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Application of Whittle's inequality for banach space valued martingales

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Abstract

Applications of Whittle's inequality for Banach space valued martingales are discussed generalizing recent results of Shixin (Statist. Probab. Lett. 32 (1997) 245–248) and earlier results of Rao (Theory Probab. Math. Statist. 16 (1978) 111–116) for Hilbert space valued martingales.

Keywords: Whittle's inequality; Banach space valued martingales

1. Introduction

Whittle (1969) proved an inequality for real valued random variables generalizing the Kolmogorov inequality, the inequality derived by Hajek and Renyi (1955) and the inequality of Dufresnoy (1967). An application of this result for Hilbert space valued random elements $\{Z_k,\ k \geqslant 1\}$ such that the family $\{\phi_k(Z_k),\ k \geqslant 1\}$ is a submartingale is given in Rao (1978) generalising the Hajek–Renyi type inequality for martingales with values in Hilbert space due to Konakov (1973). Application to obtaining a lower bound for the probability of a simultaneous confidence region in multivariate analysis is given in Rao (1978) sharpening the bound given in Sen (1971).

Recently Shixin (1997) proved the Hajek–Renyi type inequality for Banach space valued martingales. We now derive a Whittle type inequality for Banach space valued martingales from which the results in Shixin (1997) follow as special cases.

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2. Whittle's inequality

Theorem 2.1. *Let B be a Banach space. Let $\{S_n = \sum_{i=1}^n D_i, \mathscr{F}_n, n \geqslant 1\}$ be a B-valued martingale. Let $\phi(\cdot)$ be nonnegative valued function defined on B such that $\phi(S_0) = 0$ and $\{\phi(S_n), \mathscr{F}_n, n \geqslant 1\}$ is a real valued submartingale. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$. Let A_n be the event that $\phi(S_k) \leqslant \psi(u_k), 1 \leqslant k \leqslant n$, where $0 = u_0 < u_1 < \ldots \leqslant u_n \ldots$. Then*

$$P(A_n) \geqslant 1 - \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}.$$

If, in addition,

$$0 \leqslant E[\phi(S_k)|\mathscr{F}_{k-1}] - \phi(S_{k-1}) \leqslant \Delta_k, \quad 1 \leqslant k \leqslant n$$

and

$$\psi(u_k) \geqslant \psi(u_{k-1}) + \Delta_k, \quad 1 \leqslant k \leqslant n,$$

then

$$P(A_n) \geqslant \prod_{k=1}^{n} \left(1 - \frac{\Delta_k}{\psi(u_k)}\right).$$

Remarks. The above result is a consequence of the inequality in Whittle (1969). A version of Theorem 2.1 for a sequence of Hilbert space valued random elements D_n was given in Rao (1978). We now give a detailed proof of Theorem 2.1 for completeness.

Proof. Let χ_j be the indicator function of the event $[\phi(S_j) \leqslant \psi(u_j)]$ for $1 \leqslant j \leqslant n$.

Note that

$$\chi_n \geqslant \left(1 - \frac{\phi(S_n)}{\psi(u_n)}\right)$$

and hence

$$P(A_n) = E\left(\prod_{i=1}^{n} \chi_i\right) = E\left(\left\{\prod_{i=1}^{n-1} \chi_i\right\} \chi_n\right)$$

$$\geqslant E\left(\left\{\prod_{i=1}^{n-1} \chi_i\right\} \left(1 - \frac{\phi(S_n)}{\psi(u_n)}\right)\right).$$

Observe that

$$E\left\{\left(1 - \frac{\phi(S_n)}{\psi(u_n)}\right) \middle| \mathscr{F}_{n-1}\right\} = 1 - \frac{\phi(S_{n-1})}{\psi(u_n)} - \frac{E\{\phi(S_n)|\mathscr{F}_{n-1}\} - \phi(S_{n-1})}{\psi(u_n)}.$$

Since $\{\phi(S_j), \mathscr{F}_j, j \geqslant 1\}$ is a submartingale and since $\psi(u_j)$ is nondecreasing , it follows that

$$P(A_n) \geqslant E\left(\left\{\prod_{i=1}^{n-1}\chi_i\right\}\left(1 - \frac{\phi(S_{n-1})}{\psi(u_n)}\right)\right) - \frac{E\{\phi(S_n)\} - E\{\phi(S_{n-1})\}}{\psi(u_n)}$$

$$\geqslant E\left(\left\{\prod_{i=1}^{n-2}\chi_i\right\}\left(1 - \frac{\phi(S_{n-1})}{\psi(u_{n-1})}\right)\right) - \frac{E\{\phi(S_n)\} - E\{\phi(S_{n-1})\}}{\psi(u_n)}.$$

Applying this inequality repeatedly, we get that

$$P(A_n) \geqslant 1 - \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}.$$

Note that

$$E\left\{\left(1 - \frac{\phi(S_n)}{\psi(u_n)}\right)\middle| \mathscr{F}_{n-1}\right\} - \left(1 - \frac{\Delta_n}{\psi(u_n)}\right)\left(1 - \frac{\phi(S_{n-1})}{\psi(u_{n-1})}\right)$$

$$\geqslant \frac{\phi(S_{n-1})}{\psi(u_n)\psi(u_{n-1})}[\psi(u_n) - \psi(u_{n-1}) - \Delta_n]$$

and the last term is nonnegative by hypothesis. Hence

$$P(A_n) \geqslant \left(1 - \frac{\Delta_n}{\psi(u_n)}\right) E\left(\left\{\prod_{i=1}^{n-2}\chi_i\right\}\left(1 - \frac{\phi(S_{n-1})}{\psi(u_{n-1})}\right)\right).$$

Applying this inequality repeatedly, we obtain that

$$P(A_n) \geqslant \prod_{k=1}^{n}\left(1 - \frac{\Delta_k}{\psi(u_k)}\right). \quad \square$$

3. Applications

(1) Let B a Banach space which is p-smoothable where $1 \leqslant p \leqslant 2$. Let $\{S_n = \sum_{i=1}^{n} D_i, \mathscr{F}_n, n \geqslant 1\}$ be a B-valued martingale. Then $\{\|S_n\|^p, \mathscr{F}_n, n \geqslant 1\}$ is a real valued submartingale. Let $\phi(x) = \|x\|$ and $\psi(u) = u^p$. Applying Theorem 2.1, we get that

$$P(\|S_j\| \leqslant u_j, 1 \leqslant j \leqslant n) \geqslant 1 - \sum_{j=1}^{n} \frac{E\|S_j\|^p - E\|S_{j-1}\|^p}{u_j^p}$$

and hence for every $\varepsilon > 0$,

$$P\left(\sup_{1\leqslant j\leqslant n}\frac{\|S_j\|}{u_j}\geqslant\varepsilon\right)=P\left(\sup_{1\leqslant j\leqslant n}\frac{\|S_j\|^p}{u_j^p}\geqslant\varepsilon^p\right)$$

$$\leqslant\varepsilon^{-p}\sum_{j=1}^{n}\frac{E\|S_j\|^p-E\|S_{j-1}\|^p}{u_j^p}. \tag{3.1}$$

In view of Assouad's theorem (cf. Woyczynski, 1975, Theorem 2.1), it follows that there exists an absolute constant c_p depending only on p such that

$$E(\|S_j\|^p-\|S_{j-1}\|^p\mathscr{F}_{j-1})\leqslant c_pE(\|D_j\|^p|\mathscr{F}_{j-1}),\quad j\geqslant 2$$

and hence

$$E(\|S_j\|^p-\|S_{j-1}\|^p)\leqslant c_pE(\|D_j\|^p).$$

Combining the above inequality with (3.1), we have

$$P\left(\sup_{1\leqslant j\leqslant n}\frac{\|S_j\|}{u_j}\geqslant\varepsilon\right)\leqslant c_p\varepsilon^{-p}\sum_{j=1}^{n}\frac{E(\|D_j\|^p|)}{u_j^p}.$$

Remarks. Corollaries 1 and 2 and other results in Shixin (1997) follow as special cases of the above inequality.

(2) Suppose $\{D_j,\ j\geqslant 1\}$ are independent random elements and the Banach space is 2-smoothable. Further let $\phi(x)=\|x\|^2$ and $\psi(u)=u^2$. If

$$E(\|S_j\|^2-\|S_{j-1}\|^2)\leqslant u_j^2-u_{j-1}^2$$

for $1\leqslant j\leqslant n$, then

$$P(A_n)\geqslant\prod_{j=1}^{n}\left(1-\frac{E(\|S_j\|^2)-E(\|S_{j-1}\|^2)}{u_j^2}\right),$$

which is an analogue of Dufresnoy's inequality. Applying the inequality

$$E(\|S_j\|^2)-E(\|S_{j-1}\|^2)\leqslant c_2E(\|D_j\|^2),$$

we get the weaker inequality

$$P(A_n)\geqslant\prod_{j=1}^{n}\left(1-\frac{c_2E(\|D_j\|^2|)}{u_j^2}\right).$$

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