Optimum circular fit to weighted data in multi-dimensional space

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Abstract

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The problem of optimum circular fit to weighted data points in multi-dimensional space is addressed in this paper. It is shown that the modified weighted sum of square error function can be optimized to obtain expressions for the parameters of the circular fit so that expensive iterative numerical optimization algorithms are not needed. Also, it is shown that the derived parameters are independent of uniform scaling of the data weights. Experimental results indicate that reliable circular fit can be obtained even if the data comes from a small arc rather than the complete circle. Sensitivity to noise is also studied.

Keywords. Multi-dimensional circular fit, curve fitting, shape analysis, pattern recognition, computer vision.

1. Introduction

The problem of circular shape (circle, sphere or hypersphere) detection is often encountered in Pattern Recognition and Image Processing applications. There are two stages of the problem, namely (a) localization of points in the space that approximately belong to a circular shape, and (b) optimum parameter estimation from the data obtained by localization. The localization problem can be tackled by the generalized Hough transform (Illingworth and Kittler (1987)) or some kind of cluster analysis. We are concerned here with optimum parameter estimation only.

Given a set of points in 2-D space, a modified mean square error function can be optimized to find closed form expressions for the coordinates of the center and the radius of the fitting circle (Thomas and Chan (1989)). When the optimum fit is required on 2-D or 3-D *simply connected objects*, a simpler approach due to Chaudhuri (1990) may be used. However, that approach cannot be used for scattered data points.

While normally we do not encounter physical *objects* beyond 3-D, we may encounter data points in three or higher dimensions. One example is the patterns represented in multi-dimensional feature space, as frequently experienced by the pattern recognition community. More generally, the data may be weighted because of measurement imprecision and quantization as well as to give unequal importance to all data. For example, if we choose a gradient image where no thresholding is done, the data should be weighted according to the gradient magnitudes so that points with high gradient get more weight.

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In this note we show that a modified weighted sum of error function can be optimized to obtain a closed form expression of the parameters of the circular fit in any multi-dimensional space. Also, it is shown that the derived parameters are not affected if the weights are multiplied by a constant scaling factor. This method can be seen as a generalization of Thomas and Chan (1989). Experimental results are presented to show the efficacy of the technique.

2. Optimum circular fit

Consider N data points $x_1, x_2, \ldots, x_N$ in $\mathbb{R}^n$ for any positive integer $n \geqslant 2$ where

$$x_i = (x_{i1}, x_{i2}, \ldots, x_{ij}, \ldots, x_{in})^t.$$

We want to make the circular fit so that the center

$$c = (c_1, c_2, \ldots, c_j, \ldots, c_n)^t$$

and radius r_0 optimize the modified sum of square error function

$$J = \sum_{i=1}^{N} m_i \left[\sum_{j=1}^{n} (x_{ij} - c_j)^2 - r_0^2 \right]^2 \tag{1}$$

where m_i is the weight corresponding to the ith datum. Partial derivatives of J with respect to c_k and r_0 equated to zero lead to

$$\sum_{i=1}^{N} m_i \left[\sum_{j=1}^{n} (x_{ij} - c_j)^2 - r_0^2 \right] (x_{ik} - c_k) = 0 \tag{2}$$

where $k = 1, 2, \ldots, n$, and

$$\sum_{i=1}^{N} m_i \left[\sum_{j=1}^{n} (x_{ij} - c_j)^2 - r_0^2 \right] = 0. \tag{3}$$

From equation (3)

$$r_0^2 = \frac{1}{M} \sum_{i=1}^{N} m_i \left[\sum_{j=1}^{n} (x_{ij} - c_j)^2 \right] \tag{4}$$

where $M = \sum_{i=1}^{N} m_i$. For each value of k, equation (2) can be expanded as

$$\sum_{i=1}^{N} m_i x_{ik} \left[\sum_{j=1}^{n} (x_{ij} - c_j)^2 \right] - \sum_{i=1}^{N} m_i r_0^2 x_{ik} - c_k \sum_{i=1}^{N} m_i \left[\sum_{j=1}^{n} (x_{ij} - c_j)^2 - r_0^2 \right] = 0. \tag{5}$$

By equation (3), the third term of equation (5) is zero. Then equation (5) becomes

$$\sum_{i=1}^{N} m_i x_{ik} \left[\sum_{j=1}^{n} (x_{ij} - c_j)^2 \right] - r_0^2 \sum_{i=1}^{N} m_i x_{ik} = 0. \tag{6}$$

Using equation (4) again in the second term of equation (6), we have

$$\sum_{i=1}^{N} m_i x_{ik} \left[\sum_{j=1}^{n} (x_{ij} - c_j)^2 \right] - \frac{1}{M} \left(\sum_{i=1}^{N} m_i x_{ik} \right) \sum_{i=1}^{N} m_i \left[\sum_{j=1}^{n} (x_{ij} - c_j)^2 \right] = 0,$$

i.e.,

$$\sum_{i=1}^{N} m_i \left[\sum_{j=1}^{n} (x_{ij} - c_j)^2 (x_{ik} - \bar{x}_k) \right] = 0 \tag{7}$$

where

$$\bar{x}_k = \frac{1}{M} \sum_{i=1}^{N} m_i x_{ik}, \tag{8}$$

i.e.,

$$\sum_{i=1}^{N} m_i \left[(x_{ik} - \bar{x}_k) \sum_{j=1}^{n} x_{ij}^2 - 2(x_{ik} - \bar{x}_k) \sum_{j=1}^{n} c_j x_{ij} + (x_{ik} - \bar{x}_k) \sum_{j=1}^{n} c_j^2 \right] = 0. \tag{9}$$

The third term in equation (9) is

$$\sum_{i=1}^{N} m_i x_{ik} \sum_{j=1}^{n} c_j^2 - \sum_{i=1}^{N} m_i \bar{x}_k \sum_{j=1}^{n} c_j^2 = 0$$

since by equation (8)

$$\sum_{i=1}^{N} m_i x_{ik} = \sum_{i=1}^{N} m_i \bar{x}_k = M \bar{x}_k.$$

Then equation (9) becomes

$$\sum_{i=1}^{N} m_i \left[(x_{ik} - \bar{x}_k) \sum_{j=1}^{n} x_{ij}^2 - 2(x_{ik} - \bar{x}_k) \sum_{j=1}^{n} (c_j x_{ij}) \right] = 0 \tag{10}$$

which is a linear equation in c_j's. For $k = 1, 2, \ldots$ we get n equations that can be solved by the meth[od] of determinants. Once the c_j's are solved, r_0 [can] be found using equation (4). Note that each [pa]rameter can be expressed in closed form. For example in 2-D, the center (c_1, c_2) and radius r_0 are given by

$$c_1 = \frac{B'C - BC'}{AB' - A'B}, \qquad c_2 = \frac{A'C - AC'}{A'B - AB'},$$
$$r_0 = \frac{\sum_{i=1}^{N} m_i[(x_{i1}-c_1)^2 + (x_{i2}-c_2)^2]}{\sum_{i=1}^{N} m_i} \tag{11}$$

where

$$\sum_{i=1}^{N} m_i(x_{i1} - \bar{x}_1)x_{i1} = A,$$
$$\sum_{i=1}^{N} m_i(x_{i2} - \bar{x}_2)x_{i1} = A',$$
$$\sum_{i=1}^{N} m_i(x_{i1} - \bar{x}_1)x_{i2} = B,$$
$$\sum_{i=1}^{N} m_i(x_{i2} - \bar{x}_2)x_{i2} = B', \tag{12}$$
$$\sum_{i=1}^{N} m_i(x_{i1} - \bar{x}_1)(x_{i1}^2 + x_{i2}^2) = 2C,$$
$$\sum_{i=1}^{N} m_i(x_{i2} - \bar{x}_2)(x_{i1}^2 + x_{i2}^2) = 2C',$$
$$\bar{x}_j = \frac{\sum_{i=1}^{N} m_i x_{ij}}{\sum_{i=1}^{N} m_i}, \quad j = 1, 2.$$

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3. Experimental results and discussion

To verify the effectiveness of the approach, analog data with equal weight generated from ideal circles at various radii were subjected to the parameter estimation. It was seen that the estimation error is of the same order as the precision of the machine. Similarly, data from an ideal straight line resulted in an abnormally large radius, even if the number of data points is very small. Thus, the method can distinguish between linear and circular data. When the space is discrete, as in a digital image, the digital line should be about 15 pixel long to detect the line.

Next, it was tested if parameter estimation is possible when the data comes from an arc rather than from a complete circle. Figure 1(a) shows

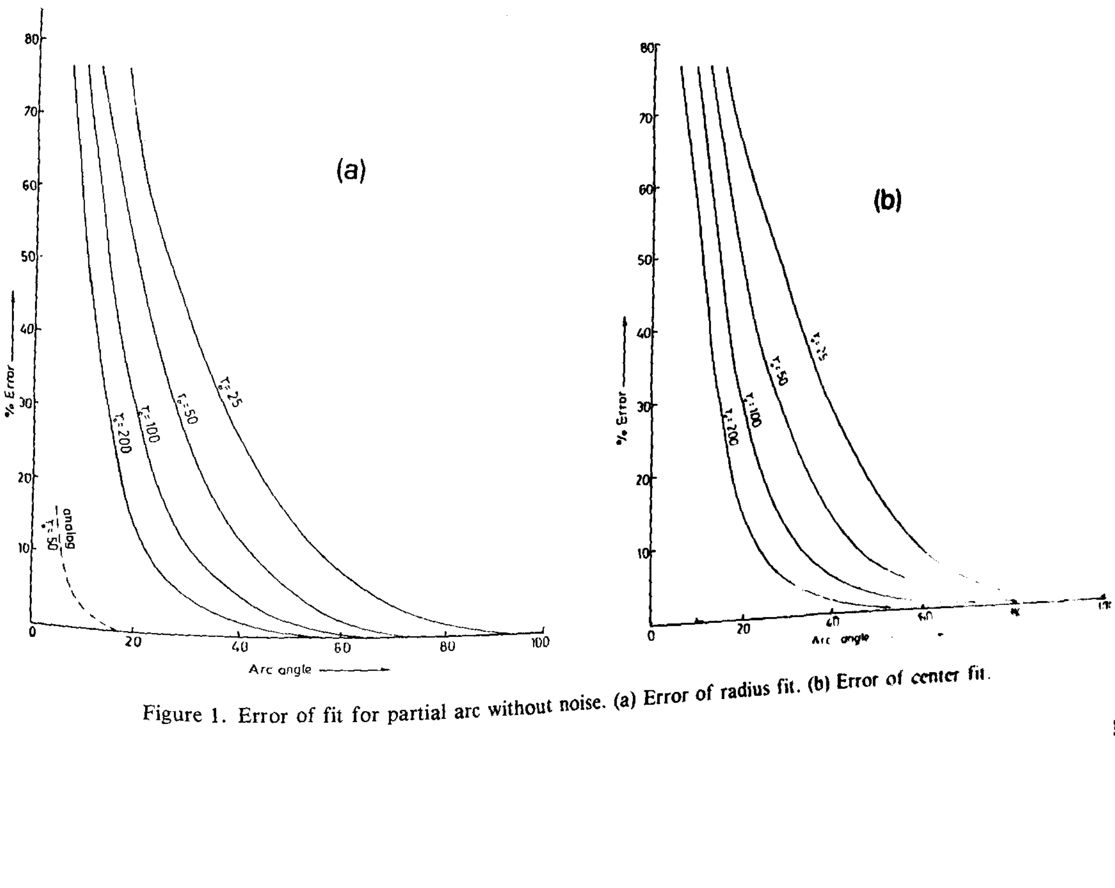

Figure 1. Error of fit for partial arc without noise. (a) Error of radius fit. (b) Error of center fit.

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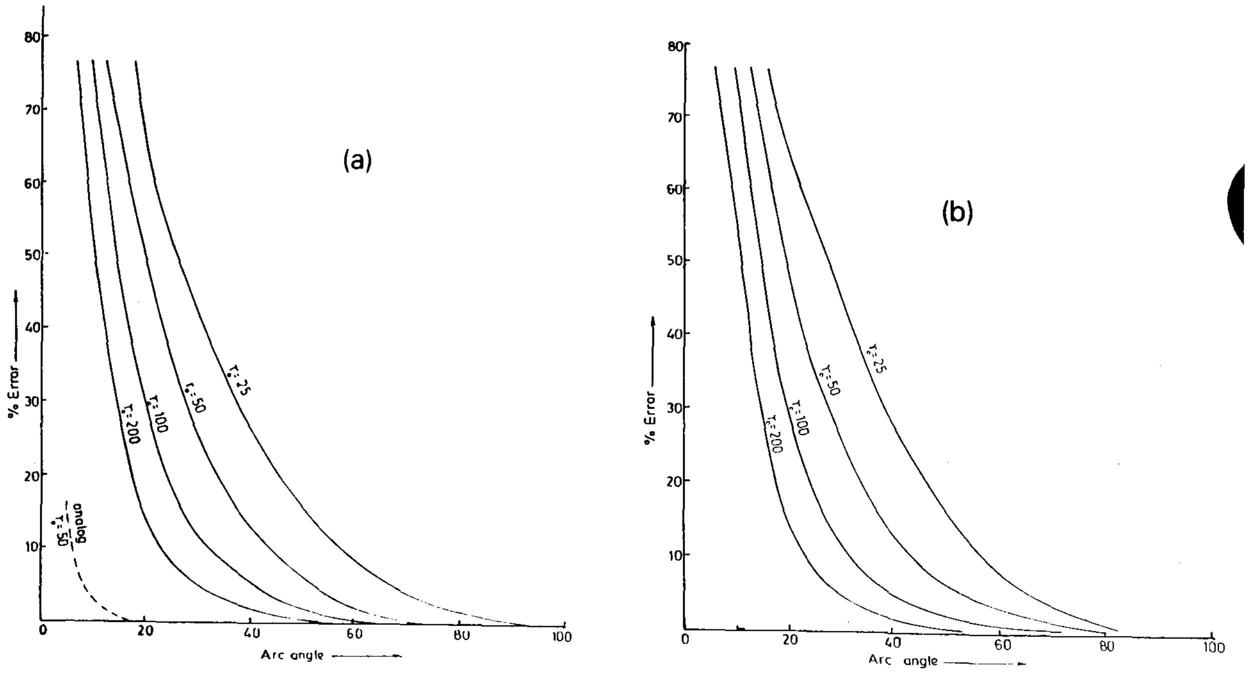

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some results where the error in estimating the radius is plotted against the angle in degrees that the endpoints of the arc subtend at the center. If r_0 and $\hat{r}_0$ are the true and estimated radius, respectively, then the error is given by

$$\frac{|r_0 - \hat{r}_0|}{r_0} \times 100\%.$$

If the data comes from an ideal *analog* circle then it is seen that an arc of 20° is enough to get the parameters with less than 1% error. The results are not sensitive to the radius value of the circle.

To get a reliable estimate of the parameters in case of a digital circle, however, the data should come from a bigger arc. It is seen that an arc angle of about 90° is necessary for the purpose. For a desired error rate the arc should be bigger if the circle radius is smaller.

The error in estimating the center is also considered. If O and $\hat{O}$ are the true and estimated center, respectively and if d is the Euclidean distance between O and $\hat{O}$, then the error is given by $(d/r_0) \times 100\%$. From Figure 1(b) it is seen that the error is of the same order as that of the radius estimation. For this reason, the error in estimating the center has not been presented in the study described below.

Next we considered the effect of noise on the estimation process. Zero-mean Gaussian pseudorandom noise has been added to the arc of fixed radius both in the analog and the digital case and the results are shown in Figure 2(a)–(b) for different standard deviations σ of noise. As expected, the noise has a more severe effect on a digital arc. However, when the arc length is comparable to the noise standard deviation, the estimated radius is very small. To the algorithm, it seems that the data come from a rectangle and the algorithm tries to make the best circular fit to the rectangle. To get a reasonably good estimate, the arc length should be of the order of 10 times more than the noise standard deviation.

To test the efficiency of the approach for weighted data, a digital image of a round object was subjected to a Sobel operator. The Sobel gradient image was subjected to the circular fit. The results are shown in Figure 3. The gradient image was partially erased to see how well the circular fit is obtained

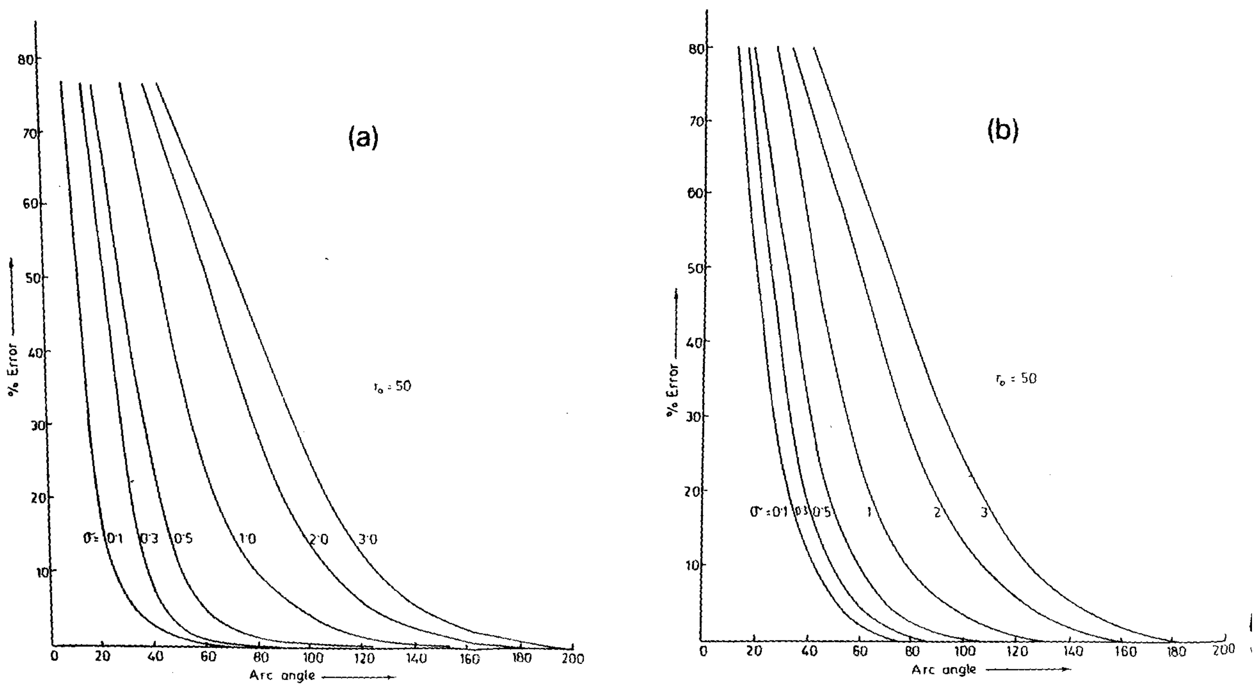

Figure 2. Error of fit for partial arc with noise. (a) Effect of noise on analog data. (b) Effect of noise on digital data.

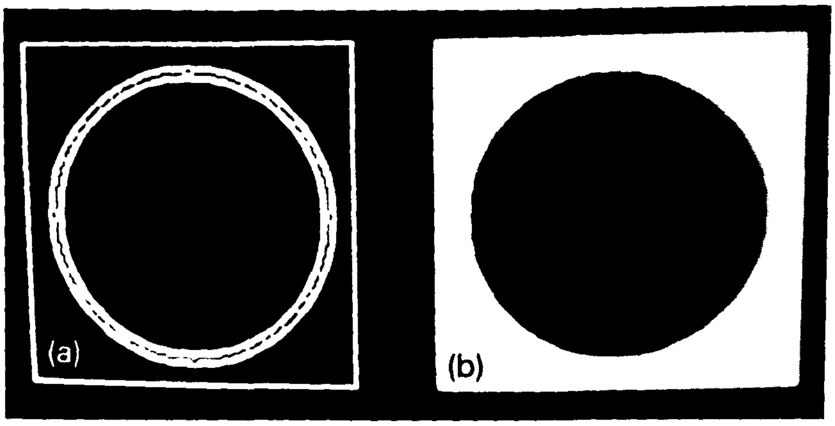

Figure 3. Circular fit on real object. (a) The object. (b) Circular fit on gradient image of the object.

from only a part. In this weighted data space also, the arc should extend about 90° to get a reasonably good fit. See Figure 4.

It may be interesting to compare the iterative techniques (Landau (1987), Bookstein (1979)) with the present method. The method due to Landau (1987) tries to minimize the error of the form $\sum (r_i - r_0)^2$ while the method due to Bookstein (1979) tries to minimize the error of the form $\sum (r_i^2 - r_0^2)^2$. Thus, the error function in our method is a generalization of that in Bookstein (1979). However, the present method, which gives a closed-

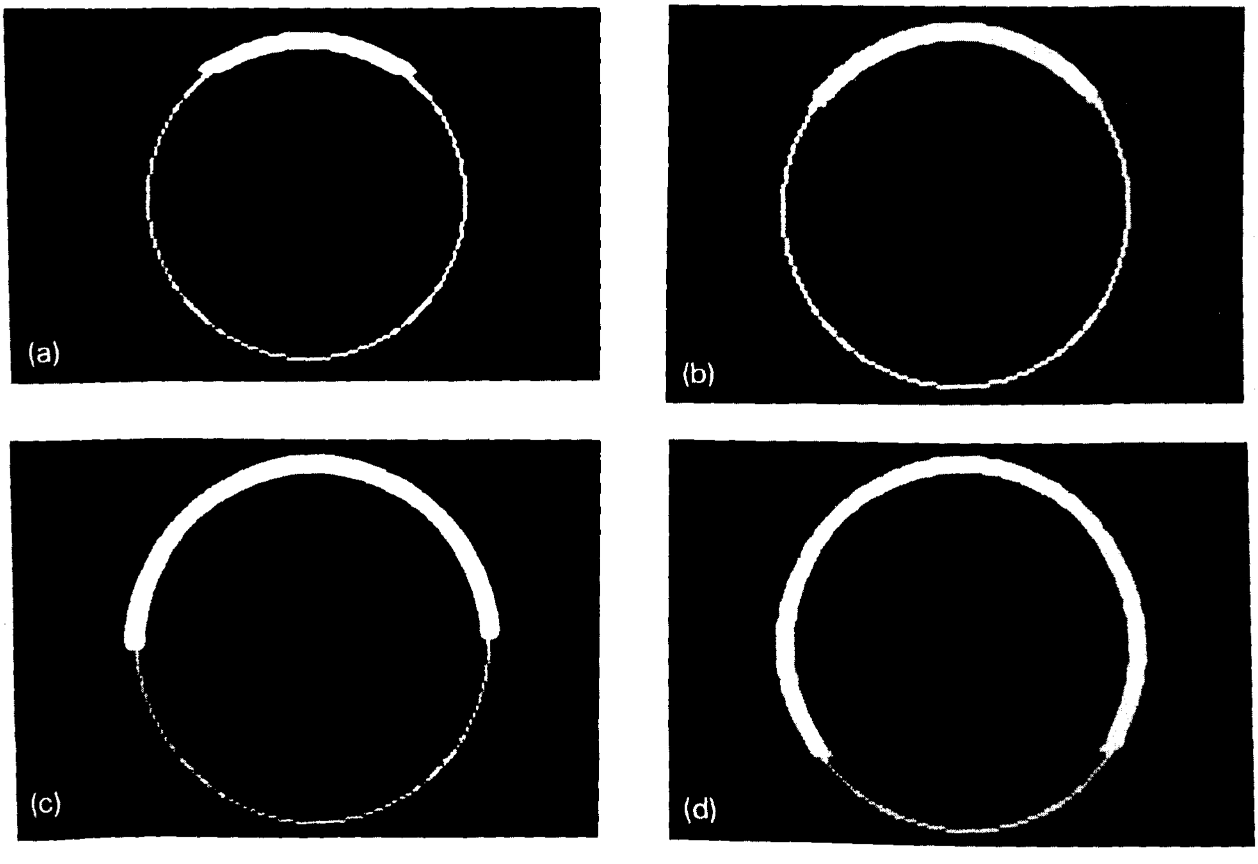

Figure 4. Circular fit on partial gradient image of the object with radius 50. (a) Arc angle = 65°, estimated radius = 45.11. (b) Arc angle = 90°, estimated radius = 49.66. (c) Arc angle = 180°, estimated radius = 50.11. (d) Arc angle = 270°, estimated radius = 50.52.

form expression for the center and radius, can be computed in one single iteration. On the other hand, the number of iterations in iterative techniques like those of Landau (1987) and Bookstein (1979) should be large to get accurate results. Another advantage of the present method over the iterative method is that the closed-form expressions of the parameters can be substituted in another expression and do further algebra, if necessary.

In the present formulation, the weight m_i does not depend on the parameters of the circular fit. However, practical situations may arise where the weight can be made dependent on, say, the position of the center of the circle. Consider, for example, the gradient image. If a pixel P with a high gradient lies on a circle, then its gradient direction would be normal to the circumference. (Conversely, the edge direction would be tangential.) Let O be the center of the circle. If OP makes an angle θ_i with the gradient direction at P and if s_i is the gradient strength, then we may consider that $m_i = s_i \cos \theta_i$. It may be interesting to examine if the objective function can be suitably modified to get separable solutions for an optimum circular fit in this situation.

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References

Bookstein, F.L. (1979). Fitting conic sections to scattered data. *Computer Vision, Graphics, and Image Processing* 9, 56–71.

Chaudhuri, B.B. (1990). Optimal circular fit to objects in two and three dimensions. *Pattern Recognition Lett.* 11, 571–574.

Illingworth, J. and J. Kittler (1987). The adaptive Hough transform. *IEEE Trans. Pattern Anal. Machine Intell.* 9, 690–698.

Landau, U.M. (1987). Estimation of circular arc center and its radius. *Computer Vision, Graphics, and Image Processing* 38, 317–326.

Thomas, S.M. and Y.T. Chan (1989). A simple approach for the estimation of circular arc center and its radius. *Computer Vision, Graphics, and Image Processing* 45, 362–370.