2.6 : A process $\{X_t,\ t \geqslant 0\}$ is said to be *ϕ-mixing stably* if, for every set E with $P(E) > 0$, and for every $A \in \mathcal{F}_t^X$ and $B \in \zeta_s^X$, $0 \leqslant t < s < \infty$,

$$|P(A \cap B|E) - P(A|E)P(B|E)| \leqslant \phi(|t-s|)P(A|E)$$

where $\phi(\cdot) \downarrow 0$ as $s \to \infty$ and $\phi(\cdot)$ not depending on E.

Example 2.6 : Suppose $\{X_t, t \geqslant 0\}$ is a progressively measurable stationary stochastic process adapted to an increasing flow $\{\mathcal{F}_t\}$ of σ-algebras defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\{\tau_n,\ n \geqslant 1\}$ be an increasing sequence of discrete-valued non-negative stopping times adapted to $\{\mathcal{F}_t\}$. Further suppose that, for all $a \geqslant 0$,

$$P(X_{\tau_k} \leqslant x | \tau_k = a) = P(X_{\tau_k} \leqslant x)$$

and

$$P(X_{\tau_{k+n}} \leqslant y\ ;\ \tau_{k+n} \geqslant \tau_k | \tau_k = a) = P(X_{\tau_{k+n}} \leqslant y)$$

Assume that $\{X_t\}$ is ϕ-mixing stably in the sense of Definition 2.6. Then $\{X_{\tau_k},\ k \geqslant 1\}$ is ϕ-mixing strongly with respect to $\{\tau_n\}$.

As in the earlier examples, let us consider

$$\begin{aligned}
&P(X_{\tau_k} \leqslant x, X_{\tau_{k+n}} \leqslant y) \\
&= \int_{R_+} P(X_{\tau_k} \leqslant x, \tau_k = a\ ;\ X_{\tau_{k+n}} \leqslant y, \tau_{k+n} \geqslant \tau_k | \tau_k = a)\, d\mu_{\tau_k}(a) \\
&\qquad \text{(here } \mu_{\tau_k}(\cdot) \text{ is the probability measure of } \tau_k) \\
&= \int_{R_+} P(X_a \leqslant x, \tau_k = a\ ;\ X_{\tau_{k+n}} \leqslant y, \tau_{k+n} \geqslant a | \tau_k = a)\, d\mu_{\tau_k}(a) \\
&= \int_{R_+} P(X_a \leqslant x, \tau_k = a | \tau_k = a) P(X_{\tau_{k+n}} \leqslant y\ ;\ \tau_{k+n} \geqslant a | \tau_k = a)\, d\mu_{\tau_k}(a) \\
&\quad + \int_{R_+} \{ \int_{R_+} O(\phi(|b-a|)) P(X_a \leqslant x, \tau_k = a)\, d\mu_{\tau_{k+n}|\tau_k = a}(b)\}\, d\mu_{\tau_k}(a) \\
&= \int_{R_+} P(X_{\tau_k} \leqslant x | \tau_k = a) P(X_{\tau_{k+n}} \leqslant y, \tau_{k+n} \geqslant \tau_k | \tau_k = a)\, d\mu_{\tau_k}(a) \\
&\quad + \int_{R_+} \{ \int_{R_+} O(\phi(|b-a|)) P(X_{\tau_k} \leqslant x | \tau_k = a)\, d\mu_{\tau_{k+n}|\tau_k = a}(b)\}\, d\mu_{\tau_k}(a) \\
&= P(X_{\tau_k} \leqslant x) P(X_{\tau_{k+n}} \leqslant y) \\
&\quad + \{ \int_{R_+} O(E[\phi(|\tau_{k+n} - a|) | \tau_k = a])\, d\mu_{\tau_k}(a)\} P(X_{\tau_k} \leqslant x). \\
&= P(X_{\tau_k} \leqslant x) P(X_{\tau_{k+n}} \leqslant y) + O(E[\phi(|\tau_{k+n} - \tau_k|)])\, P(X_{\tau_k} \leqslant x).
\end{aligned}$$

Remarks : If the process X in Example 2.6 is ϕ-mixing but not necessarily ϕ-mixing stably, it is not clear how to relate the finite dimensional distributions of stopped sequences and the original process. One expects the mixing to hold for stopped sequene if $|\tau_k| \leqslant Kc < \infty$ a.s. and $\lim \frac{\tau_{k+n}-\tau_k}{n} \geqslant d > 0$ a.s. We have not been able to formulate the result under these conditions.

3. Moment Inequalities

Theorem 3.1 : *Suppose $\{\mathcal{F}_t\}$ is an increasing flow and $\{\zeta_s\}$ is a decreasing flow of σ-algebras defined on a probability space $(\Omega, \mathcal{F}, P)$. Further assume that $\{\mathcal{F}_t\}$ and $\{\zeta_s\}$ are ϕ-mixing weakly in the sense of Definition 2.1. Let ξ be $\mathcal{F}_t$-measurable and η be ζ_s-measurable real valued random variables such that $E|\xi|^p < \infty$, $E|\eta|^q < \infty$ with $1/p+1/q = 1$, $p > 0$. Then*

$$|E(\xi\eta)-E(\xi)E(\eta)| \leqslant 2[\phi(|t-s|)]^{1/p}(E|\xi|^p)^{1/p}(E|\eta|^q)^{1/q}. \quad \ldots (3.1)$$

Remarks : Proof of this theorem is the same as the classical proof for ϕ-mixing processes as the standard proof does not make use of the fact that the σ-algebras under consideration are generated by a stochastic process $\{X_t, t \geqslant 0\}$. For ϕ-mixing processes $\{X_t, t \geqslant 0\}$, see Theorem 5.1 in Roussas and Ioannides (1987). We now give a sketch for completeness.

Proof : Let

$$\xi = \sum_{i=1}^{k} \xi_i I_{A_i}, A_i \in \mathcal{F}_t, P(A_i) > 0, 1 \leqslant i \leqslant k,$$

and
$$\eta = \sum_{j=1}^{l} \eta_j I_{B_j}, B_j \in \zeta_s, P(B_j) > 0, 1 \leqslant j \leqslant l,$$

where ξ_i and η_j are real numbers and I_A denotes the indicator function of a set A. Note that

$$\begin{aligned}
&|E(\xi\eta)-E(\xi)\, E(\eta)| \\
&= |\sum_i \sum_j \xi_i\eta_j\{P(A_i \cap B_j)-P(A_i)P(B_j)\}| \\
&= |\sum_i \sum_j \xi_i\eta_j\, P(A_i)\, [P(B_j\,|\,A_i)-P(B_j)]| \\
&= |\sum_i \xi_i y_i P(A_i)\,|\ (\text{where } y_i = \sum_j \eta_j\, [P(B_j\,|\,A_i)-P(B_j)]) \\
&= |E(\xi Y)|\,(\text{where } Y = \sum_{i=1}^{k} y_i\, I_{A_i} \\
&\leqslant E|\xi Y| \\
&\leqslant (E|\xi|^p)^{1/p}\,(E|\,Y^q)^{1/q}. \qquad \ldots (3.2)
\end{aligned}$$

It can now be checked by arguments similar to those given in Theorem 5.1 of Roussas and Ioannides (1978) that

$$(E\,|\,Y\,|^q)^{1/q} \leqslant 2[\phi(\,|\,t-s\,|\,)]^{1/p}\,(E\,|\,\eta\,|^q)^{1/q} \qquad \ldots \quad (3,3)$$

and, hence from (3.2) and (3.3), it follows that

$$|\,E(\xi\eta)-E(\xi)E(\eta)\,| \leqslant 2[\phi(\,|\,t-s\,|\,)]^{1/p}\,(E\,|\,\xi\,|^p)^{1/p}\,(E\,|\,\eta\,|^q)^{1/q}.$$

This proves the theorem a for simple $\mathcal{F}_t$-measurable random variable ξ and a simple ζ_s-measurable random variable η. The general case again follows from Lemmas 4.1 and 4.2 of Roussas and Ioannides (1987). □

Theorem 3.2 : *Suppose* $\{\mathcal{F}_t\}$ *and* $\{\zeta_s\}$ *are* ϕ*-mixing weakly as in Theorem* 3.1. *let* ξ *and* η *be* $\mathcal{F}_t$*-measurable and* ζ_s*-measurable real valued random variables respectively such that*

$$|\,\xi\,| \leqslant M_1 \text{ a.s.}, \; |\,\eta\,| \leqslant M_2 \text{ a.s.}$$

Then

$$|\,E(\xi\eta)-E(\xi)E(\eta)\,| \leqslant 2\,\phi(\,|\,t-s\,|\,)M_1M_2. \qquad \ldots \quad (3.4)$$

Remark : Proof of this theorem is same as that of Theorem 5.2 in Roussas and Ioannides (1987) by replacing $\mathcal{F}_1^k$ by $\mathcal{F}_t$ and $\mathcal{F}_{k+n}^\infty$ by ζ_s. A more general version of Theorem 3.2 is as follows.

Theorem 3.3 : *Suppose an increasing flow* $\{\mathcal{F}_t\}$ *and a decreasing flow* $\{\zeta_s\}$ *are* ϕ*-mixing weakly as in Theorem* 3.1.

Further suppose that

ξ_1 *is* $\mathcal{F}_{t_1}$*-measurable,*

ξ_i *is* $\mathcal{F}_{t_i}$*-measurable and* ζ_{s_i}*-measurable for* $2 \leqslant i \leqslant n-1$,

and ξ_n *is* ζ_{s_n}*-measurable*

where $t_i \uparrow$ *and* $s_i \uparrow$. *Assume that*

$$|\xi_i| \leqslant M_i \text{ a.s.}, \; 1 \leqslant i \leqslant n.$$

Then

$$|\,E(\xi_1\,\xi_2 \ldots \xi_n)-E(\xi_1)\,E(\xi_2) \ldots E(\xi_n)\,|$$

$$\leqslant 2\left\{\sum_{i=1}^{n-1} \phi(\,|\,s_{i+1}-t_i\,|\,\right\} \prod_{i=1}^{n} M_i. \qquad \ldots \quad (3.5)$$

Proof : The result holds for $n = 2$ by Theorem 3.2. Suppose it holds for $n-1$. Then

$$\begin{aligned}
&|E(\xi_1 \xi_2 \ldots \xi_n)-E(\xi_1)\, E(\xi_2) \ldots E(\xi_n)| \\
&\leqslant |E(\xi_1 \xi_2 \ldots \xi_n)-E(\xi_1 \xi_2 \ldots \xi_{n-1})E(\xi_n| \\
&+E|\xi_n|\, |E(\xi_1 \ldots \xi_{n-1})-E(\xi_1) \ldots E(\xi_{n-1})| \\
&= I_1+I_2 \text{ (say)}.
\end{aligned}$$

Observe that

$$\xi_1 \ldots \xi_{n-1} \text{ is } \mathcal{F}_{t_{n-1}}\text{-measurable,}$$

and $$\xi_n \text{ is } \zeta_{s_n}\text{-measurable.}$$

Hence $$\begin{aligned} I_1 &\leqslant 2\,\phi(|s_n-t_{n-1}|)E|\xi_1|E|\xi_2 \ldots \xi_n| \\ &\leqslant 2\,\phi(|s_n-t_{n-1}|)\, M_1 M_2 \ldots M_n. \qquad \ldots \quad (3.7)\end{aligned}$$

By induction argument,

$$|E(\xi_1 \ldots \xi_{n-1})-E(\xi_1) \ldots E(\xi_{n-1})| \leqslant 2\left\{\sum_{i=1}^{n-2}\phi(|s_{i+1}-t_i|)\right\} M_1 \ldots M_{n-1}. \qquad \ldots \quad (3.8)$$

Hence $$I_2 \leqslant 2\left\{\sum_{i=1}^{n-2}\phi(|s_{i+1}-t_i|)\right\} M_1 M_2 \ldots M_n. \qquad \ldots \quad (3.9)$$

Combining (3.6)—(3.9), we have

$$\begin{aligned}&|E(\xi_1 \xi_2 \ldots \xi_n)-E(\xi_1)E(\xi_2) \ldots E(\xi_n)| \\ &\leqslant 2\left\{\sum_{i=1}^{n-1}\phi(|s_{i+1}-t_i|)\right\} M_1 M_2 \ldots M_n. \;\square \qquad \ldots \quad (3.10)\end{aligned}$$

Theorem 3.4 : *Suppose the flows $\{\mathcal{F}_t\}$ and $\{\zeta_s\}$ are as defined in Theorem* 3.3. *Define ξ_i, $1 \leqslant i \leqslant n$ as before. Further suppose that*

$$E|\xi_i|^{p_i} < \infty,\; p_i > 1 \text{ and } \sum_{i=1}^{2n}\frac{1}{p_i} = 1.$$

Let $r_n = max\,(p_1 \ldots, b_n)$. Then

$$\begin{aligned}&|E(\xi_1 \ldots \xi_n)-E(\xi_1) \ldots E(\xi_n)| \\ &\leqslant 2\left\{\sum_{i=1}^{n-1}\phi(|s_{i+1}-t_i|)\right\}^{1/r_n} \prod_{i=1}^{n}\{E|\xi_i|^{p_i}\}^{1/p_i}. \qquad \ldots \quad (3.11)\end{aligned}$$

Proof : Clearly the theorem holds for $n = 2$ by Theorem 3.1. Assume that the theorem holds for $n-1$. Then

$$
\begin{aligned}
&|E(\xi_1\xi_2\ldots\xi_n)-E(\xi_1)\,E(\xi_2)\ldots E(\xi_n)| \\
&\leqslant |E(\xi_1\xi_2\ldots\xi_n)-E(\xi_1)\,E(\xi_2\ldots\xi_n)| \\
&+E|\xi_1|\,|E(\xi_2\ldots\xi_n)-E(\xi_2)\ldots E(\xi_n)| \\
&= I_3+I_4 \text{ (say)}. \qquad \ldots (3.12)
\end{aligned}
$$

Note that

$$
\begin{aligned}
I_3 &\leqslant 2\,[\phi(|s_2-t_1|)]^{1/p_1}(E|\xi_1|^{p_1})^{1/p_1}(E|\xi_2\ldots\xi_n|^{p})^{1/p}(\text{where } 1/p+1/p_1 = 1) \\
&\leqslant 2\,[\phi(|s_2-t_1|)]^{1/p_1}(E|\xi_1|^{p_1})^{1/p_1}(E|\xi_2|^{p_2})^{1/p_2}\ldots(E|\xi_n|^{p_n})^{1/p_n} \qquad \ldots (3.13)
\end{aligned}
$$

by Hölder's inequality. On the otherhand

$$
\begin{aligned}
&|E(\xi_2\ldots\xi_n)-E(\xi_2)\ldots E(\xi_n)| \\
&\leqslant 2\sum_{i=1}^{n-1}\{[\phi(|s_{i+1}-t_i|)\}^{1/r^*_{n-1}}\prod_{i=2}^{n}(E|\xi_i|^{q_i})^{1/q_i} \qquad \ldots (3.14)
\end{aligned}
$$

by induction hypothesis where $r^*_{n-1} = \max(q_2, \ldots, q_n)$, $q_i = \dfrac{p_i}{p}$, $2 \leqslant i \leqslant n$ (note that $1/q_2+\ldots+1/q_n = 1$). Observe that $r^*_{n-1} < r_n$. Furthermore

$$E|\xi_1| \leqslant (E|\xi_1|^{p_1})^{1/p_1} \qquad \ldots (3.15)$$

as $p_1 > 1$. Hence

$$I_4 \leqslant 2\left\{\sum_{i=1}^{n-1}[\phi(|s_{i+1}-t_i|)]^{1/r_n}\right\}\left(E|\xi_1|^{p_1}\right)^{1/p_1}\ldots\left(E|\xi_n|^{p_n}\right)^{1/p_n} \qquad \ldots (3.16)$$

Relations (3.12)—(3.16) prove the result since $r_n \geqslant p_1$. □

In the light of Theorems 3.1 to 3.4 obtained, it is clear that one can obtain the following results for flows of σ-algebras $\{\mathscr{F}_t\}$ and $\{\zeta_s\}$ which are ϕ-mixing strongly with respect to sequences $\{\tau_n\}$ and $\{S_m\}$ as defined in Definition 2.3. We omit the proofs. One has to replace $\phi(|t-s|)$ by $E\phi(|\tau_n-S_m|)$ at the appropriate step in the argument.

Theorem 3.5 : *Suppose $\{\tau_n, n \geqslant 1\}$ and $\{S_n, n \geqslant 1\}$ are increasing sequences of non-negative random variables and the increasing flow $\{\mathscr{F}_t\}$ is ϕ-mixing strongly with the decreasing flow $\{\zeta_s\}$ with respect to $\{\tau_n\}$ and $\{S_n\}$ in the sense of Definition 2.3.*

Let ξ be $\mathcal{F}_{\tau_n}$-measurable and η be ζ_{S_m}-measurable real-valued random variables such that

$$E|\xi|^p < \infty, E|\eta|^q < \infty, p > 0, \frac{1}{p}+\frac{1}{q}= 1.$$

Then

$$|E(\xi\eta)-E(\xi)\, E(\eta)| \leqslant 2(E[\phi(|\tau_n-S_m|)])^{1/p}\, (E|\xi|^p)^{1/p}\, (E|\eta|^q)^{1/q}.$$

Theorem 3.6 : *Suppose* $\{\mathcal{F}_t\}$ *is* ϕ*-mixing strongly with* $\{\zeta_s\}$ *with respect to sequences* $\{\tau_n\}$ *and* $\{S_m\}$ *as in Theorem* 3.5. *Further suppose that* ξ *and* η are $\mathcal{F}_{\tau_n}$ *measurable and* ζ_{S_m}*-measurable real valued random variables such that*

$$|\xi| \leqslant M_1 \text{ a.s. and } |\eta| \leqslant M_2 \text{ a.s.}$$

Then
$$|E(\xi\eta)-E(\xi)\, E(\xi)| \leqslant 2\, E\{\phi(|\tau_n-S_m|)\}\, M_1M_2.$$

Theorem 3.7 : *Suppose* $\{\mathcal{F}_t\}$ *is* ϕ*-mixing strongly with* $\{\zeta_s\}$ *with respect to* $\{\tau_n\}$ *and* $\{S_m\}$ *as in Theorem* 3.5. *Further suppose that*

ξ_1 *is* $\mathcal{F}_{\tau_1}$*-measurable,*

ξ_i *is* $\mathcal{F}_{\tau_i}$*-measurable and* ζ_{S_i}*-measurable for* $2 \leqslant i \leqslant n-1$,

and ξ_n *is* ζ_{S_n}*-measurable.*

Further suppose that

$$|\xi_i| \leqslant M_i \text{ a.s. } 1 \leqslant i \leqslant n.$$

Then
$$|E(\xi_1\xi_2\ldots\xi_n)-E(\xi_1)\ldots E(\xi_n)|$$
$$\leqslant 2\left[\sum_{i=1}^{n-1} E\{\phi(|S_{i+1}-\tau_i|)\}\right] \prod_{i=1}^{n} M_i.$$

Theorem 3.8 : *Suppose the flows* $\{\mathcal{F}_t\}$ *and* $\{\zeta_s\}$ *are as defined in Theorem* 3.7. *Define* $\{\xi_i\}$ *as in Theorem* 3.7. *Suppose that*

$$E|\xi_i|^{p_i} < \infty, p_i > 1,$$

and
$$\sum_{i=1}^{n} \frac{1}{p_i}= 1.$$

Let $r_n = max\,(p_1, \ldots, p_n)$. *Then*

$$|E(\xi_1\ldots\xi_n)-E(\xi_1)\ldots E(\xi_n)|$$
$$\leqslant 2\sum_{i=1}^{n-1} \{E\,\phi(|S_{i+1}-\tau_i|)\}^{1/r_n} \prod_{i=1}^{n} (E|\xi_i|^{p_i})^{1/p_i}.$$

4. Remarks on mixing for flows indexed by directed sets

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{\mathcal{F}_t, t \in I\}$ and $\{\zeta_s, s \in I\}$ be indexed families of sub σ-algebras of $\mathcal{F}$. Suppose I has a partial ordering $<$ such that

$$\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2} \text{ if } \tau_1 < \tau_2 \text{ and } \zeta_{\tau_1} \supset \zeta_{\tau_2} \text{ if } \tau_1 < \tau_2$$

and $d(\cdot,\cdot)$ is a metric on I. $\{\mathcal{F}_t, t\epsilon I\}$ is said to be an increasing flow and $\{\zeta_s, s\epsilon I\}$ is said to be a decreasing flow of σ-algebras.

Definition 4.1 : The increasing flow $\{\mathcal{F}_t, t\epsilon I\}$ is said to be *ϕ-mixing weakly* with the decreasing flow $\{\zeta_s, s\epsilon I\}$ if for every $A \in \mathcal{F}_t, t \in I$,

$$|P(A \cap B) - P(A)\, P(B)| \leqslant \phi(d(t, s))\, P(A)$$

for every $B \in \zeta_s$ where $\phi(d(t, s)) \downarrow 0$ as $d(t, s) \to \infty$.

Example 4.1. Let $I = \mathcal{Z}^d, d \geqslant 1$ denote the set $\{z = (z_1, \ldots, z_d) : z_i = \{0, \pm 1, \ldots\}, i = 1, \ldots, d$ equipped with the maximum norm $\|z\| = \max\limits_{1 \leqslant i \leqslant d} z_i$. For $z^{(1)} = (z_1^{(1)}, \ldots, z_d^{(1)})$ and $z^{(2)} = (z_1^{(2)}, \ldots, z_d^{(2)})$ in $\mathcal{Z}^d$, define $z^{(1)} < z^{(2)}$ if $z_i^{(1)} \leqslant z_i^{(2)}$ for $1 \leqslant i \leqslant d$. Let $\boldsymbol{X} \equiv \{X_z, z \in \mathcal{Z}^d\}$ be a family of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. $\boldsymbol{X}$ is called a d-dimensional random field. For any $z \in \mathcal{Z}^d$, define $\mathcal{F}_z$ to be the σ-algebra generated by $X_u, u < z$ and ζ_z be the σ-algebra generated by $X_v, v > z$. The d-dimensinal random field X is said to be *m-dependent* if for any finite subsets $U, V \subset \mathcal{Z}^d$, the set $\{X_u, u \in U\}$ is independent of $\{X_u, v \in V\}$ when $\|u - v\| > m$ for all $u \in U$ and $v \in V$. It is clear that $\{\mathcal{F}_z\}$ is ϕ-mixing weakly with $\{\zeta_z\}$ where

$$\phi(\|u\|) = 0 \text{ if } \|u\| > m.$$

Remarks : It is easy to see analogues of Theorem 3.1 to 3.4 hold for the flows $\{\mathcal{F}_t, t \in I\}$ and $\{\zeta_s, s \in I\}$ whenever they are ϕ-mixing weakly. In particular, one can obtain the following moment inequality for random fields. Discussion of analogues of other results is left to the reader.

Theorem 4.1 : *Define* $\{\mathcal{F}_z, Z \in \mathcal{Z}^d\}$ *and* $\{\zeta_z, z \in \mathcal{Z}^d\}$ *as in Example* 4.1. *Suppose* $\{\mathcal{F}_z, z \in \mathcal{Z}^d\}$ *is* ϕ-mixing weakly with $\{\zeta_z, z \in \mathcal{Z}^d\}$ *in the sense of Definition* 4.1. *Let* ξ *be* $\mathcal{F}_u$*-measurable and* η *be* ζ_v*-measurable such that*

$$E|\xi|^p < \infty \textit{ and } E|\eta|^q < \infty, 1/p + 1/q = 1, \; p > 0.$$

Then

$$|E(\xi\eta) - E(\xi)E(\eta)| \leqslant 2[\phi(\|u - v\|)]^{1/p} (E|\xi|^p)^{1/p} (E|\eta|^q)^{1/q}.$$

5. Remarks

We have generalized the concept of mixing and obtained some moment inequalities. The problems of obtaining moment inequalities for sums of random variables measurable with respect to σ-algebras which are ϕ-mixing strongly, central limit theorems, Berry-Esseen type bounds etc. remain open. We hope to come back to these problems in a future publication.

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