ESTIMATION OF LORENZ-RATIO FROM A FINITE POPULATION

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Introduction

In the literature, estimator of Lorenz ratio (LR) is mainly based on the assumption of a reasonable distribution on income or expenditure. e g log-normal. Pareto, Gamma. etc. which have been found to fit the income distribution well, at least for a Capitalist (or rather so called non-Communist) society. An econometrician or more generally a statistician usually assumes that the set of observed income is actually a random sample from the hypothetical infinite population following a certain distribution like the ones stated earlier. For an infinite population like log-normal or pareto etc. the assumption of 'with replacement' is redundant. This is also true because they usually take a large sample and in a large sample, variance of the estimators are almost same for both schemes. Under the assumption of a distribution and the scheme being that of simple random sampling with replacement, estimation of LR is not difficult. In fact maximum likelihood estimates of the parameters of the distribution can easily be obtained and hence of the LR. For this, explicit expression exists for each of the distributions usually assumed by the econometricians.

What Taguchi (1978) meant, we feel is that unbiasedness of the estimators usually available was not guaranteed without assuming any distribution on the population and the motivation of the

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present paper lies there in providing an unbiased estimator for LR without assuming any distribution of the population. For this a sample—theoretic approach has been taken to estimate Lorenz-Ratio from a finite population under a general sampling design. It has also been observed that the usual estimators for LR are biased under simple random sampling.

Also, an effort has been made here to provide an estimate of Lorenz-Ratio in case rank of i^{th} observation is known. Using this knowledge of rank, two alternative estimators have been proposed under probability proportional to size with replacement scheme (PPSWR). But unfortunately, we have failed to guarantee both the proposed estimators to lie in the interval [0,1], although both the estimators developed under PPSWR fare much better than that based on simple random sampling without replacement, better in the sense of having smaller variance.

Let U = (1,2,...,N) be a population of N given Units labelled 1 to N and y be a variate (a real valued function defined on U) taking value y_i for i^{th} unit of the population.

Lorenz-Ratio (LR) under discrete set up may be defined as

$$LR = \frac{1}{2N^2\bar{y}} \sum_{i=1}^{N} \sum_{j=1}^{N} | y_i - y_j | \tag{11}$$

and the continuous version of it being

$$LR = 1 - 2 \int_0^{\infty} F_1 dF$$

where,

$$F_1 = \int_0^{x} x f(x) dx \Big/ \int_0^{\infty} x f(x) dx \quad \text{and} \quad F = \int_0^{x} f(x) dx,$$

Our estimation problem here deals with of LR defined in (1.1) as uming $\bar{Y}$ is known. The case when $\bar{Y}$ is not known has not been discussed at all here,

When $\bar{Y}$ is known, two commonly used estimators for LR are

$$\hat{LR}_0^{(1)} = \frac{1}{2n^2 \bar{y}} \quad \Sigma\Sigma \quad | y_i - y_j |$$

$$\text{and } \hat{LR}_0 = \frac{1}{2n(n-1)\bar{y}} \; \Sigma\Sigma \; | y_i - y_j | \tag{1.2}$$

Both the estimators, we shall see, are biased under simple random sampling and the amount of biases have been calculated.

2. *A General Result on the Estimation of LR*

Let a sample s of fixed size n be drawn from a population U, using a sampling design D = (U, S, P,), where P is a probability measure defined on s such that $P(s) \geqslant 0$ and $\sum_{s \in S} P(s) = 1$

Let the first and second order inclusion probabilities associated with a design D, be

$$\pi_i = \sum_{s \ni i} P(s) \quad \text{and} \quad \pi_{ij} = \sum_{s \ni i, j (i \neq j)} P(s)$$

Similarly 3^{rd}, 4^{th} and higher order inclusion probabilities may be defined.

Let

$$\hat{LR} = \frac{1}{2N^2 \bar{y}} \sum_{i, j \in s}\sum \left(| y_i - y_j | / \pi_{ij} \right) \tag{2.1}$$

be an estimator for LR, when $\bar{Y}$ is known.

Theorem 2.1 : $\widehat{LR}$ is unbiased and $V(\widehat{LR})$ can be expressed as a function of LR, variance of y, and $\sum_i (\sum_j |y_i - y_j|)^2$, under simpler random sampling.

Proof :

$$E(\widehat{LR}) = \frac{1}{2N^2\bar{y}} \sum_{s \in S} \left(\sum_{i,j \in s} |y_i - y_j| / \pi_{ij} \right) P_s$$

$$= \frac{1}{2N^2\bar{y}} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{|y_i - y_j|}{\pi_{ij}} \left(\sum_{s \ni i, j} P_s \right) \qquad (2.2)$$

$$= LR.$$

Thus $\widehat{LR}$ is unbiased.

Now

$$V(\widehat{LR}) = E(\widehat{LR})^2 - \left[E(LR)\right]^2$$

$$= \left[4N^4 \bar{Y}^2\right]^{-1} \left[\sum_{i \neq j \neq k \neq l} \frac{(\pi_{ijkl} - \pi_{ij}\pi_{kl})}{\pi_{ij}\pi_{kl}} |y_i - y_j| \, |y_k - y_l| \right.$$

$$+ 8 \sum_{i \neq j \neq k} \frac{(\pi_{ijk} - \pi_{ij}\pi_{ik})}{\pi_{ij}\pi_{ik}} |y_i - y_j| \, |y_i - y_k|$$

$$\left. + 2 \sum_{i \neq j} \frac{(1 - \pi_{ij})}{\pi_{ij}} |y_i - y_j|^2 \right] \qquad (2.3)$$

In case of simple random sampling, probabilities of different orders do not depend on the units and hence can be brought out of the summation formula and in that case, the expression for $V(\hat{RL})$ takes the form as

$$V(\hat{LR}) = \left(4N^4 \bar{Y}^2 \right)^{-1} \Big(A'_1 \sum_{i \neq j \neq k \neq 1} | y_i - y_j | \; | y_k - y_1 | +$$

$$8A'_2 \sum_{i \neq j \neq k} | y_i - y_j | \; | y_i - y_k | + 2A'_3 \sum_{i \neq j} | y_i - y_j |^2 \Big) \tag{2.4}$$

where,

$$A'_1 = \frac{(\pi_{ijkl} - \pi_{ij} \; \pi_{kl})}{(\pi_{ij} \; \pi_{kl})} \; , \quad A'_2 = \frac{(\pi_{ik} - \pi_{ij} \; \pi_{ik})}{(\pi_{ij} \; \pi_{ik})} \text{ and}$$

$$A'_3 = \frac{(1 - \pi_{ij})}{\pi_{ij}}$$

Also, from Appendix A3, we have

$$\sum_{i \neq j \neq k \neq 1} | y_i - y_j | \; | y_k - y_1 | = a^2_{00} - 2 \sum a^2_{io} - 2 \sum a^2_{oj}$$

$$+ 2 \sum a^2_{ij} \sum_{i \neq j \neq k} | y_i - y_j | \; | y_i - y_k | = \sum a^2_{io} - \sum a^2_{ij}$$

and $\sum_{i \neq j} | y_i - y_j |^2 = \sum a^2_{ij}$

where $a_{oo} = \sum_i \sum_j |y_i - y_j| = 2N^2 \overline{Y} LR$

$$\Sigma a_{ij}^2 = \sum_i \sum_j |y_i - y_j|^2 = 2N^2 v(y)$$

and $\Sigma a_{io}^2 = \sum_i (\sum_j |y_i - y_j|)^2 =$

$$N^3 \int_o^\infty (\int_o^\infty (|y_i - y_j| f(y_j)) dy_j)^2 f(y_i) dy_i$$

Thus the expression in (2.4) reduces to

$$V(\hat{LR}) = \left[4N^4 \overline{Y}^2 \right]^{-1} \left[A'_1 a_{oo}^2 + 2(4A'_2 - A'_1) \Sigma a_{io}^2 - 2 \Sigma a_{oj}^2 A'_1 + 2 \Sigma a_{ij}^2 (A'_3 - 4A'_2 + A'_1) \right]$$

$$= f(LR, v(y), \Sigma a_{io}^2)$$

$$= f \left(LR, v(y), \sum_i (\sum_j |y_1 - y_j|)^2 \right) \tag{2.5}$$

and hence $\hat{v}(\hat{LR})$ may be found to be

$$\hat{V}(\hat{LR}) = f(\hat{LR}, \hat{v}(y), \Sigma \hat{a}_{io}^2) \tag{2.6}$$

Corollaries

(i) Both $\hat{LR}_0^{(1)}$ and $\hat{LR}_0^{(2)}$ of (1.2) are biased for LR and the amount of biases would be

$$B(\hat{LR}_0^{(1)}) = \sum_{i=1}^{N} \sum_{j=1}^{N} |y_i - y_j| \left(\frac{\pi_{ij}}{n^2} - \frac{1}{N^2} \right) / 2\bar{Y}$$

and
$$B(\hat{LR}_0^{(2)}) = \sum_{i=1}^{N} \sum_{j=1}^{N} |y_i - y_j| \left(\frac{\pi_{ij}}{n(n-1)} - \frac{1}{N^2} \right) / 2\bar{Y} \tag{2.7}$$

(ii) A sufficient condition for $\hat{LR}_0^{(1)}$ to be unbiased for LR would be $\pi_{ij} = n^2/N^2$ and that for $\hat{LR}_0^{(2)}$ to be unbiased for LR would be $\pi_{ij} = n(n-1)/N^2$. It may be shown that for any fixed size symmetric design $\pi_{ij} \leqslant \frac{n(n-1)}{N(N-1)}$. So π_{ij} for any fixed size symmetric design can not be equal to n^2/N^2 and hence $\hat{LR}_0^{(1)}$ can never be made unbiased just by changing the sampling design within the class of symmetric designs

3. *Comparison of the proposed estimator under simple random sampling with and without replacement*

The estimator $\hat{LR}$ in (2.1) under SRSWR takes respectively forms as

$$\hat{LR}_{(1)} = \frac{[1-2(1-p)^n + (1-2p)^n]^{-1}}{2N^2\bar{y}} \sum_i \sum_j |y_i - y_j| \tag{3.1}$$

$$\text{and } \hat{LR}_{(2)} = \frac{N(N-1)}{n(n-1)} \frac{1}{2N^2\bar{y}} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|$$

$$= \frac{(N-1)}{N} \hat{LR}_0^{(2)} \qquad (3.2)$$

$$= \frac{(N-1)n}{N(n-1)} \hat{LR}_0^{(1)} \qquad (3.3)$$

where $\hat{LR}_0^{(2)}$ and $\hat{LR}_0^{(1)}$ have been defined in (1.2).

The expressions for $\hat{LR}_{(1)}$ in (3.1) and $\hat{LR}_{(2)}$ in (3.2), (3.3) have been obtained simply substituting the value of π_{ij} under two different sampling schemes, namely SRSWR and SRSWOR. However the values of π_{ij} under two different schemes have been provided in A1 of Appendix.

Next we prove the following

Theorem 3.1 : $\hat{LR}_{(2)}$ in (3.2) will be always better than $\hat{LR}_0^{(2)}$ of (1.2) and also than $\hat{LR}_0^{(1)}$ under the assumption $\frac{n}{N} \simeq \frac{n-1}{N-1}$.

Proof : It may be shown that

$$MSE(\hat{LR}_0^{(2)}) = \frac{N^2}{(N-1)^2} V(\hat{LR}_{(2)}) + \frac{1}{(N-1)^2} (LR)$$

$$\geqslant V(\hat{LR}_{(2)})$$

Similarly,

$$\text{MES}(\hat{LR}_0^{(1)}) = \frac{N^2}{(N-1)^2}\,\frac{(n-1)^2}{n^2}\,V(\hat{LR}_{(2)}) + \frac{(N-n)^2}{n^2(N-1)^2}\,LR$$

$$\geqslant V(\hat{LR}_{(2)}) \text{ if } \frac{n}{N} \simeq \frac{n-1}{N-1}.$$

Thus follows the result that under SRSWOR $\hat{LR}_{(2)}$ is better than both $\hat{LR}_0^{(1)}$ and $\hat{LR}_0^{(2)}$.

Let π^*_{ij}, π^*_{ijk} and π^*_{ijkl} be the inclusion probabilities of different orders under SRSWOR and the corresponding inclusion probabilities under SRSWR be represented by π_{ij}, π_{ijk} and π_{ijkl}.

Now we have the following:

Theorem 3.2: A sufficient condition for $\hat{LR}_{(2)}$ to be better than $LR_{(1)}$ would be

$\pi_{ij} \leqslant \pi^*_{ij}$, $\pi^*_{ijk}/(\pi^*_{ij})^2 \leqslant \pi^*_{ijk}/(\pi_{ij})^2$ and $\pi^*_{ijkl}/(\pi^*_{ij})^2 \leqslant \pi_{ijkl}/(\pi_{ij})^2$.

Proof: The result follows simply by comaring $V(\hat{LR})$ in (2.4) under SRSWR and SRSWOR.

With the help of Lemma (1) and Lemma (2) of Appendix A4, it has been established that

always $\pi_{ij} \leqslant \pi_{ij}$ and $\pi^*_{ijk}/(\pi^*_{ij})^2 \leqslant \pi_{ijk}/(\pi_{ij})^2$ for $f \leqslant 3/13$ where f is the sampling fraction. But $\pi^*_{ijkl}/(\pi^*_{ij})^2 \leqslant \pi_{ijkl}/(\pi_{ij})^2$ may not be always true. However, with the help of a Computer, for different values of n = 4,5,10,15,20 and 50 and for each of n at different values of f = 0.001, 0.1, 0.2, 0.3 and 0.5, the values like

$$\pi_{ij},\ \pi^*_{ij},\ \pi^*_{ijk}/(\pi^*_{ij})^2,\ \pi_{ijk}/(\pi_{ijk})^2,\ \pi^*_{ijkl}/(\pi^*_{ij})^2 \text{ and } \pi_{ijkl}/(\pi_{ij})^2$$

were computed and it was found that in all the cases

$$\pi_{ij} < \pi^*_{ij} \text{ and } \pi^*_{ijk}/(\pi^*_{ij})^2 \leqslant \pi_{ijk}/(\pi_{ij})^2$$

hold good, where as except a few situations as may be seen from the following Table 3.1,

$$\pi^*_{ijkl}\ /\ (\pi^*_{ij})^2 \leqslant \pi_{ijkl}\ /\ (\pi_{ij})^2$$

hold true. However, in these cases where $\pi^*_{ijkl}/(\pi^*_{ij})^2 \geqslant \pi_{ijkl}/(\pi_{ij})^2$ hold, the difference between $\pi^*_{ijkl}/(\pi^*_{ij})^2$ and $\pi_{ijkl}/(\pi_{ij})^2$ were very small as has been reflected in the Table 3.1.

Table 3.1. Different values of $\pi^*_{ijkl}/(\pi^*_{ij})$ and $\pi_{ijkl}/(\pi_{ij})^2$

f = 0.001		
n	$\pi^*_{ijkl}/(\pi^*_{ij})^2$	$\pi_{ijkl}/(\pi_{ij})^2$
5	0.30018011	0.29983794
10	0.62240894	0.62213698
15	0 96023083	0.95902169

Remarks :

So for either $\hat{LR}^{(1)}_0$ or $\hat{LR}^{(2)}_0$ has been used as an estimator for LR and it is now observed that both the estimators are biased under simple random sampling. Thus, whether it is SRSWR or SRSWOR, the reported estimated value of LR is either under estimated or overestimated. In fact $B(\hat{LR}_0^{(1)})$ under SRSWOR may be found to be $(-\frac{(N-n)}{n(N-1)}LR)$ (Appendix A5). If $\frac{N}{N-1} \simeq 1$ and n is sufficiently large, then $B(\hat{LR}_0^{(1)}) \rightarrow 0$. Thus for large samples, there is no much harm (as for as bias is concerned) in using $\hat{LR}_0^{(1)}$, but in case of small sample, it is not advisable to use $\hat{LR}_0^{(1)}$. Instead it is better to use $\hat{LR}_{(2)}$, better in the sense of having smaller variance.

As far as the choice between $\hat{LR}_{(2)}$ and $\hat{LR}_{(1)}$ comes, it was not possible to establish the result algebrically that $V(\hat{LR})_{(2)}) \leqslant V(\hat{LR}_{(1)})$ always holds good. However through numerical evidence, it is clear that if sampling fraction 'f' is not as small as 0.001 or in other words, if $f > 0.1$, $\hat{LR}_{(2)}$ will be better than $\hat{LR}_{(1)}$. It may also be noted that on the whole $\hat{LR}_{(2)}$ will behave better than $\hat{LR}_{(1)}$ as the difference between π^*_{ijkl} / $(\pi^*_{ij})^2$ and $\pi_{ijkl}/(\pi_{ij})^2$ is very small (Ref. Table 3.1) and since the inclusion probabilities of third order will be smaller compared to that of second order.

3. The estimator proposed in (2.1) is of very general type and hence following different sampling methods like varying probability etc. similar type of estimators like $\hat{LR}_{(1)}$ and $\hat{LR}_{(2)}$ can be found for LR, wnether n is small or large–no matter.

4. Construction of two estimators under PPSWR-scheme

As may be seen in the remark (3) of the previous section that the estimator proposed in (2.1) is of very general type and hence following different sampling schemes, corresponding estimators for LR can easily be defined simply by substituting the values of π_{ij} under the respective sampling schemes. But the evaluation of $V(\hat{LR})$ and hence its comparison with other estimators will become very difficult, though not impossible. Thus to obtain an expression for $V(\hat{LR})$ under PPSWR-scheme from the general expression in (2.4) may not be an easy task. Keeping this in mind, two alternative estimators for LR under PPSWR-scheme have been proposed independently and their properties have been discussed here.

Let $y_1 < y_2 < \dots < y_N$ i. e. the rank of the i^{th} observation be known Then under

Scheme 1. Let i^{th} unit be drawn with $P_i \propto i+c$, where i is the rank and $c(\geq 0)$ is a suitably chosen constant. Then the estimator may be defined as

$$\hat{LR}' = \left[(N+1+2c)/N\right]\left[(\bar{y}/\bar{Y})-1\right] \tag{4.1}$$

and

Scheme 2. Let i^{th} unit be drawn with $P_i \propto (c-i)$, where i is the rank and c is a suitable chosen constant ($> N+1$) and let the estimator under this scheme used be

$$\hat{LR}^* = \left[(2c-N-1)/N\right]\left[1-(\bar{\bar{y}}/\bar{Y})\right] \tag{4.2}$$

Now we have following :

Theorem 4.1 : Both $\hat{LR}'$ and $\hat{LR}^*$ are unbiased for LR under the above two schemes 1 and 2 and the respective variances are

$$V(\hat{LR}') = \frac{k^2}{n}\left[(1 + c_y^2)(1+\frac{LR(y^2)}{k}) - (1 + \frac{LR(y)^2}{k})\right]$$

and $$V(\hat{LR}^*) = \frac{\lambda^2}{n}\left[(1+c_y^2)(1 - \frac{LR(y^2)}{\lambda}) - (1 - \frac{LR(y)}{\lambda})^2\right] \quad (4.3)$$

where,

$k = (N+1+2c)/N$, $\lambda = (2c-N-1)/N$, cy^2 = Square of coeffieient of variation of y, $LR(y^2)$ = Lorenz-ratio of the variable y^2 and $LR(y)$ = Lorenz-ratio of the variable y.

$V_{opt}(\hat{LR}')$ and $V_{opt}(\hat{LR}^*)$ can be obtained by minimising $V(\hat{LR}')$ and $V(\hat{LR}^*)$ with respect to k and λ respectively. In both the cases, minimum variances turn out to be negative. This is so because the optimum choice of k and λ do not ensure the $P_i \geqslant 0$ for all $i = 1,2...N$. Thus differentiating $V(\hat{LR}')$ and $V(\hat{LR}^*)$ in the permissible ranges for k and λ ($k \geqslant \frac{N-1}{N}$, $\lambda \geqslant \frac{N-1}{N}$), the minimum values have been obtained. It may be shown that in both the cases the minimum is obtained at $k = \frac{N-1}{N}$ as well as $\lambda = \frac{N-1}{N}$.

Hence,

$$NV_{min}(\hat{LR}') = \frac{1}{f}\left[1+cy^2)\left\{(\frac{N-1}{N})^2+(\frac{N-1}{N})LR(y^2)\right\}-\left\{\frac{N-1}{N}+LR(y)\right\}^2\right] \quad (4.4)$$

and

$$NV_{min}(\hat{LR}^*) = \frac{1}{f}\left((1+cy^2)\left\{(\frac{N-1}{N})^2-(\frac{N-1}{N})LR(y^2)\right\}-\left\{\frac{N-1}{N} - LR(y)\right\}^2\right].$$

Thus $V_{min}(\hat{LR}') \leqslant V_{min}(LR^*)$ if $R_0 < 0$

and $V_{min}(\hat{LR}') \geqslant V_{min}(LR^*)$ if $R_0 \leqslant 0$,

$$R_0 = (1+cy^2)(LR(y^2) - 2\ LR(y)).$$

where

$$R_0 = (1+cy^2)(LR(y^2) - 2\ LRty)).$$

In the next section, a comparison has been made between the estimators $\hat{LR}_{(2)}$, $\hat{LR}'$ and $\hat{LR}^*$. For this it has been assumed that $y \sim \Lambda(\mu, \sigma^2)$ and we observe through choice of different values of the parameter σ under $\mu = 0$

$$V(\hat{LR}^*) < V(\hat{LR}') < (\hat{LR}_{(2)})$$

always holds good. This inequality is true for any μ, since the variances of Lorenz estimators do not depend on μ.

5. An illustration on the relative performances of the proposed estimators under $y \sim \Lambda(0, \sigma^2)$

When y, the variable under study follows a log-normal distribution with parameters 0, σ^2, then can be shown that

$$LR(y) = 2\phi(\sigma/\sqrt{2})-1$$

$$LR(y^2) = 2\phi(\sqrt{2}\,\sigma)-1$$

$$\overline{Y} = e^{\sigma^2/2}$$

and $$c_y^2 = e^{\sigma^2}-1.$$

and in that case, the variances of the estimators $\hat{LR}_{(2)}$, $\hat{LR}'$ and $\hat{LR}^*$ reduce respectively, upto $0(\frac{1}{N})$, to

$$NV(\hat{LR}_{(2)}) = 2(\frac{1}{f} - 1)\left[f(\sigma) / \bar{Y}^2 - 2LR^2\right]$$

$$NV_{min}(\hat{LR}') = \frac{1}{f}\left[(1+cy^2)\ (1+LR(y^2)) - (1+LR(y))^2\right] \tag{5.1}$$

$$\text{and } NV(\hat{LR}^*) = \frac{1}{f}\left[(1+cy^2)\ (1-LR(y^2)) - (1-LR(y))^2\right]$$

where

$$f(\sigma) = \int_0^{\infty}\left[\int_0^{\infty} |\, y_i - y_j \,|\ f(y_j)\, d(y_j)\right]^2 f(y_i)\, dy_i,$$

evaluated at σ.

Table 5.1 illustrates the relative performances of $\hat{LR}'$, $\hat{LR}^*$ and $\hat{LR}_{(2)}$ for different values of σ and f.

Concluding Remarks and Discussion

It is known to us that, for a fixed size (n) design, $\sum_{i \neq j \neq 1}^{N} \pi_{ij} =$ n(n-1). Now from (2.7), a sufficient condition for $LR_0^{(1)}$ to be unbiased would be

$$\pi_{ij} = (n^2/N^2)$$

TABLE-5.1

RELATIVE PERFORMANCES OF $\widehat{LR}$, $\widehat{LR}^*$ AND $\widehat{LR}_{(2)}$

f/σ		*0.3*	*0.4*	*0.5*	*0.6*	*0.7*	*0.8*	*0.9*
	$NV(\widehat{LR}^*)$	110.3	130.9	142.6	155.3	171.0	182.6	190.4
0.001	$NV(\widehat{LR}')$	405.4	611.6	861.4	1140.1	1460.9	1823.9	2234.4
	$NV(\widehat{LR}_{(2)})$	1195.0	1674.9	2208.0	2784.2	3434.2	4144.5	4922.3
	$NV(\widehat{LR}^*)$	11.03	13.09	14.26	15.53	17.10	18.26	19.04
0.01	$NV(\widehat{LR}')$	40.54	61.16	86.14	114.01	146.08	182.39	223.43
	$NV(\widehat{LR}_{(2)})$	118.42	165.98	218.81	275.91	340.32	410.71	487.79
	$NV(\widehat{LR}^*)$	[illegible].206	2.618	2.852	3.106	3.420	3.652	3.808
0.05	$NV(\widehat{LR}')$	8.108	12.23	17.23	22.80	29.22	36.48	44.68
	$NV(\widehat{LR}_{(2)})$	22.73	31.86	41.99	52.95	65.31	78.82	93.62
	$NV(\widehat{LR}^*)$	1.103	1.309	1.426	1.553	7.110	1.826	1.904
0.1	$NV(\widehat{LR}')$	4.05	6.12	8.61	11.40	14.61	18.24	22.34
	$NV(\widehat{LR}_{(2)})$	10.77	15.09	19.89	25.08	30. 4	37.34	44.34

TABLE 5.1 (*Contd.*)

f/σ		0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.2	$NV(\widehat{LR^*})$	0.5515	0.6545	0.7130	0.7765	0.8550	0.9130	0.9520
	$NV(\widehat{LR'})$	2.027	3.060	4.312	5.701	7.304	9.123	11.170
	$NV(\widehat{LR}_{(2)})$	4.785	6.706	8.841	4.148	13.750	19.594	19.709
0.3	$NV(\widehat{LR^*})$	0.3677	0.4363	0.4753	0.5177	0.5700	0.6087	0.6347
	$NV(\widehat{LR'})$	1.349	2.037	2.868	3.796	4.821	6.019	7.373
	$NV(\widehat{LR}_{(2)})$	2.791	3.912	5.157	6.503	8.021	9.680	11.497

and the corresponding condition for $\hat{LR}_0^{(2)}$ to be unbiased would be

$$\pi_{ij} = n(n-1)/N(N-1)$$

Thus, these sufficient conditions for $\hat{LR}_0^{(1)}$ and $\hat{LR}_0^{(2)}$ to be unbiased for LR can not be satisfied for any fixed size design.

As may be seen from the Table (5.1), both the proposed rank estimators $\hat{LR}'$ and $\hat{LR}^*$ are much better than $\hat{LR}_{(2)}$, but unfortunately, as has been pointed earlier, they do not lie in the coveted interval (0.1). However, it may be observed from the Table that the variances are very small (usually $\leqslant 0.01$ or so, assuming that the sample size is at least 10 and sampling fraction is at most 0.1), so that the probability of the estimator going beyond the desired interval is also small.

These estimators can be modified once we know their distribution and hence the probability that they fall below a certain point (a_1) and above another given point (a_2) can be calculated. In that case, one can define

$$\hat{LR}^{**} = \begin{cases} \hat{LR} & \text{if } a_1 < \hat{LR} < a_2 \\ a_1 & \text{if } a_1 > \hat{LR} \\ a_2 & \text{if } a_2 < \hat{LR} \end{cases}$$

where,

a_1 and a_2 can be chosen so that

$$E(\hat{LR}^{**}) = LR$$

and $0 \leqslant a_1 \leqslant a_2 \leqslant 1$.

However, we have made no attempt here in this direction.

It is also possible to consider our proposed estimator $\widehat{LR}$ under super population approach where it is assumed that the finite population (U=1,2,...N) is a random realization from an infinite population following certain distribution and to study the property of $\widehat{LR}$ for different choices of model parameters.

REFERENCE

Tokio Taguchi (1978). On an unbiased, consistent and asymptotically efficient estimation of Gini's concentration coefficient. Metron, Vol. XXXVI, 3-4, 57-72.

APPENDIX

A1. Under SRSWOR, the inclusion probabilities of different orders are found to be

(i) $\pi^*_{ij} = n(n-1) / N(N-1) = n(n-1)p^2 / (1-p)$, where $p=1/N$

(ii) $\pi^*_{ijk} = n(n-1)(n-2) / N(N-1)(N-2) = n(n-1)(n-2)p^3 / (1-p)(1-2p)$

(iii) $\pi^*_{ijkl} = n(n-1)(n-2)(n-3)p^4 / (1-p)(1-2p)(1-3p)$

and the corresponding expressions for SRSWR turn out to be

(i)
$$\pi_{ij} = 1-(1-p_i)^n - (1-p_j)^n + (1-p_i-p_j)^n$$
$$= 1-2(1-p)^n + (1-2p)^n$$
$$= \sum_{r=2}^{n} p^r a_r \text{, where } a_r = (-1)^r \binom{n}{r} \left\{2^r-2\right\}$$

(ii)
$$\pi_{ijk} = 1-3(1-p)^n + 3(1-2p)^n - (1-3p)^n$$
$$= \sum_{r=3}^{n} p^r a'_r, \quad a'_r = (-1)^r \binom{n}{r} \left\{3-3\times 2^r + 3^r\right\}$$

and

(iii)
$$\pi_{ijkl} = 1-4(1-p)^n + 6(1-2p)^n - 4(1-3p)^n + (1-4p)^n$$
$$= \sum_{r=4}^{n} p^r a^*_r, \quad a^*_r = (-1)^r \binom{n}{r} \left\{-4+6\times 2^r - 4\times 3^r + 4^r\right\}$$

A2. *Claim* $p^r \mid a_r \mid \geqslant \mid a_{r+1} \mid p^{r+1} \; \forall \; r \geqslant 3$

Proof : To have the claim we have to show that

$$2p^r \binom{n}{r} (2^{r-1} - 1) \geqslant 2p^{r+1} \binom{n}{r+1} (2^r - 1)$$

i.e. to show that

$$p \leqslant (r+1)\,(2^r - 1) \,/\, (n-r)\,(2^{r-1} - 1)$$

Let $f(r) = (2^r - 1) / (2^{r-1} - 1)$. Since $f(r)$ is an increasing function in r and since $(2^r - 1) / (2^{r-1} - 1) \geqslant (2^r - 1) / 2^{r-1} = (2 - \frac{1}{2^{r-1}})$ thus (a) will be satisfied

$$\text{if} \quad p \leqslant \frac{(r+1)}{(n-r)} \left(2 - \frac{1}{2^{r-1}}\right) = h(r) \text{ say}$$

i.e. if $p \leqslant h(3) = 7 / (n-3)$

which is always true, as

$$\frac{1}{N} \leqslant \frac{1}{n} \leqslant \frac{1}{n-3} \leqslant \frac{7}{(n-3)}$$

Claim 2. $p^r \mid a'_r \mid \geqslant p^{r+1} \mid a'_{r+1} \mid \forall \; r \geqslant 4.$

Proof : Let $f(r) = (3 - 3.2^r + 3^r)$, then $f(r) \uparrow$ and $f(4) > 0$ and hence $f(r) > 0 \; \forall \; r \geqslant 4$.

Thus to have the claim, we have to show that

$$p \leqslant \frac{(r+1)}{(n-r)} \; \frac{(3^{r} -3.2^{r} +3)}{(3^{r+1} - 3.2^{r+1} +3)} \qquad \ldots(b)$$

Let,

$$q(r) = (3^{r} - 3.2^{r} + 3) / (3^{r+1} - 3.2^{r+1} + 3)$$

It may be shown that

$$q(r+1) - q(r) = \frac{3^{r+2}(2^{r} -4)+9.2^{r}}{(3^{r+2}-3.2^{r+2}+3)\,(3^{r+1}-3.2^{r+1}+2)} > 0$$

thus (b) will be satisfied, if

$$p \leqslant \frac{5}{(n-5} \; q(4)$$

which is always true.

Claim 3 : $p^{r} \; | a^{*}_{r} | \geqslant p^{r+1} \; | a^{*}_{r+1} | \;\; \forall \; r \geqslant 5$

Proof : We have

$$a^{*}_{r} = (-1)^{r} \binom{n}{r} \; f(r),$$

where

$f(r) = \left(-4 + 6 \times 2^{r} -4 \times 3^{r} +4^{r}\right)$. By the same logic, it my be shown that $f(r) > 0 \;\forall\; r \geqslant 5$

Now, $p^r \; |a^*_r| > p^{r+1} \; |a^*_{r+1}|$

$$\text{if} \quad p \leqslant \frac{(r+1)}{(n-r)} \quad \frac{(4^r - 4 \times 3^r + 6 \times 2^r - 4)}{(4^{r+1} - 4 \times 3^{r+1} + 6 \times 2^{r+1} - 4)}$$

i.e., if $p \leqslant \frac{(r+1)}{(n-r)} \left(\frac{1}{4} - \left(\frac{3}{4}\right)^r\right)$ { neglecting the two terms in the numerator and last three terms in denominator.

which is true for $r \geqslant 5$.

A 3 Evaluation of

$$(1) \quad \sum_{i \neq j \neq k \neq l} |y_i - y_j| \; |y_k - y_l|$$

Let $a_{ij} = |y_i - y_j|$ and $a_{kl} = |y_k - y_l|$

then

$$\sum_{i \neq j \neq k \neq l} |y_i - y_j| \; |y_k - y_l| = \sum_{i \neq j}^{N} a_{ij} \left(\sum_{k \neq l = 1}^{N} a_{kl} \right)$$

$$= \sum_{i \neq j} a_{ij} \sum_{\substack{k=1 \\ k \neq i \neq j}}^{N} \left(\sum_{l=1}^{N} a_{kl} - a_{kk} - a_{ki} - a_{kj} \right)$$

$$= \sum_{i \neq j}^{N} a_{ij} \sum_{k \neq i \neq j = 1} (a_{ko} - a_{kk} - a_{ki} - a_{kj})$$

$$= \sum_{i \neq j} a_{ij} \Big((a_{oo} - a_{io} - a_{jo}) - (\Sigma a_{kk} - a_{ii} - a_{ji})$$

$$- (\Sigma a_{ki} - a_{ii} - a_{ji}) - (\Sigma a_{kj} - a_{ij} - a_{ji}) \Big)$$

$$= \sum_{i \neq j} a_{ij} (a_{oo} - a_{io} - a_{jo} - a_{oi} + a_{ji} - a_{oj} + a_{ij})$$

$$= a_{oo} \sum_{i} (\sum_{j} a_{ij} - a_{ii}) - 2 \sum_{i} a_{io} (\sum_{j} a_{ij} - a_{ii})$$

$$- 2 \sum_{j} \sum_{i \neq j} a_{ij} a_{jo} + 2 \Sigma\Sigma a_{ij}^2 - a_{ii}^2$$

$$= a_{oo}^2 - 2\Sigma a_{io}^2 - 2\Sigma a_{oj}^2 + 2\sum_{i,j} a_{ij}^2$$

where,

$$a_{oo} = \sum_{i}^{N} \sum_{j}^{N} a_{ij} = \Sigma\Sigma \mid y_i - y_j \mid = 2N^2 \overline{\text{YLR}}.$$

$$a_{io}^2 = (\sum_{j}^{N} \mid y_i - y_j \mid)^2$$

$$\Sigma\Sigma a_{ij}^2 = 2N^2 V(y)$$

(2) $$\sum_{i \neq j \neq k = 1}^{N} \mid y_i - y_j \mid \mid y_i - y_k \mid = \sum_{i \neq j = 1} a_{ij} (\sum_{k=1}^{N} a_{ik} - a_{ii} - a_{ij})$$

$$= \sum_{i \neq j=1}^{N} a_{ij} \, (a_{io} - a_{ij}) \qquad = \sum_{i} a_{io} \, (\sum_{j} a_{ij} - a_{ii}) \; - \sum_{i,j} a_{ij}^2$$

$$= \sum_{i} a_{io}^2 - \sum_{i,j}^{N} a_{ij}^2$$

A 4.,

Lemma 1. $\pi_{ij} \leqslant \pi^*_{ij}$

Proof : $\pi_{ij} \leqslant \pi^*_{ij}$

$$\text{if } n(n-1)p^2 - n(n-1)(n-2)p^3 + \leqslant n(n-1)p^2 / (1-p)$$

by *claim 1 of A2.*,

$$\text{i.e., if } n(n-1)p^2 \leqslant n(n-1) \, p^2 / (1-p)$$

which is always true and hence the lemma 1, follows.

Lemma 2.

$$\pi^*_{ijk} \; / \; (\pi^*_{ij})^2 \leqslant \pi_{ijk} \; / \; (\pi_{ijk})^2 \;\; \forall \; r \geqslant 4 \text{ and } f \leqslant 3/13$$

Proof : The lemma will be satisfied

$$\text{if} \quad \left(\frac{n(n-1) \;\; n-2)p^3}{(1-p) \; (1-2p)} \right) \quad \frac{(1-p)^2}{(\, n(n-1)p^2 \,)2}$$

$$\leqslant \frac{n(n-1)(n-2)p^3 - \frac{3}{2}n(n-1)(n-2)(n-3)p^4 + \ldots}{\{ n(n-1)p^2 - n(n-1)(n-2)p^3 + \frac{7}{12}n(n-1)(n-2)(n-3)p^4 \ldots \}^2}$$

Now by *Claim 2* of A2,

$$\text{i.e., if } \frac{(1-p)}{(1-2p)} \leqslant \frac{1-(3(n-3)p/2)}{\{ 1 (n-2)p + (7(n-2)(n-3)/12)p^2 \}^2}$$

$$\text{or if } (1-p) \Big[1-2(n-2)p + \Big((n-2)^2 + (7(n-2)(n-3)/6)\Big)p^2$$

$$- (7(n-2)^2(n-3)/6)p^3 + (7(n-2)(n-3)/12)^2 p^4 \Big]$$

$$\leqslant 1 - ((3(n-3)/2)+2)p + (3(n-3)/2)2p^2 \qquad \ldots(c)$$

Now, since,

$$\frac{7(n-2)^2(n-3)p^3}{6} \geqslant \left(\frac{7}{12}\right)^2 (n-2)^2 (n-3)^2 p^4$$

if $p \leq \frac{24}{7} \frac{1}{n-3}$, which is always true.

Then inequality (c) will be satisfied

$$\text{if } (1-p) \Big[1-2(n-2)p + \Big((n-2)^2 + (7(n-2)(n-3)/6)\Big)p^2$$

$$\leq 1 - \Big((3(n-3)/2)+2\Big)p + 3(n-3)p^2$$

or if,

$$1-(2n-3)p + (n-2)\Big((n-2) + (7(n-3)/6) + 2\Big)p^2$$

$$-\Big((n-2)^2 + (7(n-2)(n-3)/6)\Big)p^3$$

$$\leq 1 - \Big((3(n-3)/2) + 2\Big) p + 3(n-3)p^2$$

or if $-(2n-3-\frac{3}{2}n + \frac{5}{2})p +$

$$\Big((n-2)^2 + \frac{7}{6}(n-2)(n-3) + 2(n-2) - 3(n-3)\Big)p^2 \leqslant 0$$

or if $p \leq \dfrac{(1/2)}{(n-3)\frac{13}{6} + (3/(n-2))}$ ...(d)

The Second factor in the denominator of (d), i. e., $3/(n-2)$ is maximum when $n = 4$, given that $n \geqslant 4$

Hence, or if $p \leq 3/(13n-30)$

or if $p \leq \dfrac{3}{13n}$

or if $f \leq 3/13$, where f is the sampling fraction. Thus follows the lemma :

A5. Bias of LR_0, the usual estimator for LR, i e. $B(\widehat{LR}_0)$

$= -\dfrac{N-n}{Nn}$ LR. under SRSWOR, if $N \simeq (N-1)$.

Proof : $B(LR_o)_{SRSWOR} = \frac{n(n-1)}{N(N-1)n^2} \frac{1}{2Y} \sum_i^N \sum_j^N | y_i - y_j |$

$$- \frac{1}{2N^2\bar{Y}} \sum_i \sum_j | y_i - y_j |$$

$$= \frac{1}{2N\bar{Y}} \left[\frac{(n-1)}{n(N-1)} - \frac{1}{N} \right] \sum_i^N \sum_j^N | y_i - y_j |$$

$$= - \frac{(N-n)}{n(N-n)} \frac{1}{2N^2\bar{Y}} \sum_i \sum_j | y_i - y_j |$$

$$= - \frac{(N-n)}{(N-n)n} LR$$

$$= - \frac{(N-n)}{Nn} LR \quad \text{if} \quad N \simeq (N-1).$$

A 6.

i) $$V(\hat{LR}) = \left[4N^4 \bar{Y}^2 \right]^{-1} \left[A_1' (a_{oo}^2 - 2\Sigma a_{io}^2 - 2\Sigma a_{oj}^2 + 2\Sigma a_{ij}^2) + 8A_2' (\Sigma a_{io}^2 - \Sigma a_{ij}^2) + 2A_3' \Sigma a_{ij}^2 \right]$$

$$= \left[4N^4 \bar{Y}^2 \right]^{-1} \left[A_1' a_{oo}^2 + 2 (4A_2' - A_1') \Sigma a_{io}^2 - 2A_1' \Sigma a_{oj}^2 + 2 (A_3' - 4A_2' + A_1') \Sigma a_{ij}^2 \right]$$

where

$$A_1' = \frac{\pi_{ijkl} - \pi_{ij}\,\pi_{kl}}{\pi_{ij}\,\pi_{kl}} = \frac{\pi_{ijkl}}{\pi_{ij}\,\pi_{kl}} - 1$$

$$= \frac{n(n-1)(n-2)(n-3)}{n(n-1)(n-3)(n-3)} \; \frac{(N(N-1))^2}{(N(N-1))^2} - 1$$

$$= \frac{(n-2)(n-3)\,N(N-1)}{(N-2)(N-3)\,n(n-1)} - 1 = \frac{-2\,(N-n)(2Nn-3N-3n+3)}{n(n-1)\,(N-2)\,N-3}$$

$$= -\frac{1}{N} \; \frac{2(1-f)\left(2f - \frac{3}{N} - \frac{3f}{N} + \frac{3}{2N}\right)}{f(f-1)\left(1-\frac{2}{N}\right)\left(1-\frac{3}{N}\right)} = -\frac{1}{N}\,\frac{4f(1-f)}{f^2} \text{ upto } 0\left(\frac{1}{N}\right)$$

$$= \frac{4}{N}\left(1-\frac{1}{f}\right) \text{ upto } 0\left(\frac{1}{N}\right)$$

$$A_2' = \frac{\pi_{ijk}}{\pi_{ij}\;\pi_{ik}} - 1 = \frac{n(n-1)(n-2)}{N(N-1)\,N(N-2)} \; \frac{(N(N-1))^2}{(n(n-1))^2} - 1$$

$$= \frac{(n-2)(N^2-N)}{(N-2)\,n(n-1)} - 1 = \frac{N^2 n - 2N^2 + 2N - Nn^2 + 2n^2 - 2n}{(N-2)\,n(n-1)}$$

$$= \frac{Nn(N-n) - 2(N-n)(N+n) + 2(N-n)}{(N-2)\,n(n-1)}$$

$$= \frac{(N-n)(Nn-2N-2n+2)}{(N-2)\,n(n-1)} = \frac{f(1-f)}{f^2} = -\left(1-\frac{1}{f}\right)$$

and

$$A_3' = \frac{1}{\pi_{ij}} - 1 = \frac{N(N-1)}{n(n-1)} - 1 = \frac{N^2 - N\; n^2 + n}{n(n-1)} = \frac{(N-n)(N+n)-(N-n)}{n(n-1)}$$

$$= \frac{N-n)\,(N+n-1)}{n(n-1)} = \frac{(1-f)\,1+f)}{{}^2} = \frac{1-f^2}{f^2}$$

$$= -\,(1-\frac{1}{f})\,(1+\frac{1}{f}).$$

Thus,

$$V(\widehat{LR}) = \left[4N^4\ \bar{Y}^2\right]^{-1} \left[\frac{4}{N}(1-\frac{1}{f})\,(4N^4\ \bar{Y}^2 LR^2 - 4f(\sigma)\,N^3\right.$$

$$+\,4N^2\,V(y)\,) - 8(1-\frac{1}{f})\,(f(\sigma)\,N^3 - 2N^2\,V(y)\,)$$

$$\left. -\,4(1-\frac{1}{f})\,(1+\frac{1}{f})N^2\,V(y)\right]$$

where,

$$\Sigma a_{io}^2 = N^2 \int_0^\infty (\int_0^\infty (\,|\,y_i - y_j\,|\ f(y_j)\ dy_j\,)^2\ f(y_i)\ dy_i$$

being evaluated assuming $y \sim (0, \sigma^2)$ and let the integral be denoted by $f(\sigma)$

Thus $\Sigma a_{io}^2 = N^2\ f(\sigma)$

hus $V(\widehat{LR}) = \dfrac{4N^4\ \bar{Y}^2\ LR^2\ (1-1/f)\ 4}{N\ 4N^4\ \bar{Y}^2} - \dfrac{B(1-1/f)\ f(\sigma)N^3}{4N^4\ \bar{Y}^2}$

upto $0(\frac{1}{N})$

$$= \frac{4LR^2\ (1-1/f)}{N} - \frac{2\,(1-1/f)f(\sigma)}{N\ \bar{Y}^2}$$

$$= \frac{1}{N}(1 - 1/f) \left[4LR^2 - (2f(\sigma)/\bar{Y}^2)\right]$$

$$= \frac{2}{N}(1 - 1/f) \left[2LR^2 - (f(\sigma)/\bar{Y}^2)\right]$$

$$= NV(\hat{LR}) = 2(1 - 1/f) \left[2LR^2 - (f(\sigma)/\bar{Y}^2)\right]$$

Corollary : $V(LR) > 0 \Rightarrow 2LR^2 \leqslant f(\sigma)/\bar{Y}^2$

2 (a)

(i) $$E(\hat{LR}') = \frac{(N+1+2C)}{N\bar{Y}} (E(\bar{Y}) - \bar{Y})$$

$$= \frac{(N+1+2C)}{N\bar{Y}} \quad \frac{2\Sigma yi}{N(N+1+2C)} + \frac{2C\bar{Y}}{N+1+2C} - \bar{Y}$$

$$= \frac{N+1+2C}{N\bar{Y}} \quad \frac{LRN\bar{Y}}{(N+1+2C)} + \frac{(N+1)\bar{Y}}{(N+1+2C)} - \bar{Y}$$

$$+ \frac{2C\bar{Y}}{N+1+2C} \quad \text{since } \Sigma y_i = (LR + \frac{N+1}{N}) \frac{N^2 \bar{Y}}{2}$$

$$= \frac{N+1+2C}{N\bar{Y}} \quad \frac{N\bar{Y}}{N+1+2C} LR = LR.$$

(ii) $$E(L\hat{R}^*) = E\,\frac{(2C-N-1)}{NY}\,(\bar{Y}-\bar{y})$$

$$= \frac{2C-N-1}{N\bar{Y}}\,E\,(\bar{Y}-\bar{y})$$

$$= \frac{2C-N-1}{N\bar{Y}}\left(\bar{Y} - \frac{2C\bar{Y}}{(2C-N-1)} + \frac{LRN\bar{Y}}{(2C-N-1)} + \frac{(N+1)\,\bar{Y}}{(2C-N-1)}\right)$$

$$= \frac{2C-N-1}{N\bar{Y}}\left(-\frac{(N+1)\,\bar{Y}}{(2C-N-1)} + \frac{(N+1)\,\bar{Y}}{(2C-N-1)} + \frac{N\bar{Y}}{2C-N-1}\,LR\right)$$

$$= LR.$$

2(b)

(i) $$V(LR') = E(LR')^2 - (LR)^2$$

$$= \frac{K^2}{\bar{Y}^2}\,E\,(\bar{Y}^2 - 2\bar{y}\,\bar{Y} + \bar{Y}^2)\,(LR),$$

where $K = (N+1+2C)/N$.

$$= \frac{K^2}{\bar{Y}^2}\left(V(\bar{y}) + (E\,\bar{y})^2 - 2\bar{Y}\,E(\bar{y}) + \bar{Y}^2\right) - (LR)^2$$

$$= \frac{K^2}{\bar{Y}^2} V(\bar{y}) + (LR)^2 - (LR)^2 = \frac{K^2}{\bar{Y}^2} V(\bar{y})$$

Now,

$$V(\bar{y}) = \frac{1}{n^2} \sum_{i=1}^{n} V(y_i)$$

$$\text{and } V(y_i) = \sum_{i=1}^{N} y_i^2\, p_i - \left(\sum_{i=1}^{N} y_i\, p_i\right)^2$$

$$= \frac{2\Sigma y_i^2}{N(N+1+2C)} + \frac{2C\Sigma y_i^2}{(N+1+2C)} - \left(\sum_{i=1}^{N} y_i\, p_i\right)^2$$

$$= \frac{2}{N(N+1+2C)} \left(LR(y^2) + \frac{N+1}{N}\right) \frac{N^2 E\,(y^2)}{2} + \frac{2CN(\sigma^2 + \bar{Y}^2)}{N(N+1+2C)}$$

$$- \bar{Y}^2 \left(1 + \frac{LR(y)}{K}\right)^2$$

$$= \frac{LR(y^2) N (\sigma^2 + \bar{Y}^2)}{(N+1+2C)} + (\sigma^2 + \bar{Y}^2) - \bar{Y}^2 \left(1 + \frac{LR(y)^2}{K}\right)$$

$$= \left((\sigma^2 + \bar{Y}^2)\left(1 + \frac{LR(y^2)}{K}\right) - \bar{Y}^2 \left(1 + \frac{LR(y)}{K}\right)^2\right)$$

thus,

$$V(LR') = \left[\frac{K^2}{n} (1 + 2Cy^2)\left(1 + \frac{LR(y)^2}{K}\right) - \left(1 + \frac{LR(y)^2}{K}\right)\right]$$

(ii) Similarly, it may be shown that

$$V(\widehat{LR^*}) = \frac{\lambda}{\bar{Y}^2} V(\bar{y}), \quad \text{where } \lambda = (2C-N-1)/N$$

$$\text{and } V(\bar{y}) = \frac{1}{n^2} \sum_{i=1}^{n} V(y_i), \text{ where } V(y_i) = \Sigma y_i^2 p_i - (\Sigma y_i p_i)^2$$

thus,

$$V(y_i) = \bar{Y}^2 \ (1+Cy^2)\left(1-\frac{LR(y)^2}{\lambda}\right) - \bar{Y}^2\left(1-\frac{LR(y)^2}{\lambda}\right)$$

$$\text{Hence, } V(\bar{y}) = \frac{\bar{Y}^2}{n}\left[(1+Cy^2)\ (1-\frac{LR(y^2)}{\lambda}) - (1-\frac{LR(y)}{\lambda})^2\right]$$

$$\text{Which gives } V(\widehat{LR^*}) = \frac{\lambda^2}{n}\left[(1+Cy^2)(1-\frac{LR(y^2)}{\lambda}) - (1-\frac{LR(y)^2}{\lambda})\right]$$

Anvesak

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(ii) **Collective Works** : Bardhan, P. K. (1974). The Pattern of Income Distribution in India : A Review, in : T. N. Srinivasan and P. K. Bardhan (eds), *Poverty and Income Distribution in India*, Statistical Publishing Society, Culcutta.

(iii) **Journals** : Chakravarty, S. (1985). Methodology and Economics, *Journal of Quantitative Economics* 1, No. 1, 1-9.