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Optimal designs for diallel cross experiments

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Abstract

Using nested balanced incomplete block designs, new families of optimal block designs for a certain type of diallel cross experiments are obtained. It is further shown that triangular partially balanced incomplete block designs satisfying a certain parametric condition also lead to optimal designs for diallel crosses. These results unify and extend some of the earlier results on optimality of block designs for diallel crosses.

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1. Introduction

The diallel cross is a type of mating design used in plant breeding to study the genetic properties of a set of inbred lines. Suppose there are p inbred lines and it is desired to perform a diallel cross experiment involving $p(p-1)/2$ crosses of the type $(i \times j)$, $i<j$, $i,j=1,2,\ldots,p$. The lines are to be compared with respect to their general combining abilities. Customarily, diallel cross experiments of the type mentioned above have been conducted using a completely randomised design or a randomised complete block design. However, with increase in the number of lines p, the number of crosses in the experiment increases rapidly, and in such a situation, adoption of a complete block design is not appropriate. Gupta and Kageyama (1994) and Dey and Midha (1996) have recently obtained some optimal incomplete block designs for diallel crosses. A related paper is by Singh and Hinkelmann (1995). Gupta and Kageyama (1994) make use of nested balanced incomplete block designs of Preece (1967) to arrive at two series of optimal block designs for diallel crosses. Dey and Midha (1996) show how a certain sub-class of triangular partially balanced incomplete block designs can be used to derive some optimal designs for diallel crosses.

The purpose of this communication is to investigate further the problem of obtaining optimal designs for diallel crosses. The optimality criterion chosen is the universal optimality criterion of Kiefer (1975) which, in

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particular, includes the criterion of minimization of the average variance of the best linear unbiased estimators of all elementary comparisons between the general combining ability effects of the lines involved. In Section 2, we show how in general, nested balanced incomplete block designs with sub-block size two can be used to obtain universally optimal designs for diallel crosses. We also show that nested balanced incomplete block designs satisfying a certain parametric condition lead to optimal designs for diallel crosses with a minimal number of experimental units. Several new families of optimal designs with minimal number of experimental units are presented in Section 3.

In Section 4, we show that triangular partially balanced incomplete block designs satisfying a certain parametric condition also lead to optimal designs for diallel crosses. This generalizes the result of Dey and Midha (1996) and settles the question of optimality of some designs left open by them.

2. Nested designs and optimality

Let d be a block design for a diallel cross experiment of the type mentioned in Section 1 involving p inbred lines, b blocks each of size $k(\geqslant 2)$. This means that there are k crosses in each of the blocks of d. Further, let r_{di} denote the number of times the ith cross appears in d, $i=1,2,\dots,p(p-1)/2$, and similarly, let s_{dj} denote the number of times the jth line occurs in crosses in the whole design d, $j=1,2,\dots,p$. It is then easy to see that

$$\sum_{i=1}^{p(p-1)/2} r_{di}=bk \quad \text{and} \quad \sum_{j=1}^{p} s_{dj}=2bk.$$

We also let $n=bk$ denote the number of observations generated by d. For the data obtained from the design d, we postulate the model

$$Y=\mu 1_n+\Delta_1 g+\Delta_2\beta+\varepsilon, \tag{2.1}$$

where Y is the $n\times 1$ vector of observed responses, μ is a general mean effect, 1_n denotes an n-component column vector of all ones, g and β are vectors of p general combining ability effects and b block effects, respectively, Δ_1,Δ_2 are the corresponding design matrices, that is, the (s,t)th element of Δ_1 is 1 if the sth observation pertains to the tth line, and is zero, otherwise; similarly the (s,t)th element of Δ_2 is 1 if the sth observation comes from the tth block, and is zero, otherwise; ε is the vector of random error components, these components being distributed with mean zero and constant variance σ^2. In (2.1), we have not included the specific combining ability effects. Under the model (2.1), it can be shown that the coefficient matrix of the reduced normal equations for estimating linear functions of general combining ability effects using a design d is

$$C_d=G_d-N_dN_d'/k, \tag{2.2}$$

where $G_d=(g_{dii'})$, $N_d=(n_{dij})$, $g_{dii}=s_{di}$, and for $i\neq i'$, $g_{dii'}$ is the number of times the cross $(i\times i')$ appears in d; n_{dij} is the number of times the line i occurs in block j of d.

A design d is called connected if and only if $Rank(C_d)=p-1$, or equivalently, if and only if all elementary comparisons among general combining ability effects are estimable using d. We denote by $\mathscr{D}(p,b,k)$ the class of all such connected block designs $\{d\}$ with p lines, b blocks each of size k. We need the following well-known result [(see, e.g., Cheng, 1978, p. 1246).]

Lemma 2.1. *For given positive integers s and t, the minimum of* $n_1^2+n_2^2+\cdots+n_s^2$ *subject to* $n_1+n_2+\cdots+n_s=t$, *where* n_i*'s are non-negative integers, is obtained when* $t-s[t/s]$ *of the* n_i*'s are equal to* $[t/s]+1$ *and* $s-t+s[t/s]$

are equal to $[t/s]$, *where* $[z]$ *denotes the largest integer not exceeding* z. *The corresponding minimum of* $n_1^2 + n_2^2 + \cdots + n_s^2$ *is* $t(2[t/s]+1) - s[t/s]([t/s]+1)$.

We then have

Theorem 2.1. *For any design* $d \in \mathscr{D}(p,b,k)$,

$$\mathrm{tr}(C_d) \leqslant k^{-1}b\{2k(k-1-2x) + px(x+1)\},$$

where $x = [2k/p]$ *and for a square matrix* A, $\mathrm{tr}(A)$ *stands for the trace. Equality holds if and only if* $n_{dij} = x$ *or* $x+1$ *for all* $i = 1,2,\ldots,p$, $j = 1,2,\ldots,b$.

Proof. For any $d \in \mathscr{D}(p,b,k)$, we have

$$\mathrm{tr}(C_d) = \sum_{i=1}^{p} s_{di} - k^{-1}\sum_{i=1}^{p}\sum_{j=1}^{b} n_{dij}{}^2$$
$$= 2bk - k^{-1}\sum_{i=1}^{p}\sum_{j=1}^{b} n_{dij}{}^2.$$

Now, $\sum_{i=1}^{p}\sum_{j=1}^{b} n_{dij} = 2bk$. Therefore, using Lemma 2.1,

$$\sum_{i=1}^{p}\sum_{j=1}^{b} n_{dij}^2 \geqslant b\{2k(2x+1) - px(x+1)\},$$

where $x = [2k/p]$. Hence,

$$\mathrm{tr}(C_d) \leqslant 2bk - k^{-1}b\{2k(2x+1) - px(x+1)\}$$
$$= k^{-1}b\{2k(k-1-2x) + px(x+1)\}.$$

By Lemma 2.1, equality above is attained if and only if $n_{dij} = x$ or $x+1$, for $i = 1,2,\ldots,p$; $j = 1,2,\ldots,b$. This completes the proof. □

Note that if $2k < p$ then $x = 0$ and in that case we have

$$\mathrm{tr}(C_d) \leqslant 2b(k-1), \quad d \in \mathscr{D}(p,b,k). \tag{2.3}$$

Kiefer (1975) showed that a design is universally optimal in a relevant class of competing designs if:

(i) the information matrix C of the design is completely symmetric in the sense that C has all its diagonal elements equal and all its off-diagonal elements equal, and

(ii) the matrix C has maximum trace over all designs in the class of competing designs.

Recall that a universally optimal design is, in particular, also A-optimal, that is, such a design minimizes the average variance of the best linear unbiased estimators of all elementary contrasts among the parameters of interest, that is, the general combining ability effects. Making an appeal to this result of Kiefer (1975) and to Theorem 2.1, we have the following result.

Theorem 2.2. *Let* $d^* \in \mathscr{D}(p,b,k)$ *be a block design for diallel crosses, and suppose* C_{d^*} *satisfies*

(i) $\mathrm{tr}(C_{d^*}) = k^{-1}b\{2k(k-1-2x) + px(x+1)\}$, *and*

(ii) C_{d^*} *is completely symmetric.*

Then d^* is universally optimal in $\mathscr{D}(p,b,k)$, and in particular minimizes the average variance of the best linear unbiased estimators of all elementary contrasts among the general combining ability effects.

We now show a connection between nested balanced incomplete block design of Preece (1967) and optimal designs for diallel crosses. For completeness, we recall the definition of a nested balanced incomplete block design.

Definition 2.1. A nested balanced incomplete block design with parameters $(v, b_1, k_1, r^*, \lambda_1, b_2, k_2, \lambda_2, m)$ is a design for v treatments, each replicated r^* times with two systems of blocks such that:

(a) the second system is nested within the first, with each block from the first system, called henceforth as 'block' containing exactly m blocks from the second system, called hereafter as 'sub-blocks';

(b) ignoring the second system leaves a balanced incomplete block design with usual parameters $v, b_1, k_1, r^*, \lambda_1$;

(c) ignoring the first system leaves a balanced incomplete block design with parameters $v, b_2, k_2, r^*, \lambda_2$.

From the well-known parametric relations for a balanced incomplete block design, it is easy to see that the following parametric relations hold for a nested balanced incomplete block design:

$$vr^* = b_1k_1 = mb_1k_2 = b_2k_2, \quad (v-1)\lambda_1 = (k_1 - 1)r^*, \quad (v-1)\lambda_2 = (k_2-1)r^*.$$

Consider now a nested balanced incomplete block design d with parameters $v = p, b_1, k_1, k_2 = 2, r^*$. If we identify the treatments of d as lines of a diallel experiment and perform crosses among the lines appearing in the same sub-block of d, we get a block design d^* for a diallel experiment involving p lines with $v_c = p(p-1)/2$ crosses, each replicated $r = 2b_2/\{p(p-1)\}$ times, and $b = b_1$ blocks, each of size $k = k_1/2$. Such a design $d^* \in \mathscr{D}(p,b,k)$; also, for such a design, $n_{d^*ij} = 0$ or 1 for $i = 1,2,\ldots,p$, $j = 1,2,\ldots,b$ and

$$C_{d^*} = (p-1)^{-1}2b(k-1)(I_p - p^{-1}J_p), \tag{2.4}$$

where I_p is an identity matrix of order p and J_p is a $p \times p$ matrix of all ones. Clearly, C_{d^*} given by (2.4) is completely symmetric and $\mathrm{tr}(C_{d^*}) = 2b(k-1)$ which equals the upper bound for $\mathrm{tr}(C_d)$ given by (2.3). Thus, from Theorem 2.2, the design d^* is universally optimal in $\mathscr{D}(p,b,k)$. It is also easy to see that using d^*, each elementary contrast among general combining ability effects is estimated with a variance

$$(p-1)\sigma^2/\{b(k-1)\}. \tag{2.5}$$

Further, if the nested balanced incomplete block design with parameters $v = p, b_1, k_1, b_2 = b_1k_1/2, k_2 = 2$ is such that $\lambda_2 = 1$ or equivalently if,

$$b_1k_1 = p(p-1), \tag{2.6}$$

then the optimal design d^* for diallel crosses derived from this design has each cross replicated just once and hence uses the minimal number of experimental units. Summarizing, therefore, we have

Theorem 2.3. *The existence of a nested balanced incomplete block design d with parameters $v = p$, $b_1 = b$, $b_2 = bk$, $k_1 = 2k$, $k_2 = 2$ implies the existence of a universally optimal incomplete block design d^* for diallel crosses. Further, if the parameters of d satisfy* (2.6), *then d^* has the minimal number of experimental units.*

Instead of the design $d^* \in \mathscr{D}(p,b,k)$ based on the nested balanced incomplete block design d, if one adopts a randomized complete block design with $r = 2bk/\{p(p-1)\}$ blocks, each block having all the $p(p-1)/2$ crosses, the C-matrix can easily be shown to be

$$C_R = r(p-2)(I_p - p^{-1}J_p),$$

so that the variance of the best linear unbiased estimator of any elementary contrast among the general combining ability effects is $2\sigma_1^2/\{r(p-2)\}$, where σ_1^2 is the per observation variance in the case of randomized block experiment. Thus, the efficiency factor of the design $d^* \in \mathscr{D}(p,b,k)$, relative to a randomized complete block design is given by

$$e = \frac{2b(k-1)}{r(p-2)(p-1)} = \frac{p(k-1)}{k(p-2)}. \tag{2.7}$$

3. New families of optimal designs

Gupta and Kageyama (1994) obtained two families of nested balanced incomplete block designs, leading to optimal designs for diallel crosses. These families, in our notation have the following parameters:

Series 1: $v = p = 2t+1 = b_1, b_2 = t(2t+1),\ k_1 = 2t,\ k_2 = 2;$

Series 2: $v = p = 2t, b_1 = 2t-1, b_2 = t(2t-1),\ k_1 = 2t,\ k_2 = 2.$

It is easy to verify that the designs in Series 1 and 2 above satisfy (2.6) and, hence, use the minimal number of experimental units. In this section, we show that several other families of nested balanced incomplete block designs satisfying the condition (2.6) exist and can therefore be used to derive optimal designs for diallel crosses with minimal number of experimental units. Henceforth, we denote the parameters of the design for diallel crosses by v_c, b, k, r where $v_c = p(p-1)/2$ is the number of crosses, b, the number of blocks, k, the number of crosses per block or the block size and r is the number of times each cross is replicated in the design.

Family 1: Let $v = p = 4t+1$, $t \geqslant 1$ be a prime or a prime power and x be a primitive element of the Galois field of order v, GF(v). Consider the t initial blocks

$$\{(x^i, x^{i+2t}), (x^{i+t}, x^{i+3t})\}, \quad i = 0, 1, 2, \ldots, t-1.$$

As shown by Dey et al. (1986), these initial blocks, when developed in the sense of Bose (1939), give rise to a nested balanced incomplete block design with parameters $v = p = 4t+1$, $k_1 = 4$, $b_1 = t(4t+1)$, $k_2 = 2$. Using this design, one can get an optimal design for diallel crosses with minimal number of experimental units and parameters $v_c = 2t(4t+1)$, $b = t(4t+1)$, $k = 2$, $r = 1$. It is interesting to note that this family of designs has the smallest block size, $k = 2$.

Example 3.1. Let $t = 2$ in Family 1. Then a nested balanced incomplete block design with parameters $v = p = 9$, $b_1 = 18$, $k_1 = 4$, $k_2 = 2$, $\lambda_2 = 1$ can be constructed by developing the following initial blocks over GF(3^2):

$$\{(1,2),(2x+1,x+2)\}; \quad \{(x,2x),(2x+2,x+1)\},$$

where x is a primitive element of GF(3^2) and the elements of GF(3^2) are $0,1,2,x,x+1,x+2,2x,2x+1,2x+2$. Adding successively the non-zero elements of GF(3^2) to the contents of the initial blocks, the full nested design is obtained. The design for diallel crosses is exhibited below, where the lines have been relabelled 1–9, using the correspondence $0\to1,\ 1\to2,\ 2\to3,\ x\to4,\ x+1\to5,\ x+2\to6,\ 2x\to7,\ 2x+1\to8,\ 2x+2\to9$:

$[2\times3, 6\times8]$;	$[1\times3, 4\times9]$;	$[1\times2, 5\times7]$;	$[5\times6, 2\times9]$;	$[4\times6, 3\times7]$;	$[4\times5, 1\times8]$;
$[8\times9, 3\times5]$;	$[7\times9, 1\times6]$;	$[7\times8, 2\times4]$;	$[4\times7, 5\times9]$;	$[5\times8, 6\times7]$;	$[6\times9, 4\times8]$;
$[1\times7, 3\times8]$;	$[2\times8, 1\times9]$;	$[3\times9, 2\times7]$;	$[1\times4, 2\times6]$;	$[2\times5, 3\times4]$;	$[3\times6, 1\times5]$.

This is a design for a diallel cross experiment for $p=9$ lines in 18 blocks each of size two; each cross appears in the design just once. Two designs for $p=9$ have been reported by Gupta and Kageyama (1994); both these designs have blocks of size larger than two. Further, no nested design listed by Preece (1967) leads to an optimal design for diallel crosses with $p=9$ lines in blocks of size two.

Family 2. Let $v=p=6t+1$, $t\geqslant 1$ be a prime or a prime power and x be a primitive element of GF(v). Consider the initial blocks

$$\{(x^i,x^{i+3t}),(x^{i+t},x^{i+4t}),(x^{i+2t},x^{i+5t})\},\quad i=0,1,2,\ldots,t-1.$$

Dey et al. (1986) show that these initial blocks, when developed give a solution of a nested balanced incomplete block design with parameters $v=p=6t+1$, $b_1=t(6t+1)$, $k_1=6$, $k_2=2$, $\lambda_2=1$. Hence, using this series of nested balanced incomplete block designs, we get a solution for an optimal design for diallel crosses with parameters $v_c=3t(6t+1)$, $b=t(6t+1)$, $k=3$, $r=1$.

Example 3.2. Let $t=2$ in Family 2. Then a nested balanced incomplete block design with parameters $v=p=13$, $b_1=26$, $k_1=6$, $k_2=2$, $\lambda_2=1$ is obtained by developing over GF(13) the following two initial blocks:

$$\{(1,12),(4,9),(3,10)\};\quad \{(2,11),(8,5),(6,7)\}.$$

Using this nested design, an optimal design for diallel crosses with minimal number of experimental units and parameters $v_c=78$, $b=26$, $k=3$ can be constructed.

Family 3. Let $12t+7$, $t\geqslant 0$ be a prime or a prime power and suppose $x=3$ is a primitive element of GF($12t+7$). Then, as shown by Dey et al. (1986), one can get a nested balanced incomplete block design with parameters $v=p=12t+8$, $b_1=(3t+2)(12t+7)$, $k_1=4$, $k_2=2$ by developing the following $3t+2$ initial blocks:

$$\{(1,\infty),(x^{3t+2},x^{6t+3})\},\quad \{(x^i,x^{i+3t+1})(x^{i+3t+2},x^{i+6t+3})\},\ i=1,2,\ldots,3t+1,$$

here ∞ is an invariant variety. Using this family of nested designs, one can get a family of optimal designs for diallel crosses with minimal number of experimental units and parameters $v_c=(12t+8)(12t+7)$, $b=(3t+2)(12t+7)$, $k=2, r=1$.

The next family of nested designs has $\lambda_2=2$ and, hence, in the design for diallel crosses derived from this family, each cross is replicated twice. However, this family of designs is of practical utility as the optimal designs for diallel crosses derived from this family of nested balanced incomplete block designs have a block size, $k=2$.

Family 4: Let $v=p=2t+1$, $t\geqslant 1$ be a prime or a prime power and x be a primitive element of GF($2t+1$). Then as shown by Dey et al. (1986), a nested balanced incomplete block design with parameters $v=2t+1, b_1=t(2t+1)$, $k_1=4$, $k_2=2$, $\lambda_2=2$ can be constructed by developing the following initial blocks over GF($2t+1$):

$$\{(0,x^{i-1}),(x^i,x^{i+1})\},\quad i=1,2,\ldots,t.$$

Using this family of nested designs, a family of optimal designs for diallel crosses with parameters $v_c=t(2t+1)=b$, $k=2=r$ can be constructed.

In particular, for $t=3,5$ we get optimal designs for diallel crosses with parameters

$$p=7,\quad v_c=21=b,\quad k=2=r \quad\text{and}\quad p=11,\quad v_c=55=b,\quad k=2=r.$$

For these values of p, no designs with block size two are available in Gupta and Kageyama (1994).

Example 3.3. Let $t=3$ in Family 4. Then a nested balanced incomplete block design with parameters $v=p=7$, $b_1=21$, $k_1=4$, $k_2=2$, $\lambda_2=2$ is obtained by developing over GF(7) the following three initial blocks:

$$\{(0,1),(3,2)\};\quad \{(0,3),(2,6)\};\quad \{(0,2),(6,4)\}.$$

Using this nested design, an optimal design for diallel crosses with parameters $v_c=21=b$, $k=2=r$ can be constructed and is shown below:

$$\begin{array}{lll}
[0\times 1, 2\times 3]; & [0\times 3, 2\times 6] & [0\times 2, 4\times 6]; \\
[1\times 2, 3\times 4]; & [1\times 4, 0\times 3] & [1\times 3, 0\times 5]; \\
[2\times 3, 4\times 5]; & [2\times 5, 1\times 4] & [2\times 4, 1\times 6]; \\
[3\times 4, 5\times 6]; & [3\times 6, 2\times 5] & [3\times 5, 0\times 2]; \\
[4\times 5, 0\times 6]; & [0\times 4, 3\times 6] & [4\times 6, 1\times 3]; \\
[5\times 6, 0\times 1]; & [1\times 5, 0\times 4] & [0\times 5, 2\times 4]; \\
[0\times 6, 1\times 2]; & [2\times 6, 1\times 5] & [1\times 6, 3\times 5].
\end{array}$$

Here the lines are numbered 0–6.

Remark 3.1. In Section 2, a connection between nested balanced incomplete block designs and optimal designs for diallel crosses was shown. Nested balanced incomplete block designs can be generalized to a wider class of nested designs, which may be called nested balanced block designs in the same manner as balanced incomplete block designs have been generalized to balanced block designs. Nested balanced block designs with sub-block size two can be used to derive optimal block designs for diallel crosses. One such family of designs, leading to optimal designs with minimal number of experimental units is reported below. Several other families of such designs will be reported elsewhere.

Family 5: Let $p=2t+1$, where $t\geqslant 1$ is an integer. Then a nested balanced block design with parameters $v=p=2t+1$, $k_1=2(2t+1)$, $b_1=t$, $k_2=2$, $\lambda_2=1$ can be constructed. The blocks are

$$\{(j,2t+1-j),(1+j,1-j),(2+j,2-j),\ldots,(2t+j,2t-j)\},\quad j=1,2,\ldots,t,$$

where parentheses include sub-blocks, and the symbols are reduced modulo p. Making crosses among lines appearing in the same sub-block, one gets a solution of a block design for diallel crosses with parameters $v_c=t(2t+1)$, $b=t$, $k=2t+1$, $r=1$. If d is a design for diallel crosses derived from this family of nested designs, then the C-matrix of d can be shown to be

$$C_d=(4t-2)(I_p-p^{-1}J_p). \tag{3.1}$$

Clearly, C_d given by (3.1) is completely symmetric. Also, $\mathrm{tr}(C_d)=2t(4t-2)$, which equals the upper bound given by Theorem 2.1 for $x=[2k/p]=2$. Hence, the design d is optimal and has each cross replicated just once.

4. Optimal designs based on triangular PBIB designs

It has recently been shown by Dey and Midha (1996) that triangular partially balanced incomplete block designs with two associate classes can be used to derive block designs for diallel crosses and, in particular, triangular designs satisfying $\lambda_1 = 0$ lead to optimal designs. The optimality of designs derived from triangular designs not satisfying $\lambda_1 = 0$ was left unsettled by Dey and Midha (1996). In this section, we give a general parametric condition on triangular designs, leading to optimal block designs for diallel crosses. This condition includes the condition of Dey and Midha (1996) as a special case and helps in settling the question of optimality of some designs left open by them.

To begin with let us recall the definition of a triangular design.

Definition 4.1. A binary block design with $v = p(p-1)/2$ treatments and b blocks, each of size k is called a triangular design if

(i) each treatment is replicated r times,

(ii) the treatments can be indexed by a set of two labels (i,j), $i<j$, $i,j = 1,2,\ldots,p$; two treatments, say (α,β) and (γ,δ) occur together in λ_1 blocks if either $\alpha=\gamma, \beta \neq \delta$, or $\alpha \neq \gamma, \beta=\delta$ or, $\alpha=\delta, \beta \neq \gamma$ or, $\alpha \neq \delta, \beta=\gamma$; otherwise, they occur together in λ_2 blocks.

Observe that all triangular designs with parameters $v = p(p-1)/2, b, r, k, \lambda_1, \lambda_2$ and treatments indexed by (i,j) can be viewed as nested incomplete block designs with p treatments, b blocks of size $2k$ and sub-blocks of size two. Now, following Dey and Midha (1996), we derive a block design $d \in \mathscr{D}(p,b,k)$ for diallel crosses from a triangular design d_1 with parameters $v = p(p-1)/2, b, r, k, \lambda_1, \lambda_2$, by replacing a treatment (i,j) in d_1 with the cross $(i \times j)$, $i<j$, $i,j = 1,2,\ldots,p$. Then, using Lemma 3.1 of Dey and Midha (1996) it can be shown that

$$C_d = \theta(I_p - p^{-1}J_p), \tag{4.1}$$

where $\theta = pk^{-1}\{r(k-1) - (p-2)\lambda_1\}$. Therefore, using the design d, any elementary comparison among general combining ability effects is estimated with a variance $2\sigma^2/\theta$, and the efficiency factor of the design relative to a randomized complete block design is $\theta/\{r(p-2)\}$. Further, from (4.1), it follows that

$$\mathrm{tr}(C_d) = k^{-1}p(p-1)\{r(k-1) - (p-2)\lambda_1\}. \tag{4.2}$$

Also, as shown in Theorem 2.1, for any design in $\mathscr{D}(p,b,k)$, the trace of the C-matrix is bounded above by

$$k^{-1}b\{2k(k-1-2x) + px(x+1)\}, \tag{4.3}$$

where $x = [2k/p]$. Equating (4.2) and (4.3), we have the following result.

Theorem 4.1. *A block design for diallel crosses derived from a triangular design with parameters $v = p(p-1)/2, b, r, k, \lambda_1, \lambda_2$ is universally optimal over $\mathscr{D}(p,b,k)$ if*

$$p(p-1)(p-2)\lambda_1 = bx\{4k - p(x+1)\}, \tag{4.4}$$

where $x = [2k/p]$. Further, when the condition in (4.4) holds, the efficiency factor is given by

$$e = p\{2k(k-1-2x) + px(x+1)\}/\{2k^2(p-2)\}. \tag{4.5}$$

We now prove the following result.

Lemma 4.1. *For a triangular design with parameters $v = p(p-1)/2, b, r, k, \lambda_1 = 0, \lambda_2$, the inequality $2k \leqslant p$ holds.*

Table 1
Optimal designs for diallel crosses based on triangular designs

No.	p	k	b	e	Ref.
1	5	3	30	0.92	$T13$
2	5	4	10	0.94	$T33$
3	5	5	6	1.00	$T44$
4	5	6	10	0.97	$T60$
5	6	6	10	1.00	$T62$
6	8	7	28	0.98	$T77$
7	6	9	10	1.00	$T83$
8	9	9	28	1.00	$T85$
9	10	9	45	0.92	$T91$

Proof. Since $\lambda_1 = 0$, from the basic identity $r(k-1) = n_1\lambda_1 + n_2\lambda_2$ for a partially balanced design with two associate classes (see, e.g., Dey, 1985, p. 159), we arrive at

$$r(k-1) = n_2\lambda_2 = (p-2)(p-3)\lambda_2/2,$$

so that

$$2k = (p-2)(p-3)\lambda_2/r + 2.$$

Hence,

$$p - 2k = (p-2)r^{-1}\{r - (p-3)\lambda_2\}.$$

Also, it is well known (see, e.g., Dey, 1985, p. 180) that if N is the incidence matrix of a triangular design, then one of the eigenvalues of NN', say $\theta_1 = r + (p-4)\lambda_1 - (p-3)\lambda_2 = r - (p-3)\lambda_2$, as $\lambda_1 = 0$ by hypothesis. Since NN' is non-negative definite, $\theta_1 \geqslant 0$ and we have $p \geqslant 2k$, completing the proof. □

It follows from Lemma 4.1 that for a triangular design with $\lambda_1 = 0$, $x = [2k/p] = 0$ if $2k < p$ and $4k - p(x+1) = 0$ if $2k = p$. Hence, for triangular designs with $\lambda_1 = 0$, the condition in (4.4) is always satisfied. Thus, we have the following corollary to Theorem 4.1, obtained earlier by Dey and Midha (1996).

Corollary 4.1. *A triangular design with parameters $v = p(p-1)/2, b, r, k, \lambda_1, \lambda_2$ satisfying $\lambda_1 = 0$ leads to a universally optimal design for diallel crosses.*

Dey and Midha (1996, Table 2) reported several efficient designs for diallel crosses, without claiming the optimality of these designs. Out of the designs reported in Dey and Midha (1996, Table 2), the designs listed in Table 1 are indeed optimal, as for these designs, the condition in (4.4) is satisfied. Designs obtained by repeating the blocks of a smaller design have been left out in Table 1, because the repeated design has the same efficiency factor as the smaller design. In Table 1, e denotes the efficiency factor of the design for diallel crosses given by (4.5) and the entries under Ref. are the design numbers from the catalogue of Clatworthy (1973).

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