Computing the shape of a point set in digital images

S.K. Parui, S. Sarkar and B.B. Chaudhuri

Electronics and Communication Sciences Unit, Indian Statistical Institute, Calcutta 700 035, India

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Abstract

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The point set here consists of pixels from a digital image. First, the digital Voronoi diagram of the set is constructed using the Euclidean distance. From this diagram a certain planar graph is found which is a subgraph of the Delaunay triangulation of the point set. Finally, the shape of the point set is computed as a certain subgraph of the planar graph.

Keywords. Discrete points, digital geometry, Voronoi diagram, planar graph, planar shape.

1. Introduction

One common way to define the shape of a finite set of 2-D points is its convex hull. There is a host of algorithms to compute convex hulls. But in many cases the underlying shape from which the points emerge is not convex. Edelsbrunner et al. (1983) extended one definition of the convex hull and proposed a general definition of the shape (convex or otherwise) of a finite planar set. This is called α-shape which will find applications in pattern recognition.

The present paper proposes an algorithm to find the α-shape of a finite set of digital points. In image processing problems, it is sometimes necessary to reconstruct a shape from a set of pixels or lattice points (whose coordinates are only integers). Since our aim here is to find the shape boundary of a point set including boundary concavities, we will consider α-shapes for negative α only. In Section 2 the definitions and results that are relevant for α-shape (for arbitrary α) computation in the continuous case, are given. In Section 3 we present some definitions and results on digital geometry that are useful for computing α-shapes (for negative α) in the digital case. Computational techniques are explained in Section 4. Results and conclusions are given in Section 5.

Correspondence to: Dr. Swapan Kr. Parui, Electronics and Communication Sciences Unit, Indian Statistical Institute, 203 B.T. Road, Calcutta 700 035, India.

2. Definitions and results in continuous case (Edelsbrunner et al. (1983))

Let α be an arbitrary real number. A *generalized disc of radius* $1/\alpha$ is defined as a closed disc of radius $1/\alpha$ if $\alpha>0$, the closed complement of a disc of radius $-1/\alpha$ if $\alpha<0$, and a closed halfplane if $\alpha=0$. For a set S of 2-D points, the *α-hull of S* is the intersection of all generalized discs of radius $1/\alpha$ that contain S. A point P in S is *α-extreme* in S if there exists a generalized disc of radius $1/\alpha$ containing S such that P lies on its boundary. Two

α-extreme points P and Q of S are *α-neighbours* if there exists a generalized disc of radius $1/\alpha$ containing S such that both P and Q lie on its boundary. The *α-shape* of S is the planar straight line graph whose vertices are the α-extreme points and whose edges connect the respective α-neighbours. As α approaches zero, the α-shape tends to coincide with the convex hull of S.

Proposition 2.1. *The α-shape of S is a subgraph of the Delaunay triangulation of S which can be computed from the closest point Voronoi diagram (for $\alpha<0$) or the furthest point Voronoi diagram (for $\alpha>0$) of S.*

Proposition 2.2. *For every edge e in the Delaunay triangulation of S, there exist real numbers $\alpha_{\min}(e)$ and $\alpha_{\max}(e)$ where $\alpha_{\min}(e)\leqslant\alpha_{max}(e)$ such that e is an edge of the α-shape of S if and only if $\alpha_{\min}(e)\leqslant\alpha\leqslant\alpha_{\max}(e)$.*

Thus, the edges of the α-shape can be identified after examining the edges of the Delaunay triangulation of S (Edelsbrunner et al. (1983)).

3. Definitions and results in digital case

Let $S=\{P_i, i=1,\ldots,n\}$ be a set of pixels in a

1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	$\underline{1}$	1	1	1	2	2	2	$\underline{2}$	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	5
3	2	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	5	5	5
3	3	3	3	1	1	1	1	1	4	4	2	2	2	2	2	5	5	5	5	5
3	3	3	3	3	3	1	1	4	4	4	4	4	2	2	5	5	5	5	5	5
3	3	3	3	3	3	3	4	4	4	4	4	4	4	5	5	5	5	5	5	5
3	3	3	3	3	3	3	4	4	4	4	4	4	4	5	5	5	5	5	5	5
3	3	3	$\underline{3}$	3	3	3	4	4	4	$\underline{4}$	4	4	4	5	5	5	$\underline{5}$	5	5	5
3	3	3	3	3	3	3	4	4	4	4	4	4	4	5	5	5	5	5	5	5
3	3	3	3	3	3	3	4	4	4	4	4	4	4	5	5	5	5	5	5	5
3	3	3	3	3	3	3	4	4	4	4	4	4	4	5	5	5	5	5	5	5
6	6	6	6	6	6	6	7	7	7	7	7	7	7	8	8	8	8	8	8	8
6	6	6	6	6	6	6	7	7	7	7	7	7	7	8	8	8	8	8	8	8
6	6	6	6	6	6	6	7	7	7	7	7	7	7	8	8	8	8	8	8	8
6	6	6	$\underline{6}$	6	6	6	7	7	7	$\underline{7}$	7	7	7	8	8	8	$\underline{8}$	8	8	8
6	6	6	6	6	6	6	7	7	7	7	7	7	7	8	8	8	8	8	8	8
6	6	6	6	6	6	6	7	7	7	7	7	7	7	8	8	8	8	8	8	8
6	6	6	6	6	6	6	7	7	7	7	7	7	7	8	8	8	8	8	8	8

Figure 1. Digital Dirichlet Tesselation of S which is a set of eight pixels (indicated by underlined labels). Labels 'i' mark the ith Voronoi tile for $i=1,2,\ldots,8$.

1	1	1	1	1	1	1	1	1	$\underline{1}$	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	$\underline{1}$	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	$\underline{1}$	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	*	1	1	$\underline{1}$	2	2	2	*	2	2	2	2	2	2	$\underline{2}$
$\underline{1}$	$\underline{1}$	1	1	1	1	1	1	1	$\underline{1}$	2	2	2	2	2	2	2	2	$\underline{2}$	$\underline{2}$	5
3	3	$\underline{1}$	$\underline{1}$	1	1	1	1	1	$\underline{1}$	$\underline{2}$	2	2	2	2	2	$\underline{2}$	$\underline{2}$	5	5	5
3	3	3	3	$\underline{1}$	$\underline{1}$	1	1	$\underline{1}$	4	4	$\underline{2}$	$\underline{2}$	2	2	$\underline{2}$	5	5	5	5	5
3	3	3	3	3	3	$\underline{1}$	$\underline{1}$	4	4	4	4	4	$\underline{2}$	$\underline{2}$	5	5	5	5	5	5
3	3	3	3	3	3	$\underline{3}$	4	4	4	4	4	4	$\underline{4}$	5	5	5	5	5	5	5
3	3	3	3	3	3	$\underline{3}$	4	4	4	4	4	4	$\underline{4}$	5	5	5	5	5	5	5
3	3	3	*	3	3	$\underline{3}$	4	4	4	*	4	4	$\underline{4}$	5	5	5	*	5	5	5
3	3	3	3	3	3	$\underline{3}$	4	4	4	4	4	4	$\underline{4}$	5	5	5	5	5	5	5
3	3	3	3	3	3	$\underline{3}$	4	4	4	4	4	4	$\underline{4}$	5	5	5	5	5	5	5
$\underline{3}$	$\underline{3}$	$\underline{3}$	$\underline{3}$	$\underline{3}$	$\underline{3}$	$\underline{3}$	$\underline{4}$	$\underline{4}$	$\underline{4}$	$\underline{4}$	$\underline{4}$	$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{5}$	$\underline{5}$	$\underline{5}$	$\underline{5}$	$\underline{5}$	$\underline{5}$
6	6	6	6	6	6	$\underline{6}$	7	7	7	7	7	7	$\underline{7}$	8	8	8	8	8	8	8
6	6	6	6	6	6	$\underline{6}$	7	7	7	7	7	7	$\underline{7}$	8	8	8	8	8	8	8
6	6	6	6	6	6	$\underline{6}$	7	7	7	7	7	7	$\underline{7}$	8	8	8	8	8	8	8
6	6	6	*	6	6	$\underline{6}$	7	7	7	*	7	7	$\underline{7}$	8	8	8	*	8	8	8
6	6	6	6	6	6	$\underline{6}$	7	7	7	7	7	7	$\underline{7}$	8	8	8	8	8	8	8
6	6	6	6	6	6	$\underline{6}$	7	7	7	7	7	7	$\underline{7}$	8	8	8	8	8	8	8
6	6	6	6	6	6	$\underline{6}$	7	7	7	7	7	7	$\underline{7}$	8	8	8	8	8	8	8

Figure 2. The pixels with underlined labels indicate the Digital Voronoi Diagram of S. Pixels of S have labels '*'.

digital image I of size $m\times m$ where $P_i=(r_i, c_i)$ and r_i, c_i are integers indicating the row and column positions respectively of the pixel P_i. Let, for any pixel P with coordinates (r, c) in I, $d(P, P_i)$ denote the Euclidean distance $\{(r-r_i)^2+(c-c_i)^2\}^{1/2}$. A matrix M of size $m\times m$ is defined as follows. $M(r, c)=i$ if (1) $d(P, P_i)\leqslant d(P, P_j)$ for all $j\neq i$ and (2) $d(P, P_i)<d(P, P_j)$ for all $j<i$. In other words, a pixel P in I gets as its label the index of its nearest pixel in S. If P has multiple nearest pixels in S, then the minimum index among these nearest pixels becomes the label of P (Figure 1).

Clearly, $M(r_i, c_i)=i$ for all i.

Definition 3.1. $C(i)=\{(r, c)\text{: } M(r, c)=i\}$ for $i=1,\ldots,n$.

Note that the $C(i)$'s are non-empty and disjoint and their union is the whole image I.

Definition 3.2. $C(i)$ is the *digital Voronoi polygon* or *tile* corresponding to P_i.

Definition 3.3. The set of all n tiles is the *digital Dirichlet Tessellation* (DDT) of S (Figure 1).

The *digital Voronoi diagram* (DVD) of S is defined as follows. It consists of pixels which lie on the boundary of the tiles such that it is 8-connected and it has unit thickness. More formally, a pixel

(r,c) belongs to DVD if there is a 4-neighbour (r_1,c_1) of (r,c) such that $M(r,c)<M(r_1,c_1)$. Note that any pixel P which has more than one nearest neighbour in S belongs to DVD. Also, for any P in DVD, its distances from its two nearest neighbours in S differ by at most 1 (Figure 2). This is because of the digital nature of the geometry. A pixel in DDT is called *interior* if it is not in DVD.

Definition 3.4. $B(i)=\{(r,c)\in \text{DVD}$ such that there is a 4-neighbouring interior pixel (r',c') of (r,c) satisfying $M(r',c')=i\}$ is the *boundary of the ith tile* (Figure 3).

Note that $B(i)$ may not be a subset of $C(i)$. This is because $B(i)$ may belong to an adjacent $C(j)$ due to the digital nature of the geometry.

Proposition 3.1. *$B(i)$ is a digital curve.*

Definition 3.5. $C'(i)=C(i)-B(i)$ is the *interior of the ith tile.*

Definition 3.6. $E(i,j)=B(i)\cap B(j)$ is the *Voronoi edge* shared by the ith and jth tiles (Figure 3).

Proposition 3.2. *A pixel P belongs to $E(i,j)$ if and only if P has at least one 4-neighbour in both $C'(i)$ and $C'(j)$.*

```
                  3
                  3
                  3
            *     3         *                 2
1 1               3                       2 2
    1 1           1 2                 2 2
        1 1     1     2 2           2
            1 1             2 2
            x               x
            x               x
      *     x       *       x       *
            x               x
            x               x
x x x x x x x x x x x x x x x x x x x x x
            x               x
            x               x
            x               x
      *     x       *       x       *
            x               x
            x               x
            x               x
```

Figure 3. P_1 and P_2 are the two pixels of S on the top left and on the top right respectively. $B(i)$ is the set of pixels with labels 'i' and '3' for $i=1,2$. $E(1,2)$, the set of pixels with labels '3', is a Voronoi edge. Hence P_1 is a Voronoi neighbour of P_2.

Proposition 3.3. *If $E(i,j)$ is non-null, it is a digital straight line segment. Also, $(r,c)\in E(i,j)$ implies $M(r,c)=\min\{i,j\}$.*

Definition 3.7. P_j is a *Voronoi neighbour* of P_i if $E(i,j)$ is non-empty (Figure 3).

Proposition 3.4. *P_i and P_j are Voronoi neighbours if and only if there exist two 4-neighbouring pixels (r_1,c_1) and (r_2,c_2) such that $M(r_1,c_1)=i$ and $M(r_2,c_2)=j$.*

Definition 3.8. $G(S)$ is a planar straight line graph where S is the set of vertices and P_iP_j forms an edge if P_i and P_j are Voronoi neighbours.

Proposition 3.5. *If no four points of S are cocircular, $G(S)$ is the same as the Delaunay triangulation of S. Otherwise, $G(S)$ is a subgraph of any Delaunay triangulation of S.*

Let $r=-1/\alpha$ $(\alpha<0)$. We define the α-shape of S as a planar straight line graph $A_\alpha(S)$ in the following way.

Definition 3.9. For $P_i,P_j\in S$, P_iP_j is an edge of A_α if there is a disc D of radius r such that

(i) P_i and P_j are on the boundary of D,

(ii) there is no point of S falling on the open arc P_iP_j, and

(iii) there is no point of S in the interior of D.

Proposition 3.6. *$A_\alpha(S)$ is a subgraph of $G(S)$.*

Proposition 3.7. *For every edge e of $G(S)$, there exist real numbers $\alpha_{\min}(e)$ and $\alpha_{\max}(e)$ where $\alpha_{\min}(e)\leqslant\alpha_{\max}(e)$ such that e is an edge of $A_\alpha(S)$ if and only if $\alpha_{\min}(e)\leqslant\alpha\leqslant\alpha_{\max}(e)$.*

We write $r_{\min}(e)=-1/\alpha_{\min}(e)$ and $r_{\max}(e)=-1/\alpha_{\max}(e)$. Thus, the edges of the α-shape can be identified after examining the edges of $G(S)$ and their $r_{\min}$ and $r_{\max}$ values.

4. Computation of α-shape

The computational techniques for α-shapes in the digital case are described in this section. We assume that no two points or pixels of $S=\{P_1,P_2,\dots,P_n\}$ are 8-neighbours of each other. That is, there is a gap of at least one pixel between any pair of pixels of S. This can be achieved by multiplying by two the coordinates of each pixel of an arbitrary set S.

Computation of DDT(S)

Suppose the pixels of S belong to an image I of size $m\times m$. For every pixel $p=(r,c)$ in I, we compute the Euclidean distance d_i between p and P_i. Find P_j such that $d_j=\min d_i$. If P_j is not unique, we choose the one with minimum j. We label the pixel (r,c) as j. That is, $M(r,c)=j$. The computational complexity in this step is $O(nm^2)$. Parallel computation is possible here since each of the m^2 pixels of I can be simultaneously given a label. In Figure 1, $n=8$ and $m=21$.

Table 1

Pixel labels	Coordinates (row,column)
1	4, 7
2	4, 14
3	11, 4
4	11, 11
5	11, 18
6	18, 4
7	18, 11
8	18, 18

Neighbouring triplets			
(1,2,4),	(2,1,4),	(3,1,4),	(3,4,6),
(4,1,3),	(4,2,5),	(4,3,7),	(4,5,7),
(5,2,4),	(5,4,8),	(6,3,7),	(7,4,6),
(7,4,8),	(8,5,7)		

Edges of $G(S)$	Centres of Delaunay circles	$r_{\min}$	$r_{\max}$
(1,2)	(6.64, 10.50)	3.50	∞
(1,4)	(6.64, 10.50), (8.36, 7.50)	4.03	4.39
(2,4)	(6.64, 10.50), (8.36, 14.50)	3.81	4.39
(3,1)	(8.36, 7.50)	3.81	∞
(3,4)	(8.36, 7.50), (14.50, 7.50)	3.50	4.95
(4,5)	(8.36, 14.50), (14.50, 14.50)	3.50	4.95
(5,2)	(8.36, 14.50)	4.03	∞
(3,6)	(14.50, 7.50)	3.50	∞
(4,7)	(14.50, 7.50), (14.50, 14.50)	3.50	4.95
(5,8)	(14.50, 14.50)	3.50	∞
(6,7)	(14.50, 7.50)	3.50	∞
(7,8)	(14.50, 14.50)	3.50	∞

Computation of edges of $G(S)$ *and* $r_{\min}$ *and* $r_{\max}$

Definition 4.1. A circle that passes through at least three pixels of S but does not contain any pixel of S in its interior, is called a *Delaunay circle*.

For computing $r_{\min}/r_{\max}$ values of an edge of $G(S)$ we will use the following result.

Proposition 4.1. *For every edge P_iP_j of $G(S)$, there will be exactly one (for a convex hull edge) or exactly two (for an interior edge) Delaunay circles passing through P_i and P_j.*

For every pixel $p=(r,c)$ in I, its 4-neighbourhood is considered. Suppose $M(r,c)=i$. If there are two pixels $p_1=(r_1,c_1)$ and $p_2=(r_2,c_2)$ in the neighbourhood such that p_1 and p_2 are 8-neighbours of each other and $M(r_1,c_1)=j$ and $M(r_2,c_2)=k$ with $i\neq j\neq k$, then (P_i,P_j,P_k) is called a *neighbouring triplet*. Note that for every edge of $G(S)$, there is a neighbouring triplet containing its two vertices. Thus to find all edges of $G(S)$, it is sufficient to find all neighbouring triplets. Each pixel of I is examined to see if it gives rise to a neighbouring

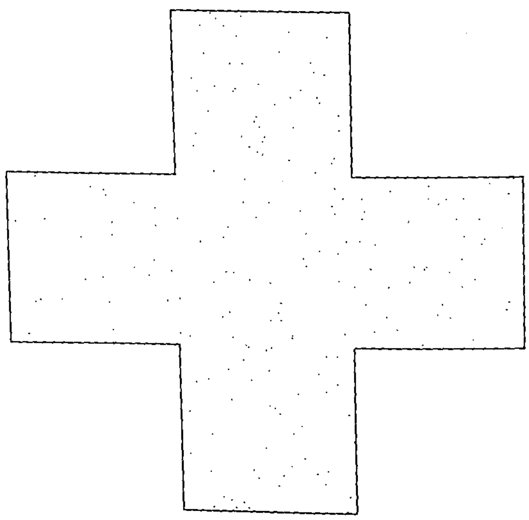

Figure 4. A '+' pattern is shown by dark lines. The dots are a random set S of 200 pixels drawn from the pattern.

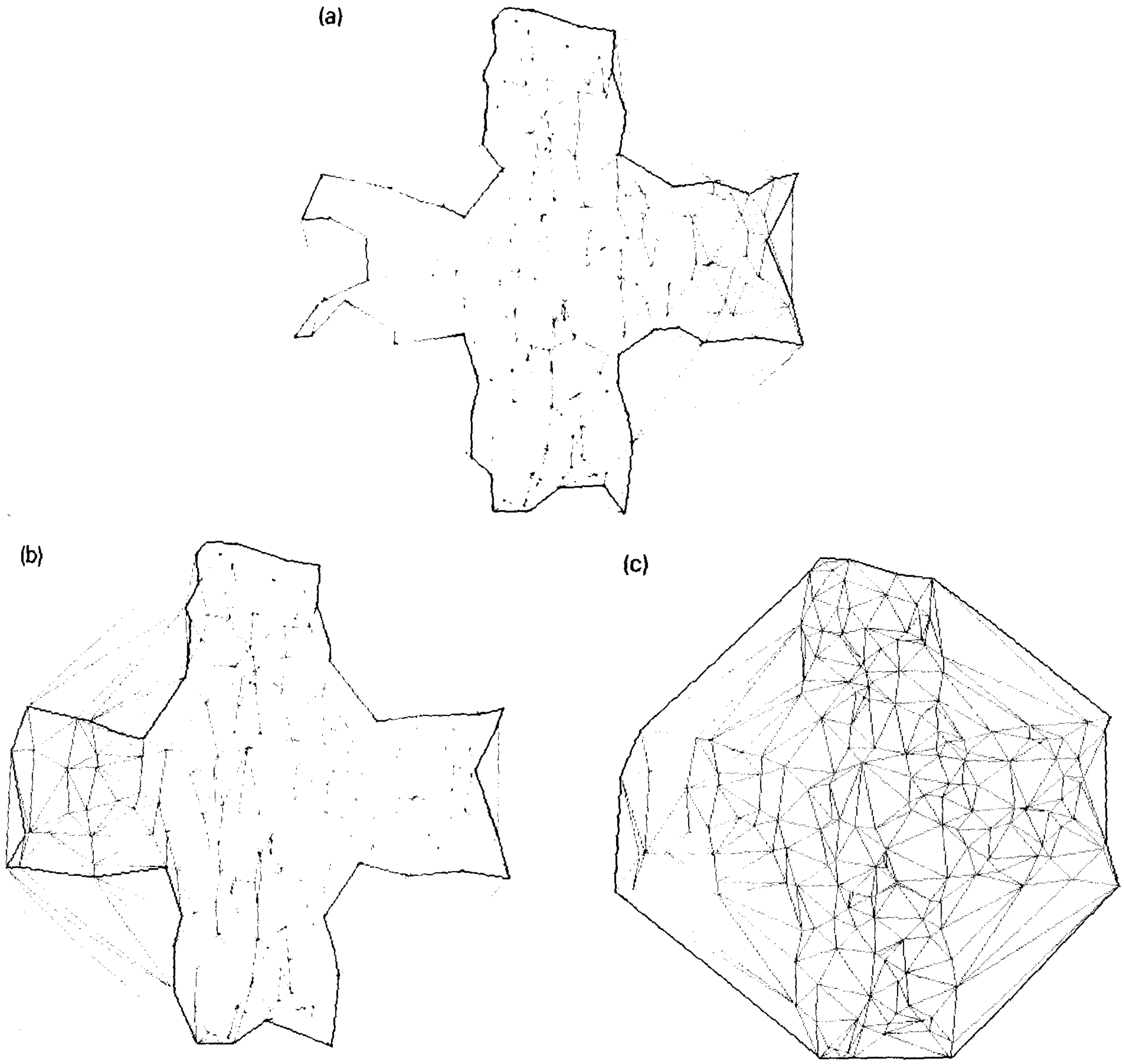

Figure 5. The α-shape of S is indicated by dark lines and the rest of the Delaunay triangulation by light lines. (a), (b) and (c) show the α-shapes for $r=29$, 35 and 189 respectively.

triplet. For such a triplet, P_i and P_j (as well as P_i and P_k) are Voronoi neighbours. Thus, P_iP_j and P_iP_k are listed as two edges of $G(S)$. Note that the circle passing through P_i, P_j and P_k is a Delaunay circle. The centre of this circle is computed and stored alongwith each of the two edges P_iP_j and P_iP_k. From Proposition 4.1 it is clear that either one or two such centres will be stored with each edge of $G(S)$.

For an edge P_iP_j of $G(S)$, its $r_{\min}$ and $r_{\max}$ values are computed in the following way. If there are two centres c_1 and c_2 stored against the edge, then $r_{min}=$ half of the length of the edge P_iP_j and $r_{\max}=\max(d_1,d_2)$ where $d_t=$ Euclidean distance between P_i and c_t for $t=1,2$. If there is only one centre c associated with the edge P_iP_j, then $r_{\max}=\infty$. If c and P_k fall on the same side of P_iP_j, $r_{\min}=$ half of the length of P_iP_j. Otherwise, $r_{\min}=$ Euclidean distance between P_i and c.

Given the DDT(S), the time complexity to find the edges of $G(S)$ and their $r_{\min}$ and $r_{\max}$ values is of the order $O(m^2)$. The number of the edges of

$G(S)$ are of the order of $O(n)$ (Preparata and Shamos (1985)).

Computation of $A_\alpha(S)$

Above we have found all edges of $G(S)$ and the $r_{\min}$ and $r_{\max}$ values for each of them. For any fixed value of α, let $r=-1/\alpha$. To get the edges of $A_\alpha(S)$, we find only those edges e of $G(S)$ such that $r_{\min} \leqslant r \leqslant r_{\max}$. Given $G(S)$, the time needed to find A_α from it is of the order of $O(n)$.

5. Results and conclusions

The output after various computational steps is given in Table 1 for the set of eight pixels shown in Figure 1. For a sufficiently large r, say 5.0, there are seven α-shape edges which form the convex hull of the eight pixels. We next take a cross ('+') pattern to demonstrate how our algorithm works. A random sample S (with uniform distribution) of 200 pixels are drawn from a digital cross pattern (Figure 4). In Figure 5 the α-shapes of S are shown for $r=-1/\alpha=29$, 35 and 189. The choice of an optimal α is still an open problem (Toussaint (1988)). We are currently working on this problem with the assumption of a uniform distribution.

Among the steps needed to compute A_α from S, the most time consuming step is to find DDT(S) from S. However, a parallel implementation of this computation is possible. In fact, a SIMD machine will be quite suitable for the purpose. Such a machine will also be appropriate for computing $G(S)$ from DDT(S).

Earlier Toriwaki and Yokoi (1988) presented an algorithm to compute Voronoi diagrams on a digital plane using the L_1 metric. But we have used the L_2 metric for the computation of Voronoi diagrams.

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