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Complementarity forms of theorems of Lyapunov and Stein, and related results

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Abstract

The well-known Lyapunov's theorem in matrix theory / continuous dynamical systems asserts that a (complex) square matrix A is positive stable (i.e., all eigenvalues lie in the open right-half plane) if and only if there exists a positive definite matrix X such that $AX + XA^*$ is positive definite. In this paper, we prove a complementarity form of this theorem: A is positive stable if and only if for any Hermitian matrix Q, there exists a positive semidefinite matrix X such that $AX + XA^* + Q$ is positive semidefinite and $X[AX + XA^* + Q] = 0$. By considering cone complementarity problems corresponding to linear transformations of the form $I - S$, we show that a (complex) matrix A has all eigenvalues in the open unit disk of the complex plane if and only if for every Hermitian matrix Q, there exists a positive semidefinite matrix X such that $X - AXA^* + Q$ is positive semidefinite and $X[X - AXA^* + Q] = 0$. By specializing Q (to $-I$), we deduce the well-known Stein's theorem in discrete linear dynamical systems: A has all eigenvalues in the open unit disk if and only if there exists a positive definite matrix X such that $X - AXA^*$ is positive definite. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Lyapunov; Stein; Positive stable; Complementarity; Degree

1. Introduction

In connection with the (asymptotic) stability of the linear dynamical system

$$\dot{x}(t) = Ax,$$

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Lyapunov [16] has proved that a (complex) matrix A is negative stable (i.e., all the eigenvalues of A lie in the open left-half plane) if and only if there exists a positive definite matrix X such that $XA + A^*X$ is negative definite. Gantmacher's equivalent reformulation of the above result [4, Chapter XV, Theorem 3′], now known as *Lyapunov's theorem*, is the following:

Theorem 1. *Let A be a complex square matrix and K be positive definite. Then A is positive stable if and only if there exists a (Hermitian) positive definite matrix X such that*

$$XA + A^*X = K.$$

One of the objectives of the paper is to prove the following complementarity form of the above theorem:

A is positive stable if and only if for each Hermitian matrix Q, there exists a (Hermitian) positive semidefinite matrix X such that $AX + XA^* + Q$ is positive semidefinite and

$$X[AX + XA^* + Q] = 0.$$

Such a result for a real matrix A with both Q and X real and symmetric was proved in [8] by considering a (linear) complementarity problem over the cone of symmetric positive semidefinite matrices and by showing the equivalence of positive stability of A and the so-called **P**-property of the linear transformation $L_A(X) := AX + XA^{\mathrm{T}}$:

$$X \text{ symmetric, } XL_A(X) \text{ symmetric and negative semidefinite} \quad \Rightarrow \quad X = 0.$$

In this paper, we extend this analysis to the general case by considering linear complementarity problem over the cone of (Hermitian) positive semidefinite matrices and by showing the equivalence of positive stability of A and the so-called $\mathbf{P}_1$-property of the linear transformation $L_A(X) := AX + XA^*$:

$$X \text{ Hermitian, } XL_A(X) + L_A(X)X \text{ negative semidefinite} \quad \Rightarrow \quad X = 0.$$

In matrix theory / discrete dynamical systems, the following result is known as *Stein's theorem.*

Theorem 2. *Let A be a complex square matrix and K be (Hermitian and) positive definite. Then all the eigenvalues of A lie in the open unit disk if and only if there exists a (Hermitian) positive definite matrix X such that*

$$X - AXA^* = K.$$

By considering cone complementarity problems specialized to the cone of Hermitian positive semidefinite matrices, we deduce the above result of Stein from its complementarity form:

A has all its eigenvalues in the open unit disk if and only if for each Hermitian matrix Q, there exists a (Hermitian) positive semidefinite matrix X such that $X - AXA^* + Q$ is positive semidefinite and

$$X[X - AXA^* + Q] = 0.$$

2. Preliminaries

For a matrix $A \in \mathbb{C}^{n\times n}$, we denote the operator norm (on $\mathbb{C}^n$) and the spectral radius by $\|A\|$ and $\rho(A)$, respectively. Throughout this paper, $\mathscr{H}^n$ denotes the space of all $n \times n$ complex Hermitian matrices. For $X, Y \in \mathscr{H}^n$, we define

$$\langle X, Y\rangle := \mathrm{tr}(XY),$$

where $\mathrm{tr}(XY)$ denotes the trace of the product XY. (Note that $\mathrm{tr}(XY)$ is real since X and Y are Hermitian.) We recall that a complex matrix M is said to be *positive semidefinite* (*definite*) if $\langle Mx, x\rangle \geqslant 0$ (> 0) for all $0 \neq x \in \mathbb{C}^n$ (where we assume that $\mathbb{C}^n$ carries the usual complex inner product). Note that positive semidefinite matrices on $\mathbb{C}^n$ are Hermitian, i.e., $X^* = X$, where X^* denotes the adjoint (= conjugate transpose) of X. For a linear operator $L : \mathscr{H}^n \to \mathscr{H}^n$, we denote its norm and the spectral radius (with respect to the above inner product on $\mathscr{H}^n$) by $\|L\|$ and $\rho(L)$, respectively. Let

$$\mathscr{H}^n_+ := \{X \in \mathscr{H}^n : X \text{ is positive semidefinite}\}.$$

We use the symbol

$$X \succeq (\succ) 0$$

to say that X is Hermitian and positive semidefinite (positive definite); the symbol $X \preceq 0$ means that $-X \succeq 0$. We list below some well-known matrix theoretic properties that are needed in the paper:

(a) $X \succeq 0 \Rightarrow UXU^* \succeq 0$ for any matrix U of appropriate size.
(b) $X \succeq 0,\ Y \succeq 0 \Rightarrow \langle X, Y\rangle \geqslant 0$.
(c) $X \succeq 0,\ Y \succeq 0, \langle X, Y\rangle = 0 \Rightarrow XY = YX = 0$.
(d) The cone $\mathscr{H}^n_+$ is self-dual, i.e., if $X \in \mathscr{H}^n$ and $\langle X, Y\rangle \geqslant 0$ for all $Y \succeq 0$, then $X \succeq 0$.
(e) Given X and Y in $\mathscr{H}^n$ with $XY = YX$, there exist a unitary matrix U, real diagonal matrices D and E such that $X = UDU^*$ and $Y = UEU^*$.

For two matrices X and Y in $\mathscr{H}^n$, we define the *Jordan product* by

$$X \circ Y := XY + YX.$$

2.1. *Semidefinite linear complementarity problems:* **P**- *and* **P**$_1$*-properties*

Given a finite dimensional real Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$, a closed convex cone $\mathscr{K}$ in $\mathscr{H}$, a linear transformation $T : \mathscr{H} \to \mathscr{H}$, and a vector $q \in \mathscr{H}$, we define the (cone) *complementarity problem* $\mathrm{CP}(T, \mathcal{K}, q)$ as the problem of finding a vector $x \in \mathcal{H}$ such that

$$x \in \mathcal{K}, \quad y := Tx + q \in \mathcal{K}^* \quad \text{and} \quad \langle x, y \rangle = 0,$$

where $\mathcal{K}^*$, called the dual cone, is defined by

$$\mathcal{K}^* := \{u \in \mathcal{H} : \langle u, z \rangle \geqslant 0 \; \forall z \in \mathcal{K}\}.$$

When $\mathcal{H} = \mathbb{R}^n$, $\mathcal{K} = \mathbb{R}^n_+$ (the non-negative orthant), and $\langle x, y \rangle$ is the usual inner product between vectors in $\mathbb{R}^n$, the above complementarity problem reduces, for a matrix $M \in \mathbb{R}^{n \times n}$, to the *linear complementarity problem* $\mathrm{LCP}(M, q)$: Find $x \in \mathbb{R}^n$ such that

$$x \geqslant 0, \quad y := Mx + q \geqslant 0 \quad \text{and} \quad \langle x, y \rangle = 0,$$

where the inequalities are defined in the componentwise sense. With numerous applications to optimization, engineering, economics, etc., see [3,6], the study of linear complementarity problem has received wide attention in the optimization community. In the LCP theory [3], the uniqueness of solution in $\mathrm{LCP}(M, q)$ for all q is addressed via the following equivalent conditions [3, Theorems 3.3.4 and 3.3.7]:

(i) Every principal minor of M is positive.

(ii) For every non-zero vector $x \in \mathbb{R}^n$, there exists an index i such that $x_i(Mx)_i > 0$.

(iii) The implication

$$x \in \mathbb{R}^n, \quad x * (Mx) \leqslant 0 \;\Rightarrow\; x = 0 \tag{1}$$

holds, where $x * (Mx)$ is the componentwise product of vectors x and Mx.

(iv) For every $q \in \mathbb{R}^n$, $\mathrm{LCP}(M, q)$ has a unique solution.

We recall from [5] that a matrix $M \in \mathbb{R}^{n \times n}$ is a **P**-matrix (or is said to have the **P**-property) if it satisfies condition (i) (or equivalently, either condition (ii) or condition (iii)). Thus, in the LCP setting, the uniqueness of solution in $\mathrm{LCP}(M, q)$ is described by the **P**-property of the matrix M.

In this paper, we consider another important instance of the cone complementarity problem obtained by putting $\mathcal{H} = \mathcal{H}^n$ and $\mathcal{K} = \mathcal{H}^n_+$. Corresponding to a linear transformation $L : \mathcal{H}^n \to \mathcal{H}^n$ and a matrix $Q \in \mathcal{H}^n$, the *semidefinite linear complementarity problem*, $\mathrm{SDLCP}(L, Q)$, is to find a matrix $X \in \mathcal{H}^n$ such that

$$X \succeq 0, \quad Y := L(X) + Q \succeq 0 \quad \text{and} \quad \langle X, Y \rangle = 0 \text{ (equivalently } XY = 0).$$

Dealing with the space $\mathcal{S}^n$ of real symmetric $n \times n$ matrices (and the cone $\mathcal{S}^n_+$ of real symmetric $n \times n$ positive semidefinite matrices), a similar complementarity problem was formally introduced, in a slightly different form, by Kojima et al. [14] to describe a model unifying various problems arising from systems and control theory and combinatorial optimization. In this setting (of $\mathcal{S}^n$), to address the uniqueness issue—when does $\mathrm{SDLCP}(L, Q)$ have a unique solution for all Q—two analogs of condition (iii) above, called the **P**- and the $\mathbf{P}_1$-properties, were introduced in Definitions 2 and 6 of [8] by replacing the componentwise product $x * (Mx)$ by the Jordan product $X \circ L(X)$ and the cone $\mathbb{R}^n_+$ by the cone $\mathcal{S}^n_+$. It was shown in [8] that the uniqueness in SDLCP(L, Q) for all Q implies the **P**-property and that the converse holds under an additional condition. Now, since our focus here is the space $\mathscr{H}^n$, we slightly modify the definitions in [8] and state the following.

Definition 3. For a linear transformation $L : \mathscr{H}^n \to \mathscr{H}^n$, we say that:
(i) L has the $\mathbf{P}_1$-property if

$$X \circ L(X) \preceq 0 \;\Rightarrow\; X = 0, \tag{2}$$

(ii) L has the **P**-property if

$$X \text{ and } L(X) \text{ commute}, \quad XL(X) \preceq 0 \;\Rightarrow\; X = 0. \tag{3}$$

Thus, (2) and (3) can be considered as non-commutative and commutative analogs of (1) for $\mathscr{H}^n$ with respect to the cone $\mathscr{H}^n_+$. We note here that (1) and (2) are two instances of a more general $\mathbf{P}_1$-property that can be defined on any Euclidean Jordan Algebra [8].

It is clear that the $\mathbf{P}_1$-property implies the **P**-property. While we show that these two properties are the same in some particular instances (see Theorems 6 and 11), in general they are different. This can be seen in the following example.

Example 1. Define the transformation L on $\mathscr{H}^2$, by

$$X = \begin{bmatrix} a & u+\mathrm{i}v \\ u-\mathrm{i}v & b \end{bmatrix} \quad \text{and} \quad L(X) = \begin{bmatrix} a-u & a+\mathrm{i}v \\ a-\mathrm{i}v & u+b \end{bmatrix},$$

where a, b, u, and v are real. In the particular case,

$$Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad L(Z) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

we have $Z \circ L(Z) \preceq 0$. Hence L does not have the $\mathbf{P}_1$-property. Now to check the **P**-property, suppose $XL(X) = L(X)X \preceq 0$. Then the $(1, 1)$ component of $XL(X)$, being real and non-positive, yields $a = v = 0$. Simple algebraic manipulations show that $b = u = 0$. Thus $X = 0$ proving the **P**-property of L.

It can be shown, as in Theorem 7 of [8], that if SDLCP(L, Q) has a unique solution for all $Q \in \mathscr{H}^n$, then L has the **P**-property. Since our focus here is the study complementarity forms of theorems of Lyapunov and Stein, we consider the solvability of SDLCP(L, Q). In this regard, we recall a result of Karamardian [13] specialized to the cone $\mathscr{H}^n_+$.

Theorem 4. *Consider a linear transformation $L : \mathscr{H}^n \to \mathscr{H}^n$. If the problems* SDLCP$(L, 0)$ *and* SDLCP(L, E), *for some positive definite $E \in \mathscr{H}^n$, have unique solutions (namely zero), then for all $Q \in \mathscr{H}^n$,* SDLCP(L, Q) *has a solution.*

As a consequence, we have:

Corollary 5. *If* $L : \mathcal{H}^n \to \mathcal{H}^n$ *has the* **P**-*property, then* SDLCP(L, Q) *has a solution for all* $Q \in \mathcal{H}^n$.

Proof. Suppose X is a solution of SDLCP(L, tI), where I denotes the identity matrix in $\mathcal{H}^n$ and $t \in \{0, 1\}$. Then

$$X \succeq 0, \quad X[L(X) + tI] = [L(X) + tI]X = 0,$$

and hence

$$XL(X) = L(X)X = -tX \preceq 0.$$

By the **P**-property, $X = 0$. Thus the problems SDLCP$(L, 0)$ and SDLCP(L, I) have unique solutions (namely zero). The result now follows from the above theorem. □

3. A complementarity form of Lyapunov's theorem

For a matrix $A \in \mathbb{C}^{n\times n}$, we define the transformation $L_A : \mathcal{H}^n \to \mathcal{H}^n$ by

$$L_A(X) = AX + XA^*. \tag{4}$$

In the following theorem, we refer to the equivalence of (a) and (d) as the *complementarity form of Lyapunov's theorem.*

Theorem 6. *For* $A \in \mathbb{C}^{n\times n}$ *and the corresponding* L_A, *the following are equivalent:*
(a) *A is positive stable.*
(b) *L_A has the* $\mathbf{P}_1$-*property.*
(c) *L_A has the* **P**-*property.*
(d) *For each* $Q \in \mathcal{H}^n$, SDLCP(L_A, Q) *has a solution.*
(e) *For each positive definite* $Q \in \mathcal{H}^n$, *there exists a positive definite* $X \in \mathcal{H}^n$ *such that* $AX + XA^* = Q$.
(f) *There exists a positive definite X in* $\mathcal{H}^n$ *such that* $AX + XA^*$ *is positive definite.*

Proof. (a) ⇒ (b): Let A be positive stable and suppose that there is a non-zero $X \in \mathcal{H}^n$ such that $X \circ L(X) \preceq 0$. We write $X = UDU^*$, where U is a unitary matrix and D is a real, non-zero, diagonal matrix. We define

$$E := U^*[L_A(X)]U \quad \text{and} \quad B := U^*AU$$

so that

$$E = BD + DB^*, \quad E \circ D \preceq 0 \quad \text{and} \quad B \text{ is positive stable.} \tag{5}$$

We first consider the case when D is invertible. Then we have $D^{-1}(E \circ D)D^{-1} \preceq 0$ and so

$$B + B^* + (D^{-1}BD) + (D^{-1}BD)^* \preceq 0.$$

This implies that $4\,\mathrm{Re}\,\mathrm{tr}(B) \leqslant 0$ contradicting the fact that B is positive stable. Now, we consider the case of D being singular (but still non-zero). In this case, we may write without loss of generality

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix},$$

where D_1 is invertible, the sizes of B_1 and E_1 agree with that of D_1, and $E_3 = (E_2)^*$. Now $E = BD + DB^*$ leads to

$$E_1 = B_1 D_1 + D_1 B_1^* \quad \text{and} \quad E_2 = D_1 B_3{}^*$$

while $E \circ D \preceq 0$ gives

$$E_1 \circ D_1 \preceq 0 \quad \text{and} \quad E_2 = 0.$$

We see that $B_3 = 0$; since B is positive stable, B_1 is also positive stable. We thus have

$$E_1 = B_1 D_1 + D_1 B_1^*, \quad E_1 \circ D_1 \preceq 0 \quad \text{and} \quad B_1 \text{ is positive stable.} \tag{6}$$

As in the first case, (6) leads to a contradiction. Thus $X = 0$ proving (b).
The implication (b) $\Rightarrow$ (c) is obvious. The implication (c) $\Rightarrow$ (d) follows from Corollary 5.

(d) $\Rightarrow$ (e): A proof of this can be given along the lines of the proof of Theorem 5 in [8]. For the sake of completeness, we provide an alternate, shorter proof (due to a referee). Assume that (d) holds and let $Q \in \mathscr{H}^n$ be positive definite. Let X be a solution of SDLCP$(L_A, -Q)$ so that $X \succeq 0$, $Y = AX + XA^* - Q \succeq 0$ and $XY = 0$. We claim that X is non-singular. In fact, if $Xu = 0$ with $u \in \mathbb{C}^n$, then

$$0 \leqslant \langle Yu, u \rangle = -\langle Qu, u \rangle.$$

Since Q is positive definite, we have $u = 0$. Now from the non-singularity of X and the equality $XY = 0$, we get $Y = 0$ proving $AX + XA^* = Q$. Thus we have (e).

The implication (e) $\Rightarrow$ (f) follows immediately from putting $Q = I$.

The implication (f) $\Rightarrow$ (a) is well known, see [11, Theorem 2.2.1]. □

Remark 1. By working with a real A, the space $\mathscr{S}^n$ of symmetric $n \times n$ matrices, and the cone of $n \times n$ symmetric positive semidefinite matrices, we can modify the proof of (a) $\Rightarrow$ (b) and show that *A is positive stable if and only if $L_A(X) = AX + XA^{\mathrm{T}}$ has the* $\mathbf{P}_1$*-property*:

$$X \in \mathscr{S}^n, \quad X \circ L_A(X) \preceq 0 \;\Rightarrow\; X = 0.$$

This result improves Theorem 5 in [8], where it was shown that A being positive stable is equivalent to the **P**-property of L_A.

Remark 2. We note that item (e) asserts the existence of a positive definite X. When A is positive stable, because of the $\mathbf{P}_1$-property, the transformation L_A is invertible. Thus the solution X in (e) is unique. If one is interested in a positive semidefinite solution of $AX + XA^* = Q$, there is a shorter, elegant proof due to Osman Guler: Assume that (d) holds and let $Q \in \mathscr{H}^n$ be positive definite. Let X be a solution of SDLCP$(L_A, -Q)$. Since $X \succeq 0$, $Y = AX + XA^* - Q \succeq 0$ and $XY = 0$, we easily verify that $Y^3 = YYY = -YQY$. Since Y is positive semidefinite and Q is positive definite, It follows that $Y^3 = 0$ implying $Y = 0$.

A slight modification of the above argument shows the following (known fact):

When A is positive stable, for any positive semidefinite Q, there exists a unique positive semidefinite X such that $AX + XA^* = Q$.

4. A cone complementarity result

Before considering a complementarity form of Stein's theorem, in this section we consider a few results on cone complementarity problems. First we recall two results of Schneider, who in [18], unifies and extends the results of Lyapunov and Stein.

Lemma 7 [18]. *Let $\mathscr{K}$ be a (pointed) closed convex cone with interior $\mathscr{K}^\circ$ in a finite dimensional real (topological) vector space V. Let R and S be linear transformations on V satisfying the conditions $S(\mathscr{K}) \subseteq \mathscr{K}$ and either $R(\mathscr{K}^\circ) \supseteq \mathscr{K}^\circ$ or $R(\mathscr{K}^\circ) \cap \mathscr{K}^\circ = \emptyset$. If $T = R - S$, then the following are equivalent:*
(i) *R is non-singular, $R^{-1}(\mathscr{K}) \subseteq \mathscr{K}$ and $\rho(R^{-1}S) < 1$,*
(ii) *T is non-singular and $T^{-1}(\mathscr{K}^\circ) \subseteq \mathscr{K}^\circ$,*
(iii) *There exists $u \in \mathscr{K}^\circ$ such that $Tu \in \mathscr{K}^\circ$.*

Theorem 8 [18]. *Let A, C_k $(k = 1, 2, \ldots, s)$ be complex $n \times n$ matrices, which can be simultaneously triangulated. Suppose the eigenvalues of A and C_k under a natural correspondence are α_i, $\gamma_i^{(k)}$, $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, s$. For $X \in \mathscr{H}^n$, let*

$$T(X) := AXA^* - \sum_1^s C_k X C_k^*.$$

Then the following are equivalent:
(1) $|\alpha_i|^2 - \sum_1^s |\gamma_i^{(k)}|^2 > 0$ *for all* $i = 1, 2, \ldots, n$.
(2) *For every positive definite $Q \in \mathscr{H}^n$, there exists a unique $X \in \mathscr{H}^n$ such that $T(X) = Q$.*
(3) *There exists a positive definite $X \in \mathscr{H}^n$ such that $T(X)$ is positive definite.*

Motivated by the complementarity version of Lyapunov's theorem, we may ask whether it is possible to have complementarity versions of Lemma 7 and Theorem 8. The following example answers this in the negative. It also shows that various generalizations of Schneider's theorem given in [19] do not have complementarity

Example 2. Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Let $\mathscr{K} = \mathscr{H}^2_+$ and define R, S, and T on $\mathscr{H}^2$ by

$$R(X) = AXA^*, \quad S = 0 \quad \text{and} \quad T = R - S.$$

We easily verify that
(1) R is invertible (because $R^{-1}(X) = A^{-1}X(A^*)^{-1}$) and $\rho(R^{-1}S) < 1$.
(2) $R(\mathscr{K}^\circ) \supseteq \mathscr{K}^\circ$ (because $A^{-1}X(A^*)^{-1}$ is positive definite whenever X is).
(3) $T^{-1}(\mathscr{K}^\circ) \subseteq \mathscr{K}^\circ$.
Clearly, condition (i) in Lemma 7 and condition (1) in Theorem 8 hold. However, routine, simple algebraic manipulations show that SDLCP(T, Q) has no solution. In spite of this drawback, we present a positive result for transformations of the form $\alpha I - S$.

We now turn our attention to cone complementarity problems. We recall from the complementarity theory [9] that solutions of CP($T, \mathscr{K}, q$) are precisely the zeros of (the so-called fixed-point map)

$$F(x) := x - \Pi_{\mathscr{K}}(x - [Tx + q]),$$

where $\Pi_{\mathscr{K}}(x)$ denotes the projection of x onto $\mathscr{K}$. For some recent literature on the vast area of complementarity theory, see [6].

Theorem 9. *Let $\mathscr{K}$ be a closed convex cone in a finite dimensional real Hilbert space $\mathscr{H}$. Let $S : \mathscr{H} \to \mathscr{H}$ be a linear transformation satisfying the conditions:*

$$S(\mathscr{K} \cup \mathscr{K}^*) \subseteq \mathscr{K} \quad \text{and} \quad \rho(S) < \alpha.$$

Then for all $q \in \mathscr{H}$, the complementarity problem CP($\alpha I - S, \mathscr{K}, q$) has a solution.

Proof. We assume without loss of generality that $\alpha = 1$. We first show that zero is the only solution of CP($I - S, \mathscr{K}, 0$). Clearly, zero is a solution of this problem. Suppose that x is any solution of CP($I - S, \mathscr{K}, 0$) so that

$$x \in \mathscr{K}, \quad y := (I - S)x \in \mathscr{K}^* \quad \text{and} \quad \langle x, y\rangle = 0.$$

Since $y = (I - S)x$ and $\rho(S) < 1$, we may write

$$x = (I - S)^{-1}y = \left(\sum_0^\infty S^n\right) y = y + Ay,$$

where $A := \sum_1^\infty S^n$. Since $S(\mathscr{K} \cup \mathscr{K}^*) \subseteq \mathscr{K}$ and $y \in \mathscr{K}^*$, we have $Sy \in \mathscr{K}$ and hence $S^n y \in \mathscr{K}$ for all $n \geqslant 1$; we see that $Ay \in \mathscr{K}$. Now

$$0 = \langle x, y\rangle = \langle y + Ay, y\rangle = \|y\|^2 + \langle Ay, y\rangle \geqslant \|y\|^2$$

since $\langle Ay, y\rangle \geqslant 0$ (which follows from $y \in \mathscr{K}^*$ and $Ay \in K$). We see that $y = 0$ and hence $x = 0$. For $t \in [0, 1]$, by working with tS instead of S, we see that zero is the only solution of $\mathrm{CP}(I - tS, \mathscr{K}, 0)$.

We now fix a $q \in \mathscr{H}$. To show that $\mathrm{CP}(I - S, \mathscr{K}, q)$ has a solution, we consider the corresponding fixed-point map

$$F(x) := x - \Pi_{\mathscr{K}}[x - \{(I - S)x + q\}]$$

and show that $F(x)$ has a zero in H via some standard degree theoretic arguments [7,15]. Define, for $x \in \mathscr{H}$ and $t \in [0, 1]$,

$$\phi(x, t) := x - \Pi_{\mathscr{K}}[x - \{(I - tS)x + tq\}].$$

Clearly, $\phi(x, t)$ defines a homotopy between the identity map $\phi(x, 0) = x$ and

$$\phi(x, 1) = F(x).$$

We now claim that as t varies over $[0, 1]$, the zero sets of $\phi(\cdot, t)$ are (uniformly) bounded. Suppose there exist sequences $\{x^k\}$ in $\mathscr{H}$ and $\{t_k\}$ in $[0, 1]$ such that $\phi(x^k, t_k) = 0$ for all natural numbers k, and $\|x^k\| \rightarrow \infty$. Then x^k solves $\mathrm{CP}(I - t_kS, \mathscr{K}, t_kq)$ and so

$$x^k \in \mathscr{K}, \quad y^k := (I - t_kS)x^k + t_kq \in \mathscr{K}^* \quad \text{and} \quad \langle x^k, y^k\rangle = 0. \tag{7}$$

We may assume that $t_k \rightarrow t^*$ and $(x^k/\|x^k\|) \rightarrow x^*$. It follows from (7) that

$$x^* \in \mathscr{K}, \quad y^* := (I - t^*S)x^* \in \mathscr{K}^*, \quad \text{and} \quad \langle x^*, y^*\rangle = 0.$$

Since x^* has norm one, it is a non-zero solution of $\mathrm{CP}(I - t^*S, \mathscr{K}, 0)$ contradicting an earlier observation. Thus we have the claim. Now let Ω be a bounded open set in $\mathscr{H}$ containing the zeros of $\phi(\cdot, t)$ as t varies in $[0, 1]$. (Note that $0 \in \Omega$.) By the homotopy invariance property of degree [15, Theorem 2.1.2,], we conclude

$$\deg(F(x), \Omega, 0) = \deg(\phi(\cdot, 1), \Omega, 0) = \deg(\phi(\cdot, 0), \Omega, 0).$$

Since $\phi(\cdot, 0)$ is the identity map and zero belongs to Ω, we see that $\deg(\phi(\cdot, 0), \Omega, 0) = 1$ by Theorem 1.1.4 of [15]. Thus,

$$\deg(F(x), \Omega, 0) = 1,$$

and by Theorem 2.1.1 of [15], F has a zero in Ω. This zero solves $\mathrm{CP}(I - S, \mathscr{K}, q)$. $\square$

Corollary 10. *Let $\mathscr{K}$ be a pointed (meaning $\mathscr{K} \cap -\mathscr{K} = \{0\}$), self-dual (meaning $\mathscr{K}^* = \mathscr{K}$) closed convex cone in a finite dimensional real Hilbert space $\mathscr{H}$, $S : \mathscr{H} \rightarrow \mathscr{H}$ be linear with $S(\mathscr{K}) \subseteq \mathscr{K}$. Then for $T = \alpha I - S$, the following are equivalent:*

(1) $\rho(S) < \alpha$.

(2) $CP(T, \mathscr{K}, q)$ has a solution for all $q \in \mathscr{H}$.

(3) There exists $u \in \mathscr{K}^\circ$ such that $Tu \in \mathscr{K}^\circ$.

Proof. The implication (1) $\Rightarrow$ (2) follows from the previous theorem. To see (2) $\Rightarrow$ (3), first note that $\mathscr{K}^\circ \neq \emptyset$ (since $\mathscr{K}$ is pointed and self-dual). Take $e \in \mathscr{K}^\circ$ and consider a solution $\bar{x}$ of CP$(T, \mathcal{K}, -e)$. Then $\bar{x} \in \mathcal{K}$ and $T\bar{x} - e \in \mathcal{K}$. We see that $T\bar{x} \in \mathcal{K} + \mathcal{K}^\circ \subseteq \mathcal{K}^\circ$; by perturbing $\bar{x}$ we produce $u \in \mathcal{K}^\circ$ with $Tu \in \mathcal{K}^\circ$. The implication (3) ⇒ (1) is immediate from Lemma 7. □

Remark 3. When $\mathcal{H} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}^n_+$, the above corollary gives some well-known equivalent properties of non-singular M-matrices (which are matrices of the form $A = sI - B$, where $B \geqslant 0$ and $s > \rho(B)$), see Chapter 6 in [2].

5. A complementarity form of Stein's theorem

In this section, we fix a matrix $A \in \mathbb{C}^{n\times n}$ and define transformations S and S_A on $\mathcal{H}^n$ by

$$S(X) = AXA^* \quad \text{and} \quad S_A(X) = X - AXA^*.$$

Theorem 11. *The following are equivalent:*
(a) $\rho(S) < 1$.
(b) $\rho(A) < 1$, *i.e., all eigenvalues of A lie in the open unit disk.*
(c) S_A *has the* $\mathbf{P}_1$*-property.*
(d) S_A *has the* **P***-property.*
(e) *For each* $Q \in \mathcal{H}^n$, SDLCP(S_A, Q) *has a solution.*
(f) *For each positive definite* $Q \in \mathcal{H}^n$, *there exists a positive definite* $X \in \mathcal{H}^n$ *such that* $X - AXA^* = Q$.
(g) *There exists a positive definite* X *in* $\mathcal{H}^n$ *such that* $X - AXA^*$ *is positive definite.*

Before giving a proof of this result, we note that the equivalence of (a), (e) and (g) follows from Corollary 10. The (perhaps well known) equivalence of (a) and (b) is easy to see. Thus we have *the complementarity form of Stein's theorem*, namely, the equivalence of (b) and (e). However, in the proof below, we derive the complementarity form via the $\mathbf{P}_1$- and **P**-properties.

Proof. (a) ⇒ (b): If $Au = \lambda u$ for some non-zero vector u and a scalar λ, then for the matrix $X := uu^*$, we have $S(X) = A(uu^*)A^* = (Au)(Au)^* = |\lambda|^2 X$. Thus, when $\rho(S) < 1$, we must have $\rho(A) < 1$.

(b) ⇒ (c): Suppose (b) is true and $X \in \mathcal{H}^n$ is a non-zero matrix satisfying $X \circ S_A(X) \preceq 0$. We write $X = UDU^*$, where U is unitary and D is real, non-zero, and diagonal. Letting

$$E := U^* S_A(X) U \quad \text{and} \quad B := U^* A U,$$

we get

$$E = D - BDB^*, \quad E \circ D \preceq 0 \quad \text{and} \quad \rho(B) < 1. \tag{8}$$

We first consider the case when D is invertible. Then from $D^{-1}(E \circ D)D^{-1} \preceq 0$, we get

$$2I - [BDB^*D^{-1} + D^{-1}BDB^*] \preceq 0. \tag{9}$$

Since $\det(BDB^*D^{-1}) < 1$, there is an eigenvalue μ with $|\mu| < 1$ and a non-zero vector v such that $(BDB^*D^{-1})v = \mu v$. From (9), we get $2\|v\|^2 - 2\,\mathrm{Re}(\mu)\|v\|^2 \leqslant 0$ which contradicts $|\mu| < 1$. Thus D is not invertible. We write without loss of generality,

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix},$$

where D_1 is invertible. Now $D \circ E \preceq 0$ gives

$$D_1 \circ E_1 \preceq 0 \quad \text{and} \quad E_2 = 0. \tag{10}$$

On the other hand, the equation $E = D - BDB^*$ gives

$$E_1 = D_1 - B_1 D_1 B_1^* \quad \text{and} \quad B_1 D_1 B_3^* = 0. \tag{11}$$

We claim that B_1 is non-singular. If this is not the case, let $B_1^* u = 0$ for some non-zero u. Then $E_1 u = D_1 u$ and so $0 < \langle D_1^2 u, u\rangle = \langle D_1 E_1 u, u\rangle = \frac{1}{2}\langle (D_1 \circ E_1)u, u\rangle \leqslant 0$ leading to a contradiction. Hence B_1 is non-singular and by (11), $B_3 = 0$. It follows (from $\rho(B) < 1$) that $\rho(B_1) < 1$. This, together with (10) and (11), as in the first case, leads to a contradiction. Thus D and hence X must be zero proving the $\mathbf{P}_1$-property of S_A. The implications (c) ⇒ (d) ⇒ (e) ⇒ (f) ⇒ (g) are similar to the corresponding ones in the proof of Theorem 6. Finally, the proof of the implication (g) ⇒ (a) follows from Corollary 10. □

Remark 4. By working with a real matrix A and the cone of real, symmetric, positive semidefinite $n \times n$ matrices in the space $\mathcal{S}^n$ of real symmetric $n \times n$ matrices, one can state a result similar to Theorem 11 for the transformation $X \mapsto X - AXA^{\mathrm{T}} : \mathcal{S}^n \to \mathcal{S}^n$. We omit the details.

Remark 5. In matrix theory, it is well known that the theorems of Lyapunov and Stein are equivalent in the sense that each can be deduced from the other by means of certain transformations. In fact, the transformation $A \mapsto B := (I + A)^{-1}(I - A)$ converts matrices with all eigenvalues in the open unit disk to matrices with all eigenvalues in the open right-half plane and shows that the equations

$$Y = BX + XB^* + Q$$

and

$$\tfrac{1}{2}(I + A)Y(I + A^*) = \overline{Y} = X - AXA^* + \overline{Q}$$

(with $\overline{Q} = (I + A)Q(I + A^*)$) are derivable from each other. Moreover, $\overline{Y}$ is positive definite if and only if Y is so.

Unfortunately, this transformation does not allow us to transform the complementarity form of Lyapunov's theorem into a complementarity form of Stein's theorem because under this transformation, we cannot go from the condition $XY = 0$ to $X\overline{Y} = 0$. We do not know if this conversion can be done by some other means.

Motivated by the equivalence of (a) and (c) in the previous theorem, *we may ask whether such an equivalence holds for any linear transformation* $S : \mathscr{H}^n \to \mathscr{H}^n$ *with* $S(\mathscr{H}^n_+) \subseteq \mathscr{H}^n_+$. Note that in view of Corollary 10, this amounts to asking *whether* $I - S$ *has the* $\mathbf{P}_1$*-property when* $S(\mathscr{H}^n_+) \subseteq \mathscr{H}^n_+$ *and* $\rho(S) < 1$. While we do not have an answer for this question, we have the following:

Proposition 12. *Suppose* $S : \mathscr{H}^n \to \mathscr{H}^n$ *is linear with* $\|S\| < 1$. *Then* $I - S$ *has the* $\mathbf{P}_1$*-property.*

Proof. Suppose X is non-zero and $X \circ [X - S(X)] \preceq 0$. Then

$$\langle X, X\rangle \leqslant \langle X, S(X)\rangle \leqslant \|X\|\,\|S(X)\|$$

by the Cauchy–Schwarz inequality. We see that $\|X\| \leqslant \|S(X)\|$ and hence $\|S\| \geqslant 1$. This contradicts our assumption and so $X = 0$, proving the $\mathbf{P}_1$-property. □

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For further reading

The following references are also of interest to the reader: [1,10,12,17,20].

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