Statistics & Probability Letters 27 (1996) 177–180

On the coefficient of variation of the $\mathscr{L}$- and $\bar{\mathscr{L}}$-classes

Arnab Bhattacharjee, Debasis Sengupta*

Applied Statistics, Surveys & Computer Science Unit, Indian Statistical Institute, 203 Barrackpore Trunk Road, Calcutta 700 035, India

Received February 1994; revised February 1994

Abstract

The coefficient of variation of a life distribution is no more than 1 if it belongs to the $\mathscr{L}$-class and no less than 1 if it belongs to the $\bar{\mathscr{L}}$-class. However, there are nonexponential distributions in each of these classes that have coefficient of variation equal to 1.

Keywords: $\mathscr{L}$-class; $\bar{\mathscr{L}}$-class; NBUE; HNBUE; HNWUE; Coefficient of variation

1. Introduction and summary

A distribution F with a finite mean μ and support on $[0,\infty)$ is said to belong to the $\mathscr{L}$-class if

$$\int_0^\infty e^{-st}\bar{F}(t)\,dt \geqslant \frac{\mu}{1+\mu s} \quad \text{for all } s \geqslant 0,$$

where $\bar{F} = 1 - F$. The dual $\bar{\mathscr{L}}$-class is defined by reversing the above inequality. Since the introduction of the $\mathscr{L}$- and $\bar{\mathscr{L}}$-classes by Klefsjö (1983), several important questions regarding these classes (particularly the $\mathscr{L}$-class) have remained unanswered. We deal with the following conjectures:

(A) The coefficient of variation (CV) of a distribution belonging to the $\mathscr{L}$-class is less than or equal to 1.
(A′) The CV of a distribution belonging to the $\bar{\mathscr{L}}$-class is greater than or equal to 1.
(B) The exponential distribution is characterized within the $\mathscr{L}$-class through the equality CV = 1.
(B′) The exponential distribution is characterized within the $\bar{\mathscr{L}}$-class through the equality CV = 1.

Results (A) and (A′) hold for the HNBUE and HNWUE classes, respectively (see Klefsjö, 1982). On the other hand, Basu and Bhattacharjee (1984) showed that (B) holds for the HNBUE class. It may be recalled that the $\mathscr{L}$- and $\bar{\mathscr{L}}$-classes include the HNBUE and HNWUE classes, respectively. Further, only the exponential distribution belongs to both the $\mathscr{L}$- and $\bar{\mathscr{L}}$-classes.

If proved, the above statements would have several important consequences. To begin with, (A) would imply the existence of the second moment of a distribution in the $\mathscr{L}$-class. The result (A) would also provide a simple proof of the closure of the $\mathscr{L}$-class under limits of distributions, through a uniform integrability argument (see Chaudhuri, 1993b). The results (A) and (B) together would lead to a test of exponentiality within the $\mathscr{L}$-class, based on the first and second moment estimates (see Klefsjö, 1986). The results (A) and (B) together may also establish the convergence of suitably scaled $\mathscr{L}$-distributions to the unit exponential distribution along the lines of Basu and Bhattacharjee (1984). Such results are useful in characterizing the limiting life distribution of a system having independent components with cold standby redundancy. The results (A′) and (B′) together would have similar implications as above for the $\overline{\mathscr{L}}$-class.

To the best of our knowledge, no attempt has been made to settle the above conjectures till the recent past. Chaudhuri (1993a) gave proofs of (A), implicitly assuming the existence of the second moment. In this paper we give a simple, yet rigorous proof of (A), proving in the process the existence of the second moment. The proof of (A′) follows along the lines of that of (A). We generalize the inequality in (A) to the case of a pair of Laplace ordered distributions. Subsequently, we disprove (B) and (B′) through two counterexamples and point out a mistake in Chaudhuri's 'proof' of (B).

2. Proofs of (A) and (A′)

Proof of (A). For a given distribution F with finite mean μ, define

$$G_F(s) = \int_0^\infty (1 - e^{-st})[e^{-t/\mu} - \bar{F}(t)]\,dt \quad \text{for all } s \geqslant 0.$$

It is easy to see that a distribution F belongs to the $\mathscr{L}$-class if and only if G_F is a nonnegative function over $[0, \infty)$. Since $\lim_{s\downarrow 0}(1 - e^{-st})/s = t$ and $\int_0^\infty te^{-t/\mu}\,dt = \mu^2$, it suffices to show that $\lim_{s\downarrow 0} G_F(s)/s$ exists and is a finite and nonnegative number. Let

$$H_F(s) = \int_0^\infty \left(\frac{1 - e^{-st}}{s}\right) \bar{F}(t)\,dt,$$

such that $H_F(s) = \mu^2(1 + \mu s)^{-1} - G_F(s)/s$, which is also a nonnegative function on $(0, \infty)$. Consider a positive sequence $\{s_n\}$ converging to 0. Clearly, $\lim_{n\to\infty} H_F(s_n)$ cannot be $-\infty$. On the other hand, $\lim_{n\to\infty} H_F(s_n) = \infty$ if and only if $\lim_{n\to\infty} G_F(s_n)/s_n = -\infty$, which is also not possible (G_F being nonnegative). The only way that $\lim_{s\downarrow 0} H_F(s)$ may not exist is when there are two positive sequences $\{s_n^{(1)}\}$ and $\{s_n^{(2)}\}$ converging to 0 such that $\lim_{n\to\infty} H_F(s_n^{(1)})$ and $\lim_{n\to\infty} H_F(s_n^{(2)})$ are finite but different. However, this possibility is ruled out by the Monotone Convergence Theorem. Therefore, each of the limits $\lim_{s\downarrow 0} H_F(s)$ and $\lim_{s\downarrow 0} G_F(s)/s$ exist and is in $[0, \infty)$.

Remark 2.1. The existence of $\lim_{s\downarrow 0} H_F(s)$ ensures the finiteness of the second moment of F through the Monotone Convergence Theorem.

Remark 2.2. Suppose the distribution F is Laplace-ordered (Rolski and Stoyan, 1976) with respect to another distribution G (with identical mean) in the sense that

$$\int_0^\infty e^{-st}(\bar{F}(t) - \bar{G}(t))\,dt \geqslant 0 \quad \text{for all } s \geqslant 0.$$

Also, let $\mu_2(F)$ and $\mu_2(G)$ be the second moments of F and G, respectively. Then an argument similar to the above one shows that $\mu_2(F) \leqslant \mu_2(G)$. Thus, if G has a finite second moment, so does F.

Remark 2.3. Chaudhuri (1993a) used a function $g(s)$ which is $-s$ times the integral of G_F, and worked with its first and second derivatives at $s = 0$ in order to prove (A). The existence of the second derivative at

0, which is related to the existence of the second moment, was assumed implicitly. The above proof fills the gap. It may also be noted that the finiteness of the second moment is also crucial to the argument given by Chaudhuri (1993b) establishing the closure of the $\mathscr{L}$-class under limits of distributions.

Proof of (A′). If the second moment is not finite, the statement holds trivially. If the second moment is finite, the proof is similar to that of (A). □

3. Counterexamples to disprove (B) and (B′)

Example 3.1. Take a discrete distribution F with point masses $(1-\alpha)$ and α at a and $a+b$, respectively. If the CV is forced to be unity, then the following condition must hold:

$$b = a(1-2\alpha)^{-1}\left(1+\sqrt{1/\alpha - 1}\right).$$

We will now show that F belongs to the $\mathscr{L}$-class when $\alpha = \frac{4}{29}$, by proving that G_F is nonnegative. Substituting the appropriate values of α and b in G_F and simplifying, we obtain

$$G_F(s) = \frac{1-\mathrm{e}^{-sa}}{s} + \frac{4}{29}\frac{\mathrm{e}^{-sa}-\mathrm{e}^{-35sa/6}}{s} - \frac{1}{s+3/(5a)}.$$

Therefore,

$$\begin{aligned}
\frac{sG_F(s/a)}{a} &= (1-\mathrm{e}^{-s}) + \frac{4}{29}(\mathrm{e}^{-s}-\mathrm{e}^{-35s/6}) - \frac{s}{s+3/5}\\
&= \frac{4}{29}\left(\frac{3/5}{s+3/5} - \mathrm{e}^{-35s/6}\right) + \frac{25}{29}\left(\frac{3/5}{s+3/5} - \mathrm{e}^{-s}\right)\\
&= \frac{29\mathrm{e}^{35s/6} - 25\mathrm{e}^{29s/6} - (125s/3)\mathrm{e}^{29s/6} - 20s/3 - 4}{29(5s/3+1)\mathrm{e}^{35s/6}}\\
&= \frac{\sum_{i=2}^{\infty}\left(\frac{35s}{6}\right)^i\left(\frac{1}{i!}\right)\left[1-\left(\frac{725+150i}{841}\right)\left(\frac{29}{35}\right)^i\right]}{(5s/3+1)\mathrm{e}^{35s/6}}.
\end{aligned}$$

The term in the square brackets is an increasing function of i, taking the value $\frac{8}{49}$ for $i = 2$. Therefore, the numerator is positive for all positive s. Clearly, G_F is positive over $(0,\infty)$ and hence F belongs to the $\mathscr{L}$-class. This disproves (B). □

Remark 3.1. The 'proof' of (B) given by Chaudhuri (1993a) involves the incorrect argument that the limit of a sequence of strictly negative numbers is *strictly* negative.

Remark 3.2. Borges et al. (1984) developed a test for checking exponentiality against the NBUE alternative based on the sample coefficient of variation. Klefsjö (1986) pointed out that this test is applicable for the HNBUE alternative as well, since the equality CV = 1 characterizes the exponential distribution within the HNBUE class. However, in view of the above counterexample, this test is not appropriate for checking exponentiality within the $\mathscr{L}$-class. Convergence of suitably scaled $\mathscr{L}$-distributions to the unit exponential distribution is also unlikely to hold.

Example 3.2. If one expands the quantity $(s/b)G_F(s/b)$ for a general two-point distribution as above (without the constraint CV equal to one), then all the terms in the Taylor series are *negative* if and only if $a = 0$ and $\alpha \leqslant \frac{1}{2}$. On the other hand, the CV is equal to $1/\alpha - 1$, which is greater than or equal to one under the above constraint. Thus, the two-point distribution with equal masses at 0 and anywhere else (that is, $\alpha = \frac{1}{2}$) is an $\overline{\mathscr{L}}$-distribution with CV equal to 1. Clearly, this counterexample disproves (B′). □

Remark 3.3. The special case of the above two-point distribution with $b = 1$ was cited by Klefsjö (1983) as one which is in the $\overline{\mathscr{L}}$-class but not in the HNWUE class.

Acknowledgements

The authors wish to thank Mr. Murari Mitra of the Indian Statistical Institute for helpful discussions.

References

Basu, S.K. and M.C. Bhattacharjee (1984), On weak convergence within the HNBUE family of life distributions, *J. Appl. Probab.* **21**, 654–660.

Borges, W.S., F. Proschan and J. Rodrigues (1984), A simple test for new better than used in expectation, *Comm. Statist. Theory Methods* **13**, 3217–3223.

Chaudhuri, G. (1993a), Coefficient of variation for the $\mathscr{L}$-class of life distributions, *Comm. Statist. Theory Methods* **22**, 2619–2622.

Chaudhuri, G. (1993b), A note on the $\mathscr{L}$-class of life distributions, to be published in: *Sankhyā*. (Also, Tech. Report # 11-93, Stat-Math Division, Indian Statistical Institute, Calcutta.)

Klefsjö, B. (1982), The HNBUE and HNWUE classes of life distributions, *Naval Res. Log. Quart.* **29**, 331–344.

Klefsjö, B. (1983), A useful ageing property based on the Laplace transform, *J. Appl. Probab.* **20**, 615–626.

Klefsjö, B. (1986), Comments on "A simple test for New Better than Used in Expectation" by Borges, Proschan and Rodrigues, *Comm. Statist. Simulation Comput.* **15**, 285–286.

Rolski, T. and D. Stoyan (1976), On the comparison of waiting times in GI/G/1 queues, *Oper. Res.* **24**, 197–200.