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ON NON-NEGATIVITY OF THE NEAREST PROPORTIONAL TO SIZE SAMPLING DESIGN

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Abstract

The conditions under which the nearest proportional to size sampling design introduced by Gabler (1987) turns out to be non-negative are identified and these conditions are utilized in getting a rejective IPPS sampling plan.

1. Introduction

Consider a finite population U of size N and let $y_i(i = 1, \ldots, N)$ be the values of a variate y under enquiry. Our problem is to estimate the population total $Y = \sum_{i=1}^{N} y_i$ on the basis of a sample s of fixed size n drawn from the population with a probability $p_o(s)$.

Gabler (1987) has introduced the nearest proportional to size sampling design $p^*(s)$ defined as

$$p^*(s) = (\sum_{i \epsilon s} \lambda_i) p_o(s) \tag{1.1}$$

where $\lambda_i's(i = 1, \ldots, N)$ are all positive and are given by

$$\underset{\sim}{\pi}_o \, \underset{\sim}{\lambda} = \underset{\sim}{\pi}^* \tag{1.2}$$

$$\text{where } \underset{\sim 0}{\pi} = \begin{bmatrix} \pi_1{}^0 & \pi^0_{12} & \cdots & \pi^0_{1N} \\ \pi^0_{21} & \pi_2{}^0 & \cdots & \pi^0_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \pi^0_{N1} & \pi^0_{N2} & \cdots & \pi^0_N \end{bmatrix}$$

$\underset{\sim}{\lambda}' = (\lambda_1, \ldots, \lambda_N)$ and $\underset{\sim}{\pi}^{*\prime} = (\pi_1{}^*, \ldots, \pi_N{}^*), \pi_i{}^o(\pi_i{}^*)'s$ being the first order inclusion probabilities for the sampling design $p_0(s)(p^*(s))$ and $\pi_{ij}{}^{o}$'s being the second order inclusion probabilities for the pair of units for the design $p_0(s)$.

Gabler (1987) has also discussed how to realize $p^*(s)$ starting from an arbitrary fixed sample size (n) design $p_o(s)$ and he has called such a design a π^*ps design which satisfies $\sum_{i=1}^{N} \pi_i{}^* = n$.

The conditions under which the system of non-homogeneous linear equations (1.2) is consistent and admits of a non-negative solution for $\underset{\sim}{\lambda}$ are considered in this paper. These conditions are utilized in getting a rejective IPPS sampling plan.

2. Conditions for non-negativity of $p^*(s)$

First of all we note that the system of non-homogeneous linear equations $\underset{\sim}{\pi}_o \, \underset{\sim}{\lambda} = \underset{\sim}{\pi}^*$ is consistent if and only if Rank $(\underset{\sim}{\pi}_o)$ = Rank $(\underset{\sim}{\pi}_o \underset{\sim}{\pi}^*)$.

In case $\underset{\sim o}{\pi}$ is non-singular, the system possesses a unique solution

$$\underset{\sim}{\lambda} = \underset{\sim}{\pi}_o^{-1} \underset{\sim}{\pi}^*. \tag{2.1}$$

As a necessary and sufficient condition for non-negativity of $\underset{\sim}{\lambda}$ we make use of the Farkas' Lemma which states that if

$$\underset{\sim}{\pi}_o \underset{\sim}{y} \leq \underset{\sim}{0} \Longrightarrow (\underset{\sim}{\pi}^*, \underset{\sim}{y}) \leq 0 \text{ for any } \underset{\sim}{y}$$

then $\underset{\sim}{\pi}_o \, \underset{\sim}{\lambda} = \underset{\sim}{\pi}^*$ admits of a non-negative solution for $\underset{\sim}{\lambda}$. Here $(\underset{\sim}{\pi}^*, \underset{\sim}{y})$ denotes the inner product of the vectors $\underset{\sim}{\pi}^*$ and $\underset{\sim}{y}$.

Example 1

$$\text{Let } \underset{\sim}{\pi}_0 = \begin{bmatrix} \frac{n}{N} & \frac{n(n-1)}{N(N-1)} & & \frac{n(n-1)}{N(N-1)} \\ \vdots & \vdots & & \vdots \\ \frac{n(n-1)}{N(N-1)} & \frac{n(n-1)}{N(N-1)} & & \frac{n}{N} \end{bmatrix}$$

$$\begin{aligned}
\underset{\sim}{\pi}^{*\prime} &= (\pi_1^* \ldots \pi_N^*) \text{ where} \\
\pi_i^* &= \frac{n-1}{N-1} + \frac{N-n}{N-1} p_i, i = 1, \ldots, N. \\
\text{Now}, \underset{\sim}{\pi}_0 \underset{\sim}{y} \leq \underset{\sim}{0} &\implies \frac{n}{N} y_i + \frac{n(n-1)}{N(N-1)} \sum_{j \neq i} y_j < 0 \ \forall\, i \\
&\implies \{\frac{n}{N} - \frac{n(n-1)}{N(N-1)}\} y_i + \frac{n(n-1)}{N(N-1)} \sum_1^N y_i < 0 \ \forall\, i \\
&\implies \frac{n}{N} \cdot \frac{N-n}{N-1} y_i + \frac{n(n-1)}{N(N-1)} \sum_1^N y_i < 0 \ \forall\, i \\
&\implies \frac{N-n}{N-1} y_i + \frac{n-1}{N-1} \sum_{i=1}^N y_i < 0 \ \forall\, i \\
&\implies \frac{N-n}{N-1} \sum_{i=1}^N y_i p_i + \frac{n-1}{N-1} \sum_{i=1}^N y_i < 0 \\
&\implies (\underset{\sim}{\pi}^*, \underset{\sim}{y}) \leq 0.
\end{aligned}$$

This is true for any $\underset{\sim}{y}$. Hence by Farkas' Lemma $\underset{\sim}{\pi}_0 \lambda = \pi^*$ admits of a non-negative solution for $\underset{\sim}{\lambda}$ viz. $\lambda_i = \frac{N}{n} p_i, i = 1, \ldots, N$.

Example 2

Let $n = 2$ and $p_i \geq 0, \sum_1^N p_i = 1$. For $s = \{i, j\}$

we define $p_0(s) = p_i p_j L_2^{-1}$

where L_2^{-1} is the normalization factor. Then we have for $i = 1, \ldots, N$

$$\pi_i^0 = L_2^{-1} p_i (1 - p_i), \ \pi_i^* = p_i + p_i \sum_{j \neq i} \frac{p_j}{1 - p_j}$$

and $\pi_{ij}^0 \; = \; L_2^{-1} p_i p_j$.

Now $\underset{\sim}{\pi}_0 = L_2^{-1} \begin{bmatrix} p_1(1-p_1) & p_1p_2 & \cdots & p_1p_N \\ \vdots & \vdots & \cdots & \vdots \\ p_1p_N & p_2p_N & \cdots & p_N(1-p_N) \end{bmatrix}$ so that

$$\begin{aligned}
\underset{\sim}{\pi}_0\, \underset{\sim}{y} \leq \underset{\sim}{0} \;\; & \Rightarrow \;\; y_i p_i(1-p_i) + p_i \sum_{j\neq i} p_j y_j \;\leq\; 0 \;\forall\; i \\
& \Longrightarrow \;\; y_i p_i + \frac{p_i}{1-p_i} \sum_{j\neq i} p_j y_j \;\leq\; 0 \;\forall i \\
& \Longrightarrow \;\; y_i p_i + \frac{p_i}{1-p_i} \sum_{1}^{N} p_i y_i - \frac{p_i^2\, y_i}{1-p_i} \;\leq\; 0 \;\forall i \\
& \Longrightarrow \;\; \sum_{i=1}^{N} y_i p_i + \sum_{i=1}^{N} \frac{p_i}{1-p_i} \sum_{1}^{N} p_i y_i - \sum_{i=1}^{N} \frac{p_i^2 y_i}{1-p_i} \leq 0 \\
& \Longrightarrow \;\; (\underset{\sim}{\pi}^*, \underset{\sim}{y}) \leq 0
\end{aligned}$$

This is true for any $\underset{\sim}{y}$. Hence by Farkas' Lemma $\underset{\sim}{\pi}_0\, \underset{\sim}{\lambda} = \underset{\sim}{\pi}^*$ admits of a non-negative solution for $\underset{\sim}{\lambda}$ viz. $\lambda_i = L_2.\frac{1}{1-p_i}, i = 1, \cdots, N$.

In case $\underset{\sim}{\pi}_0\, \underset{\sim}{\lambda} = \underset{\sim}{\pi}^*$ does not possess a non-negative solution for $\underset{\sim}{\lambda}$ we can check it by means of the Theorem of Alternatives or the Duality Theorem which states that one of the following two assertions is true

(i) $\underset{\sim}{\pi}_0\, \underset{\sim}{\lambda} = \underset{\sim}{\pi}^*$ has a positive solution

(ii) $\underset{\sim}{\pi}_0\, \underset{\sim}{y} > \underset{\sim}{0}$ has a solution satisfying $(\underset{\sim}{\pi}^*, \underset{\sim}{y}) \leq 0$.

3. Rejective IPPS Sampling Plan

Let S and S_0 denote respectively the set of all possible samples and the set of arbitrary samples. We may define a sampling plan $p_0(s)$ which assigns zero probability of selection to each of the arbitrary samples belonging to S_0 just by restricting our plan to $S - S_0$ as follows:

$$p_0(s) = \begin{cases} \dfrac{p(s)}{1 - \sum\limits_{s \in S_0} p(s)} & \text{for } s \in S - S_0 \\ 0 & \text{otherwise} \end{cases}$$

where $p(s)$ is an IPPS sampling plan.

Obviously $p_0(s)$ is no longer an IPPS design. So we are now looking for the nearest proportional to size sampling design $p^*(s)$ introduced by Gabler (1987) in the sense that $p^*(s)$ minimizes the directed distance $D(p_0, p^*)$ from the design $p_0(s)$ to $p^*(s)$ defined as

$$D(p_0, p^*) = E_{p_0}\left[\frac{p^*(s)}{p_o(s)} - 1\right]^2 = \sum_s \frac{p^{*2}(s)}{p_0(s)} - 1$$

subject to the constraints $\sum_{s \ni i} p^*(s) = \pi_i^* = \pi_i, i = 1, \ldots, N$ where $\pi_i's$ (assumed positive for each i) are the first order inclusion probabilities for the IPPS sampling plan $p(s)$. So the idea is as follows:

We are trying to get rid of the arbitrary samples S_0 just by confining ourselves to $S - S_0$ and introducing a new design $p_0(s)$. As a consequence $p_0(s)$ deviates from the original IPPS design $p(s)$ so far as the inclusion probabilities are concerned. So we are now searching for a design $p^*(s)$ which is as near as possible to $p_0(s)$ and at the same time achieves the same set of first order inclusion probabilities, π_i, for the original IPPS sampling plan.

According to Gabler (1987), a solution of the above minimization problem is given by

$$p^*(s) = \left(\sum_{i \epsilon s} \lambda_i\right) .p_0(s)$$

provided $p^*(s)$ is non-negative for all $s \epsilon S - S_0$ especially when all $\lambda_i's$ are non-negative which hold in practice when all the units of the population are evenly distributed over the set of arbitrary samples. This is established in the Theorem 3.3 to follow and is also illustrated with a numerical example in the last section.

Theorem 3.1. If $i \notin S_0$, then $\underset{\sim}{\pi}_0 \underset{\sim}{\lambda} = \underset{\sim}{\pi}^*$ does not possess a non-negative solution for $\underset{\sim}{\lambda}$.

Proof. First we assume that

$$\text{Rank } \underset{\sim}{\pi}_0 = \text{ Rank } (\underset{\sim}{\pi}_0 \, \underset{\sim}{\pi}^*)$$

so that $\underset{\sim}{\pi}_0 \underset{\sim}{\lambda} = \underset{\sim}{\pi}^*$ is consistent.

$$\text{Consider } \underset{\sim}{y}' = \left[\underbrace{-\frac{1}{n}, \ldots, -\frac{1}{n}}_{i-1}, \frac{c}{n}, \underbrace{-\frac{1}{n}, \ldots, -\frac{1}{n}}_{N-i}\right].$$

$$\text{Then } (\underset{\sim}{\pi}^*, \underset{\sim}{y}) = (\underset{\sim}{\pi}, \underset{\sim}{y}) = \frac{c}{n}\pi_i - \frac{1}{n}\sum_{j\neq i}\pi_j$$
$$= \tfrac{c}{n}.np_i - \tfrac{1}{n}\sum_{j\neq i} np_j$$

where $p_i = \frac{x_i}{X}(X = \sum_1^N x_i)$ and $x_i's(i = 1, \ldots, N)$ are known size measures.

$$\begin{aligned} &= cp_i - \sum_{j\neq i} p_j \\ &= cp_i - (1 - p_i) \\ &= (c+1)pi - 1 \leq 0 \text{ if } c \leq \frac{1}{pi} - 1 \end{aligned} \quad (3.1)$$

$$\text{Consider } \underset{\sim 0}{\pi}\ \underset{\sim}{y} = \begin{bmatrix} \pi_1^0 & \pi_{12}^0 & \cdots & \pi_{1N}^0 \\ \pi_{21}^0 & \pi_2^0 & \cdots & \pi_{2N}^0 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{N1}^0 & \pi_{N2}^0 & \cdots & \pi_N^0 \end{bmatrix} \begin{bmatrix} \left.\begin{matrix} -\frac{1}{n} \\ \vdots \\ -\frac{1}{n} \end{matrix}\right\} (i-1) \\ \frac{c}{n} \\ \left.\begin{matrix} -\frac{1}{n} \\ \vdots \\ -\frac{1}{n} \end{matrix}\right\} (N-i) \end{bmatrix}$$

Consider the product of the ith row vector of $\underset{\sim}{\pi}_0$ and $\underset{\sim}{y}$.

$$\begin{aligned} &\frac{c}{n}\pi_i^0 - \frac{1}{n}\sum_{j\neq i}\pi_{ij}^0 \\ &= \frac{c}{n}\pi_i^0 - \frac{(n-1)\pi_i^0}{n} \\ &= \frac{(c+1)\pi_i^0 - n\pi_i^0}{n} \\ &= \frac{[c-(n-1)]\pi_i^0}{n} > 0 \text{ if } c > (n-1) \end{aligned} \quad (3.2)$$

Consider the product of the jth row vector of $\underset{\sim}{\pi}_0$ and $\underset{\sim}{y}(j \neq i)$.

$$\begin{aligned} &\frac{c}{n}\pi_{ij}^0 - \frac{\pi_j^0}{n} - \frac{1}{n}\sum_{k\neq j,i}\pi_{jk}^0 \\ &= \frac{c}{n}\pi_{ij}^0 - \frac{\pi_j^0}{n} - \frac{1}{n}\sum_{k\neq j}\pi_{jk}^0 + \frac{\pi_{ij}^0}{n} \end{aligned}$$

$$= \frac{(c+1)\pi_{ij}^0}{n} - \frac{\pi_j^0}{n} - \frac{(n-1)\pi_j^0}{n}$$

$$= \frac{(c+1)\pi_{ij}^0}{n} - \pi_j^0 > 0 \text{ if } c > \frac{n\pi_j^0}{\pi_{ij}^0} - 1. \qquad (3.3)$$

Thus $\underset{\sim}{\pi}_0 \underset{\sim}{y} > \underset{\sim}{0}$ if we choose

$$n - 1 < \frac{n\pi_j^0}{\pi_{ij}^0} - 1 < c \qquad (3.4)$$

Again $(\underset{\sim}{\pi}^*, \underset{\sim}{y}) \leq 0$ if $c \leq \frac{1}{p_i} - 1$.
So combining (3.1) and (3.4) we get

$$n - 1 < \frac{n\pi_j^0}{\pi_{ij}^0} - 1 < c \leq \frac{1}{p_i} - 1 \qquad (3.5)$$

Thus $\underset{\sim}{\pi}_0 \underset{\sim}{y} > \underset{\sim}{0}$ has a solution satisfying $(\underset{\sim}{\pi}^*, \underset{\sim}{y}) \leq 0$. So if $i \notin S_0$, the second assertion is true. Hence by the Theorem of Alternatives $\underset{\sim}{\pi}_0 \underset{\sim}{\lambda} = \underset{\sim}{\pi}^*$ can not admit of a non-negative solution for $\underset{\sim}{\lambda}$.

Remark 3.1: The bounds of c are consistent.

$I \quad n - 1 < \frac{n\pi_j^0}{\pi_{ij}^0} - 1 \implies \pi_{ij}^0 < \pi_j^0 \forall j \neq i$

$II \quad n - 1 < \frac{1}{p_i} - 1 \implies np_i < 1 \forall i$

$III \quad \frac{n\pi_j^0}{\pi_{ij}^0} - 1 < \frac{1}{p_i} - 1$

$\implies np_i.\pi_j^0 < \pi_{ij}^0$

$\implies np_i \sum_{j=1}^{N} \pi_j^0 < \sum_{j=1}^{N} \pi_{ij}^0$

$\implies np_i.n < n\pi_i^0$

$\implies \pi_i < \pi_i^0,$

which is true because $\pi_i^0 = \sum_{s \ni i | s \in S - S_0} \frac{p(s)}{1-x}$ where $x = \sum_{s \in S_0} p(s)$
$= \sum_{s \ni i | s \in S} \frac{p(s)}{1-x}$ as $i \notin S_0 = \frac{\pi_i}{1-x} > \pi_i$ as $(1-x) < 1$.

Theorem 3.2 : If all the units are evenly distributed over $S_0 (S - S_0)$, then Rank $\underset{\sim}{\pi}_0 = N$.

Proof.

Consider $\underset{\sim}{\pi}_0 = \begin{bmatrix} \pi_1^0 & \pi_{12}^0 & \cdots & \pi_{1N}^0 \\ \pi_{21}^0 & \pi_2^0 & \cdots & \pi_{2N}^0 \\ \cdots & \cdots & \cdots & \cdots \\ \pi_{N1}^0 & \pi_{N2}^0 & \cdots & \pi_N^0 \end{bmatrix} = (\underset{\sim}{\alpha}_1\, \underset{\sim}{\alpha}_2 \ldots, \underset{\sim}{\alpha}_N)$, say.

We will show that $\underset{\sim}{\alpha}_1, \ldots, \underset{\sim}{\alpha}_N$ are linearly independent column vectors so that the rank of the column space of $\underset{\sim}{\pi}_0$ is N. Consider a linear combination of $\underset{\sim}{\alpha}_1, \ldots, \underset{\sim}{\alpha}_N$

$$C_1\underset{\sim}{\alpha}_1 + C_2\underset{\sim}{\alpha}_2 + \cdots + C_N\underset{\sim}{\alpha}_N$$

$$= C_1[\pi_1^0\underset{\sim}{e}_1 + \pi_{21}^0\underset{\sim}{e}_2 + \cdots + \pi_{N1}^0\underset{\sim}{e}_N] + \cdots + C_N[\pi_{1N}^0\underset{\sim}{e}_1 + \pi_{2N}^0\underset{\sim}{e}_2 + \cdots + \pi_N^0\underset{\sim}{e}_N]$$

where $\underset{\sim}{e}_1' = (1\,0\,0 \cdots 0)$, $\underset{\sim}{e}_2' = (0\,1\,0 \cdots 0), \cdots,$
$\underset{\sim}{e}_N' = (0\,0\,0 \cdots 1)$ are N linearly independent vectors

$$= (C_1\pi_1^0 + \cdots + C_N\pi_{1N}^0)\underset{\sim}{e}_1 + \cdots + (C_1\pi_{1N}^0 + \cdots + C_N\pi_N^0)\underset{\sim}{e}_N. \tag{3.6}$$

If possible, let $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \cdots, \underset{\sim}{\alpha}_N$ form a linearly dependent set of vectors. Then

$$C_1\underset{\sim}{\alpha}_1 + C_2\underset{\sim}{\alpha}_2 + \cdots + C_N\underset{\sim}{\alpha}_N = \underset{\sim}{0}$$
$\Longrightarrow$ At least one of $C_1, C_2, \ldots, C_N$ is non-zero.
Suppose C_1 is non-zero and $C_2 = \cdots = C_N = 0$ (3.7)

Now from (3.6)

$$\left.\begin{aligned} C_1\underset{\sim}{\alpha}_1 + \cdots + C_N\underset{\sim}{\alpha}_N = 0 \Rightarrow C_1\pi_1^0 + \cdots + C_N\pi_{1N}^0 &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ C_1\pi_{N1}^0 + \cdots + C_N\pi_N^0 &= 0 \end{aligned}\right\} \tag{3.8}$$

as $\underset{\sim}{e}_1, \cdots, \underset{\sim}{e}_N$ are linearly independent.

Now from (3.7) and (3.8) we have

$$C_1\pi_1^0 = 0, \cdots, C_1\pi_{N1}^0 = 0 \tag{3.9}$$

From (3.9) we get $C_1(\pi_1^0 + \cdots + \pi_{N1}^0) = 0$

$$\Rightarrow nC_1\pi_1^0 = 0$$
$$\Rightarrow C_1\pi_1^0 = 0 \tag{3.10}$$

But if all the units are evenly distributed over $S_0(S - S_0)$, then $\pi_i^0 \neq 0$, $i = 1, \cdots, N$.

$$\text{So } C_1 \pi_1^0 = 0 \Rightarrow C_1 = 0. \tag{3.11}$$

Thus we arrive at a contradiction. So $\underset{\sim}{\alpha}_1, \cdots, \underset{\sim}{\alpha}_N$ are linearly independent. Hence rank $(\underset{\sim}{\pi}_0) = N$.

Remark 3.2 If all the units are evenly distributed over $S_0(S - S_0)$ then $\underset{\sim}{\pi}_0 \underset{\sim}{\lambda} = \underset{\sim}{\pi}^*$ has a unique solution for $\underset{\sim}{\lambda}$ viz $\underset{\sim}{\lambda} = \underset{\sim}{\pi}_0^{-1} \underset{\sim}{\pi}^*$ where $\underset{\sim}{\pi}_0^{-1}$ is the inverse of $\underset{\sim}{\pi}_0$.

Theorem 3.3 If all the units are evenly distributed over S_0, then $\pi_0 \lambda = \pi^*$ admits of a non-negative solution for $\underset{\sim}{\lambda}$.

Proof. Now $\underset{\sim}{\pi}_0 \underset{\sim}{\lambda} = \underset{\sim}{\pi}^*$ will be consistent if and only if

$$\text{Rank } \underset{\sim}{\pi}_0 = \text{ Rank } (\underset{\sim}{\pi}_0 \underset{\sim}{\pi}^*).$$

Now by Theorem 3.2, if all the units are evenly distributed over $S_0(S - S_0)$ then Rank $(\underset{\sim}{\pi}_0) = N$.

So $\underset{\sim}{\pi}_0 \underset{\sim}{\lambda} = \underset{\sim}{\pi}^*$ admits of a solution if and only if

$$\text{Rank } (\underset{\sim}{\pi}_0 \quad \underset{\sim}{\pi}^*) = N$$

$$\text{i.e. Rank } (\underset{\sim}{\alpha}_1 \underset{\sim}{\alpha}_2 \cdots, \underset{\sim}{\alpha}_N \underset{\sim}{\pi}^*) = N.$$

But $\underset{\sim}{\alpha}_1 \cdots, \underset{\sim}{\alpha}_N$ are linearly independent. So $\underset{\sim}{\pi}^*$ can be expressed as a linear compound of $\underset{\sim}{\alpha}_1, \cdots, \underset{\sim}{\alpha}_N$.

Let $\underset{\sim}{\pi}^* = c_1 \underset{\sim}{\alpha}_1 + \cdots + c_N \underset{\sim}{\alpha}_N$

where $c_1, \cdots, c_N$ are non-zero scalars.

$$\text{Thus } \pi_i^* = \sum_{j=1}^{N} c_j \pi_{ij}^0. \tag{3.12}$$

Consider the inequality $\underset{\sim}{\pi}_0 \underset{\sim}{y} > \underset{\sim}{0}$

$$\Rightarrow \begin{bmatrix} \pi_1^0 & \pi_{12}^0 & \cdots & \pi_{1N}^0 \\ \pi_{21}^0 & \pi_2^0 & \cdots & \pi_{2N}^0 \\ \cdots & \cdots & \cdots & \cdots \\ \pi_{N1}^0 & \pi_{N2}^0 & \cdots & \pi_N^0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} > \underset{\sim}{0}$$

$$\Rightarrow \left.\begin{array}{cccccccccc} \pi_1^0 y_1 & + & \pi_{12}^0 y_2 & + & \cdots & + & \pi_{1N}^0 y_N & > & 0 \\ \pi_{21}^0 y_1 & + & \pi_2^0 y_2 & + & \cdots & + & \pi_{2N}^0 y_N & > & 0 \\ \cdots & & \cdots & & \cdots & & \cdots & & \\ \pi_{N1}^0 y_1 & + & \pi_{N2}^0 y_2 & + & \cdots & + & \pi_N^0 y_N & > & 0. \end{array}\right\} \tag{3.13}$$

$$\text{Now } (\underset{\sim}{\pi}^\star, \underset{\sim}{y}) = \sum_{i=1}^{N} \pi_i^\star y_i = \sum_{i=1}^{N} y_i (\sum_{j=1}^{N} c_j \pi_{ij}^0)$$
$$= \sum_{j=1}^{N} c_j \sum_{i=1}^{N} y_i \pi_{ij}^0$$
$$= \sum_{j=1}^{N} c_j (\sum_{i=1}^{N} y_i \pi_{ji}^0) \quad (3.14)$$

Thus from (3.13) and (3.14) we find that the inequality $\underset{\sim}{\pi}_0 \underset{\sim}{y} > \underset{\sim}{0}$ does not have a solution satisfying $(\underset{\sim}{\pi}^\star, \underset{\sim}{y}) \leq 0$. Hence by the Theorem of Alternatives the first assertion is true i.e. $\underset{\sim}{\pi}_0 \underset{\sim}{\lambda} = \underset{\sim}{\pi}^\star$ admits of a non-negative solution for $\underset{\sim}{\lambda}$.

Remark 3.3 If $\underset{\sim}{\lambda} = (c_1 c_2 \cdots c_N)'$ be a non-negative solution for $\underset{\sim}{\lambda}$ then it is easy to check that

$$\underset{\sim}{\pi}_0 \underset{\sim}{y} \leq \underset{\sim}{0} \Rightarrow (\underset{\sim}{\pi}^\star, \underset{\sim}{y}) \leq 0 \text{ for any } \underset{\sim}{y}.$$

Thus Farkas' Lemma holds here.

4. A numerical example

Suppose the population consists of N=7 villages numbered 1 to 7. There are 35 possible samples, each of size n=3, out of which the 14 samples constitute the set S_0 of arbitrary samples:

1	2	3	2	4	5
1	2	4	2	5	6
1	3	6	2	6	7
1	3	7	3	4	5
1	4	6	3	5	7
1	4	7	4	6	7
2	3	5	5	6	7

Suppose that the following p_i values are associated with the seven villages: 0.12, 0.14, 0.15, 0.15, 0.14, 0.17, 0.13.

Since the p_i values satisfy the condition

$$\frac{1}{n} \cdot \frac{n-1}{N-1} \leq p_i \leq \frac{1}{n} \forall i$$

we apply modified Midzuno - Sen (1952, 1953) scheme to get an IPPS scheme with the revised normed size measures θ_i's given by

Table 4.1

Rejective IPPS sampling plan corresponding to Modified Midzuno - Sen Scheme

s	$p^*(s)$	s	$p^*(s)$
1 2 5	0.037025	2 3 7	0.0435515
1 2 6	0.0487859	2 4 6	0.0619369
1 2 7	0.0304937	2 4 7	0.0452762
1 3 4	0.037441	2 5 7	0.0451112
1 3 5	0.0381801	3 4 6	0.0589856
1 4 5	0.0397059	3 4 7	0.0449543
1 5 6	0.0515426	3 5 6	0.0630282
1 5 7	0.0321097	3 6 7	0.0560379
1 6 7	0.0447155	4 5 6	0.065549
2 3 4	0.048403	4 5 7	0.0477477
2 3 6	0.0594161		

$$\theta_i = \frac{N-1}{N-n}[np_i - \frac{n-1}{N-1}], i = 1, \cdots, N.$$

Applying the method described in Section 3, we obtain the rejective IPPS sampling plan $p^*(s)$ given in Table 4.1, that matches the original π_i values and makes the probability of selecting a sample belonging to the arbitrary set S_0 of samples exactly equal to zero.

Remark 4.1 In the above example, all the units are evenly distributed over the arbitrary set S_0 of samples which ensures a non-negative solution for $\underset{\sim}{\lambda}$ and this enables us to construct a nearest proportional to size sampling design $p^*(s)$ retaining the same IPPS property of the original design $p(s)$.

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