INDIAN STATISTICAL INSTITUTE

Vectors and Matrices I : B. Stat 1st year: Mid Semester Examination: 2012-13
September 4, 2012.

Maximum Marks 40 Maximum Time 2 hrs.

Answer all questions. Each question has 7 marks.

(1) Prove or disprove: Let A, B be $n \times n$ real matrices. If $AB = I$ then $BA = I$, where I is $n \times n$ identity matrix.

(2) Prove or disprove: Let V_1 and V_2 be two finite dimensional vector spaces over $\mathbb{R}$. Let T be a linear map from V_1 to V_2. Suppose for $x_1, x_2 \ldots, x_k \in V_1$, the set $\{Tx_1, Tx_2, \ldots, Tx_n\}$ is linearly independent in V_2. Then $\{x_1, x_2 \ldots, x_k\}$ is linearly independent in V_1.

(3) Prove or disprove: Let V be a vector space over $\mathbb{R}$ of dimension n and W be a subspace of V. Suppose for a given basis $\{x_1, x_2 \ldots, x_n\}$ of V, only $\{x_1, x_2 \ldots, x_k\}$ is contained in W for some $k < n$. Then W is generated by $\{x_1, x_2 \ldots, x_k\}$.

(4) Prove or disprove: Let $\{(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)\}$ be a set of linearly independent vectors in $\mathbb{R}^3$ (as a vector space over $\mathbb{R}$). Then the set $\{(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\}$ is also a linearly independent set in $\mathbb{R}^3$.

(5) Let $W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = x_2, x_3 = x_4 = x_5\}$ be the subspace of $\mathbb{R}^5$.
Prove that there does not exist a linear map T from $\mathbb{R}^5$ to $\mathbb{R}^2$, such that its null space $N(T) = W$.

(6) Let V be a finite dimensional vector space and W be a subspace of V. Let $\dim V = n$ and $\dim W = m$ with $n > m$. Find the dimension of the quotient space V/W.

INDIAN STATISTICAL INSTITUTE

Analysis I : B. Stat 1st year: Mid Semester Examination: 2012-13.
Date: September 7, 2012

Maximum Marks 40 Maximum Time 2 hrs.

Answer all questions.

(1) Find supremum and infimum of the set $S = \{\sqrt{n^2+2} - n \mid n \in \mathbb{N}\}$. [5]

(2) If $a_n \geq 0$, $n \in N$ then prove that convergence of $\sum_{n=1}^{\infty} a_n$ implies convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$. [5]

(3) Let the recursive sequences $\{x_n\}$ and $\{y_n\}$ be given by setting

$$0 < y_1 < x_1, \quad x_{n+1} = \frac{x_n + y_n}{2} \quad \text{and} \quad y_{n+1} = \frac{2x_n y_n}{x_n + y_n}.$$

Prove that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ and evaluate the limit. [7]

(4) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms and $r_n = \sum_{k=n+1}^{\infty} a_k$, $n \in \mathbb{N}$. Prove that

$$\frac{a_n}{\sqrt{r_{n-1}}} < 2(\sqrt{r_{n-1}} - \sqrt{r_n}).$$

Hence prove that $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_{n-1}}}$ converges. [7]

(5) If $\sum_{n=1}^{\infty} a_n$ converges and b_n is a bounded monotonic sequence of real numbers then prove that $\sum_{n=1}^{\infty} a_n b_n$ converges. [7]

(6) Prove that $\sum_{n=1}^{\infty} \frac{1}{(n!)^2}$ is an irrational number. [7]

(7) If A denotes a nonempty subset of $\mathbb{R}$ then prove that the function

$$d_A(x) = \text{infimum}\{|x - a| \mid a \in A\}$$

is a continuous function on $\mathbb{R}$. If $A = \mathbb{Q}$, the set of all rational numbers, then describe the set $S = \{x \in \mathbb{R} \mid d_A(x) = 0\}$. [7]

PROBABILITY THEORY I
B. STAT. IST YEAR SEMESTER 1
INDIAN STATISTICAL INSTITUTE

Mid-semestral Examination
Time: 2 Hours Full Marks: 35
Date: September 10, 2012

1. The event B is said to repel the event A if $P(A|B) < P(A)$ and is said to attract A if $P(A|B) > P(A)$. If B attracts A, then which of the following statement(s) is/are correct?
   (a) A attracts B.
   (b) B^c repels A.
   (c) A^c attracts B^c. [$2 \times 3 = 6$]

2. I have five coins, two of which are double-headed, one is double-tailed and the other two are fair coins. I shut my eyes, pick a coin and toss it. What is the probability that the lower face will be Head? On opening my eyes, if I see Head, what will be the probability that the lower face will be Head? If I toss the coin again, what is the probability that the lower face will be Head? [$3 \times 3 = 9$]

3. (a) In the Institute, there are n faculty members. A committee has to be formed using them. The committee may contain as many members as possible. (For example, all n may be part of the committee.) However, the committee must have a Chairman and a Convener, who must be distinct persons. Further, the committee may have a Deputy Chairman; if there is none, then the Chairman plays the role of the Deputy Chairman as well. Show that the number of such committees is $n^2(n-1)2^{n-3}$. [4]
   (b) Show that
   $$\sum_{k=0}^{n} \binom{n}{k} k(k-1)^2 = n^2(n-1)2^{n-3}.$$ [3]

4. For two events A and B, define their symmetric difference as the event
   $$A \triangle B = (A \setminus B) \cup (B \setminus A).$$
   Further define $d(A,B) = P(A \triangle B)$. Then, for any three events A, B and C, show that
   $$d(A,C) \leq d(A,B) + d(B,C).$$ [7]

5. A teacher has given 9 sets of practice problems to his students. Each set contains 10 problems, of which the set numbered i has i problems starred, for $i = 1, 2, \ldots, 9$. The teacher first selects a set, with the set numbered i getting selected with probability proportional to $10 - i$. Then he selects one question at random from the selected set for the examination. He informs the students that the selected question is a starred one. Then what is the probability that the selected question has come from the set numbered i? If the teacher allows the students to carry solutions of any one set to the exam, which set should the students carry along? [3+3=6]

Indian Statistical Institute
Statistical Methods I
B-I, Midsem

Date: 12.09.12 Duration: 2hrs.

Attempt all questions. The maximum you can score is 20. Justify all your steps. This is a closed book, closed notes examination. You may use your own calculator.

If copying is detected in the solution for any problem, all the students involved in the copying will get 0 for that problem. Also an additional penalty of 5 will be subtracted from the overall aggregate of each of these students.

1. The direction of wind blowing through a certain point is recorded. The data set consists of angles (in radians) measured in the counterclockwise direction from due north. Thus, wind blowing westwards is recorded as $\frac{\pi}{2} = 1.57$, while eastwards wind is recorded as $\frac{3\pi}{2} = 4.71$. The data set is

   0.37, 0.18, 6.08, 6.14, 0.24, 5.92.

   Suggest (with justification) suitable measures of central tendency and dispersion for such data. Also compute them for this data set.[5+5]

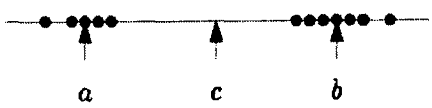
2. A data set $X_1, ..., X_n$ is known to consist of two clusters. We want to find three numbers:

   (a) a cut-off value c such that all X_i's to its left are in cluster 1, and the rest are in cluster 2,

   (b) a central tendency measure a for cluster 1,

   (c) a central tendency measure b for cluster 2.

   See the picture below.

   Suggest how you can obtain a, b, c to minimise

   $$\sum_1 (X_i - a)^2 + \sum_2 (X_i - b)^2,$$

   where $\sum_1$ is taken over all i's for which $X_i < c$, and $\sum_2$ is taken over the remaining i's. Are the optimum choices of a, b and c unique?[10]

3. Discuss each of the following arguements in light of concepts like control, confounding, blocking, randomisation, replication etc. Mention only the concepts that are relevant. Do not let your personal belief interfere in the solutions. You are required to clearly mention (with justification) if the conclusion follows logically from the given data. Assume the data mentioned to be correct.

   (a) During the last month I happened to see a solitary *shalikh* bird in the morning on exactly 5 days. On these days I had the following misfortunes: fell from my bicycle, got bad marks in an exam, lost an umbrella, had a fight with my best friend, had a sudden unaccountable fever. So I conclude that the so-called superstition that "seeing a solitary *shalikh* bird in the morning brings bad luck during the day" is actually true. I have also checked with 5 friends of mine who have the same belief, and they have also had similar experience. [3]

   (b) Average income of Indians is much less than that of Americans. In America I find most households very neat and clean, while in India most households are shabby. So I conclude that American households are neat and clean because Americans are rich. [3]

4. The parents of two IIT entrance candidates are arguing about the usefulness of the Brilliant Tutorial as follows.

   (a) Parent 1: 90% of the students who were admitted to IIT last year went to the Brilliant Tutorial. So Brilliant must be useful.

   (b) Parent 2: 99% of the students who went to Brilliant last year could not make it into IIT. So Brilliant is surely not useful.

   How would you, as a statistician, settle the arguement in an objective way? If you want to use extra data, then clearly state what data you need. Your suggestions and data requirements should be practicable. [4]

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination
B. Stat. I year: 2012–2013
C and Data Structures

Date: 14–09–2012 Marks: 75 Time: 3 Hours

The questions are of 75 marks. Answer any part of any question.
The maximum you can get is 60.

1. Write the output of each of the following statements.

   (a)
   ```
   printf("%d\n",2&&3);
   printf("%d",2&3);
   ```
   (b)
   ```
   int d = 0; while(d < 5);
   ```
   (c)
   ```
   int x=10, y=2, z=2;
   z=y=x++ + ++y*2;
   printf("%d",z);
   ```
   (d)
   ```
   void main() {
     int *p, n = 12;
     p = &n;
     printf("%d\n",p);
     printf("%d",*p++);
     printf("%d\n",p);
   }
   ```

[4x2 = 8]

2. Point out the errors in each of the following pieces of code and fix them.

   (a)
   ```
   #define GREETINGS = "Hello World!"
   int main() {
     printf("%s", GREETINGS);
   }
   ```
   (b)
   ```
   #include <stdio.h>;
   int main() {
     printf("Hello World!\n");
   }
   ```

[2x2 = 4]

3. (a) Write a program without using semicolons that prints the message “Hello World!” on the screen.
   (<u>Hint</u>: `printf` function returns the number of charaters it prints on the screen.)

   (b) Write statements to swap two integer variables a and b without using a temporary or third variable.

(c) Let n be an `int` variable. Write expressions using bitwise operators to test whether atleast 5 of the last 8 bits of `n` are equal to 1. Output 1 if this is the case and 0 otherwise. For what values of `n` will each of the two possible outputs occur. Provide examples.

[4+4+7 = 15]

4. Let the type `node` be defined as follows.

```
struct node {
  int data;
  struct node *next;
};
```

Write a function `struct node * reverselist(struct node * head)` that inputs a linked list pointed by `head`, reverses the list and returns a pointer to the resulting list. [10]

5. Write a *recursive* function `int binom(int n, int m)` that takes as input two non-negative integers n, m with $n \geq m$ and returns the binomial coefficient $\binom{n}{m}$. Make sure you write the terminating condition correctly. [8]

6. (a) What is the maximum number of divisions performed by Euclid's algorithm for computing the GCD of two given non-negative integers a and b? Justify your answer.

   (b) What are the maximum and minimum number of comparisons made by the insertion sort algorithm? Assume the input is a list of n elements. Justify your answer.

[10+8 = 18]

7. We call a list $a_0, a_1, \ldots, a_{n-1}$ a *rotated sorted list* if there is some k such that $a_k, \ldots, a_{n-1}, a_0, \ldots, a_{k-1}$ is a sorted list. Write an algorithm that takes a rotated sorted list as input and outputs k. [12]

INDIAN STATISTICAL INSTITUTE

Analysis I : B. Stat 1st year
First Semester Examination: 2012-13
19 November , 2012.

Maximum Marks 60 | Maximum Time 3 hrs.

(1) Use Sandwich theorem to find the limit of the sequence $\{x_n\}$ where

$$x_n = \left(\sqrt{2} - 2^{\frac{1}{3}}\right)\left(\sqrt{2} - 2^{\frac{1}{5}}\right)\ldots\left(\sqrt{2} - 2^{\frac{1}{2n+1}}\right).$$

[4]

(2) Let A and B be closed subsets of $\mathbb{R}$ and A is bounded. Prove that the set

$$C = \{a + b \mid a \in A, b \in B\},$$

is closed in $\mathbb{R}$. [6]

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at the point $c \in \mathbb{R}$ with $f(c) = 0$. Prove that $g(x) = |f(x)|$ is differentiable at c if and only if $f'(c) = 0$. [6]

(4) If $f : [0, 1] \to [0, 1]$ is a continuous function then prove that there exists $x_0 \in [0, 1]$ such that $f(x_0) = 1 - x_0$. [6]

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(f(x)) = -x$ for all $x \in \mathbb{R}$.
(a) Prove that f is injective. [2]
(b) Prove that f cannot be continuous. [4]

(6) Let $a_0, a_1, \ldots, a_n$ be real numbers such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0.$$

prove that the polynomial $p(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ has a real root. [6]

(7) Let f be a continuous function on $[a, b]$. If f is twice differentiable on (a, b) and $|f''(x)| \leq M$ for all $x \in (a, b)$ then prove that f is uniformly continuous on (a, b). [6]

(8) Let f be a twice differentiable function on $[a, b]$ with $f'(a) = f'(b) = 0$. Use Taylor's theorem to prove that there exists $c \in (a, b)$ such that the following inequality holds: $|f''(c)| \geq \frac{4}{(b-a)^2}|f(b) - f(a)|$. [8]

(9) Let $a > b$ be positive real numbers. Determine all possible values of a and b such that the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^a - n^b}$$

converges. Also determine all possible values of a and b for which the above series converges absolutely. [10]

(10) Prove that the function $f(x) = \frac{\sin x}{x}$ is decreasing and concave in $(0, \frac{\pi}{2})$. [10]

Indian Statistical Institute
Statistical Methods I
B-I, First Semestral Examination

Date: 21·11·12 Duration: 2hrs.

Attempt all questions. The maximum you can score is 50. Justify all your steps. This is a closed book, closed notes examination. You may use your own calculator.

If copying is detected in the solution for any problem, all the students involved in the copying will get 0 for that problem. Also an additional penalty of 10 will be subtracted from the overall aggregate of each of these students.

1. You are given a bivariate data set
$$(X_1, Y_1), ..., (X_n, Y_n).$$
Let
$$y = a + bx$$
be the OLS regression line of Y on X. Then the OLS regression line of Y on X passing through the origin is
$$y = bx.$$
Is this true of false? Justify you answer. [10]

2. Consider the following inconsistent system
$$\underbrace{\begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 3 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}}_{\mathbf{y}}$$
Find (with justification) *all* values for x, y such that $\|A\mathbf{x} - \mathbf{y}\|$ is the minimum possible. [10]

3. A vaccination against bird flu has been proposed. To assess its effectiveness a poultry farm used 20 hens in a cage. The statistician reached into the cage with his hand, and brought out the first 10 he managed to catch. These 10 were vaccinated. After a month the numbers of infected birds are counted both in the vaccinated and the non-vaccinated groups to assess the efficacy of the vaccination. Is this an observational study or a statistical experiment? Why? Discuss this procedure in light of the following desirable qualities: randomisation, control and blocking. [1+9]

4. Let $x_1, ..., x_{101}$ be any 101 real numbers. Let $f(x_1, ..., x_{101})$ denote their median. Find the minimum number k such that for every subset $\{i_1, ..., i_k\} \subseteq \{1, ..., 101\}$ of size k we have
$$\lim_{\substack{x_{i_1} \to \infty \\ \cdots \\ x_{i_k} \to \infty}} f(x_1, ..., x_{101}) = \infty.$$

What is the answer if $f(x_1, ..., x_{101})$ denotes the mean? [7+3]

5. Fit (using least squares method) an equation of the form

$$y = ax + \frac{b}{x}$$

to the data

x	7.0	3.4	1.4	$0.8+\frac{r}{10}$	8.1	2.1
y	0.5	5.1	3.2	6.2	7.0	3.6

Here r is your 2-digit roll number. Derive your formula and show your calculation. [5+10]

INDIAN STATISTICAL INSTITUTE
Semestral Examination
B. Stat. I year: 2012–2013

C and Data Structures

Date: 23-11-2012 | Marks: 120 | Time: 3 Hours

The questions are of 120 marks. Answer any part of any question.
The maximum you can get is 100.

1. Write a function to read a positive integer and print out its digits in reverse order. [10]

2. Consider a linked list where the elements are sorted. Given an element x, write a function to find out if it is present in the list and if so, delete it. [15]

3. Consider a railway reservation system for a single train, single journey, single class with no intermediate stops. Design an appropriate data structure to store information. Write a function to cancel a wait listed ticket. [20]

4. Suppose that the names 'Pamban, Vidyasagar, Vashi, Ellis, Coronation, Godavari, Rabindra' were to be stored in hashed files. Suggest a single-digit hash function on these files so that no two files map to the same digit. [15]

5. Consider a linked list with nodes $a_1, a_2, \ldots, a_n$ with a_i containing the link to a_{i+1} for $1 \leq i \leq n-1$. We say that the linked list has a loop if $\exists \ell \in \{1, 2, \ldots, n\}$ such that a_n points to a_ℓ. The *length* of the loop is defined to be $n - \ell + 1$. Provide algorithms for the following.

   (a) Detect whether or not the list has a loop.

   (b) If a loop exists, then find the length of the loop.

   Algorithms that are efficient both in terms of time and space consumed will be given more credit. [12+8 = 20]

6. Show that quicksort, on an input list of n elements, runs in time approximately $n \ln n$ on the average. Assume that the first element in the list is always chosen as the pivot. [12]

7. Suppose we have arrays `PreOrder[n]`, `InOrder[n]` and `PostOrder[n]` that give the preorder, inorder and postorder positions, repectively of each node `n` of a tree. Describe an algorithm that tells whether node `i` is the ancestor of node `j` for any pair of nodes `i` and `j`. [10]

8. A k-ary tree is a tree with every node having at most k children. A k-ary tree of height h is *complete* if it has the maximum possible number of nodes.

   (a) What is the number of nodes in a complete k-ary tree of height h? Justify your answer.

(b) The *level-order listing* of the nodes of a tree first lists the root, then all nodes at level 1, then all nodes at level 2 and so on. Nodes at the same level are listed in left-to-right order. Suppose that the nodes of a complete k-ary tree are stored in an array according to the level-order listing. Describe an algorithm that takes as input a node of the tree and outputs the index of its parent in the array.

[8+10]

First Semester Examinations (2012-13)
B.Stat – 1 yr.
Remedial English
100 marks
$1\frac{1}{2}$ hours

Date: 23.11.12

1) Write an essay *on any one* of the following topics. Five paragraphs are expected.
   a) Different forms of transportation
   b) An eventful day
   c) Autobiography of a street dog

(60 marks)

2) Fill in the blanks with appropriate prepositions:
I was travelling ________ Kolkata _______ Mumbai. The train ______which I was journeying suddenly stopped _______ Kharagpur. The gentleman _______ a white suit told the attendant that he would prefer coffee _____ tea.

______enquiry I found that he was working ______ a big industrial house. He started conversing ______ the state of our country which ended ______ ______ disagreement.

When he suggested meeting ______ lunch, I declined the offer though I know he was a man ______ rare talents and different ______ his brother whom I could connect.

He was true _______his organization and abstained ______liquor but slightly devoid _____ sense. "I am obliged _____ you _____ your kindness", said I since I found our discussion hardly relevant ______ the subject.

(20 marks)

3) Fill in the blanks with appropriate words:
_____ Tagore _____ him _____ _____ some order _____ the vast _____ of newspaper clippings _____ _____ been collected -------- his foreign --------, and in to the _____ files of correspondence, he realized _____ here _____ _____ virgin field for _____ original research. During the course _____ ______ months of hard ____ he _____ these newspaper _____ and letters.

(20 marks)

PROBABILITY THEORY I
B. STAT. IST YEAR SEMESTER 1
INDIAN STATISTICAL INSTITUTE

Semestral Examination
Time: 3 Hours Full Marks: 50
Date: November 26, 2012

1. For a collection of events $A_1, \ldots, A_n$, show that
$$P(A_1 \cdots A_n) \geq P(A_1) + \cdots + P(A_n) - (n-1).$$ [3]

2. Consider the sample space $\Omega = \{1, \ldots, p\}$, where p is prime. For any $A \subset \Omega$, define $P(A) = |A|/p$, where $|A|$ is the cardinality of the set A.
   (a) Check that P defines a probability.
   (b) If the events A, B are independent, show that at least one of the events A and B must either be $\emptyset$ or Ω. [3+4=7]

3. For three random variables X, Y and Z, show that
$$\mathrm{cov}(X, Y) = \mathrm{E}[\mathrm{cov}(X, Y|Z)] + \mathrm{cov}(\mathrm{E}[X|Z], \mathrm{E}[Y|Z]).$$
Assume that all the relevant quantities are well defined. [5]

4. Find out the probability mass function which has the generating function
$$P(t) = \exp(-\lambda(1-t^2)).$$ [5]

5. Let X_1, X_2 be two independent geometric random variables with parameters p_1 and p_2 on $\{0, 1, 2, \ldots\}$. Find $P[X_1 > X_2]$ and the probability mass functions of $\max(X_1, X_2)$ and $\min(X_1, X_2)$. [4+4+4=12]

6. Let $\{X_i\}_{i \geq 1}$ are independent random variables with X_i having Bernoulli distribution with parameter p_i, for $i \geq 1$. Show that, as $n \to \infty$,
$$\frac{1}{n}\left[\sum_{i=1}^{n}(X_i - p_i)\right] \xrightarrow{P} 0.$$ [6]

7. Consider an urn containing b black and w white balls to begin with. At each stage, we add r black balls and then withdraw r balls at random from $b+r+w$ balls. Show that expected number of white balls after t stages is $\left(\frac{b+w}{b+w+r}\right)^t w$. [6]

8. If $X_1, \ldots, X_n$ are i.i.d. positive random variables, for any $1 \leq k \leq n$, find the value of
$$\mathrm{E}\left[\frac{\sum_{i=1}^{k} X_i}{\sum_{i=1}^{n} X_i}\right].$$ [6]

INDIAN STATISTICAL INSTITUTE

Vectors and Matrices I : B. Stat 1st year: Final Semester Examination: 2012-13
November 28, 2012.

Maximum Marks 60 Maximum Time 2:30 hrs.

Answer all questions.

Notation: $\rho(A)$ is rank of the matrix A, HCF is Hermite canonical form, $\operatorname{Null} T =$ Null space of T.

(1) Give short answers to these questions.

(a) Let V be the set of all sequences of real numbers. For $i = 1, 2, \ldots$, let e_i be the sequence whose i-th entry is 1 and all other entries are 0. Is the set $\{e_i \mid i = 1, 2, \ldots\}$ a basis of V? Justify.

(b) Give an example of a vector space V and a linear transformation $T : V \to V$ such that T is onto but not $1-1$.

(c) Suppose for three real matrices A, B, C; $AA^TB = AA^TC$. Show that $A^TB = A^TC$.

(d) Consider the system of linear equations $Hx = d$ where H is in HCF. Show that the system is consistent if and only if d_i is zero whenever $h_{ii} = 0$.

5+5+5+5=20

(2) Let U and W be two subspaces of a finite dimensional vector space V. Find an isomorphism between the quotient spaces $(U + W)/W$ and $U/(U \cap W)$. 7

(3) Let V be an n-dimensional vector space and $T : V \to V$ be a linear transformation.

(a) If $\rho(T^k) = \rho(T^{k+1})$ for some $k \in \mathbb{N}$ then show that $\rho(T^{k+1}) = \rho(T^{k+2})$.

(b) Prove that there exists a $p \in \mathbb{N}$, $1 \le p \le n - 1$, such that

$$\rho(T) < \rho(T^2) < \cdots < \rho(T^p) = \rho(T^{p+1}) = \cdots .$$

(c) Using (b) show that $\operatorname{Null} T^p = \operatorname{Null} T^{p+1}$.

(d) Using (a), (b), (c) show that $V = T^pV \oplus \operatorname{Null} T^p$.

3+2+5+5=15

(4) Let V be a $\mathbb{C}$-vector space and F_1, F_2 be linear functionals on V. Let $W = \ker F_1$. If $F_2(w) = 0$ for all $w \in W$, is it possible to express F_2 in terms of F_1? Prove your assertion. 8

P. T. O.

(5) Let V_1, V_2 be two vector spaces over $\mathbb{R}$ with $\dim V_1 = n, \dim V_2 = m$. Let $T : V_1 \to V_2$ be a linear transformation. Prove that for any $r \in [\rho(T), \min\{m,n\}]$ there exists a linear transformation $S : V_2 \to V_1$ such that $ToS|_{T(V_1)}$ is identity and $\rho(S) = r$. 10

(6) Consider the system of linear equations:

$$x + y + z = 1$$
$$\alpha x + \beta y + \gamma z = \delta$$
$$\alpha^3 x + \beta^3 y + \gamma^3 z = \delta^3.$$

Consider the cases (a) $\alpha \neq \beta, \alpha \neq \gamma, \gamma \neq \beta$ and (b) $\alpha = \beta = \gamma$. For each of these cases find conditions on δ so that the system is consistent and find the general solutions when it is consistent. 10

INDIAN STATISTICAL INSTITUTE

Vectors and Matrices I : B. Stat 1st year: Back paper Examination: 2012-13

Date ..24...1..2..12

Full Marks 100 | Maximum Marks 45 | Time 3 hrs.

Notation: $\rho(A)$ is rank of the matrix A, HCF is Hermite canonical form.

(1) Give short answers to these questions.

(a) Let $T : U \to V$ be a surjective linear map and $\ker T = W$. Show that V is isomorphic to U/W.

(b) Let V be a finite dimensional vector space over $\mathbb{R}$. Let W_1 and W_2 be two subspaces of V. Give example to show that $W_1 \cup W_2$ may not be a subspace of V. What is the smallest subspace containing $W_1 \cup W_2$?

(c) Justify that $\mathbb{C}^2$ over the field $\mathbb{C}$ and $\mathbb{C}^2$ over the field $\mathbb{R}$ are two different vector spaces.

(d) Let V be a vector space with a basis $e_1, e_2, \ldots e_n$ and W be a subspace of V such that $e_i \in W$ only for $i = 1, \ldots r$ where $r < n$. Is W generated by $\{e_i \mid i = 1, \ldots, r\}$?

(e) Let V be a finite dimensional vector space over $\mathbb{R}$, V^* its dual and V^{**} its double dual. Find a natural isomorphism from $V \to V^{**}$. If $V = \mathbb{R}^n$, find a natural isomorphism from $V \to V^*$.

(f) Let A, B be two $m \times n$ matrices, P be a $m \times m$ matrix and $A = PB$. Show that P is invertible if and only if $\rho(A) = \rho(B)$.

(g) Let A be an $m \times n$ real matrix. Show that A has full column rank if and only if $A^T A$ is nonsingular.

5+7+3+4+6+5+5=35

(2) Define direct sum of two subspaces U and W of a finite dimensional vector space V. Show that $U + W$ is the direct sum of U and W if and only if for all bases A and B of U and W respectively, $A \cup B$ is a basis of $U + W$. 8

P. T. O.

(3) Consider a system of linear equations $Ax = b$ where A is an $m \times n$ matrix and $b \neq 0$.

(a) Show that $Ax = b$ has

(i) no nonzero solution if and only if $\rho(A) < \rho([A; b])$

(ii) Unique solution if and only if $\rho(A) = \rho([A; b]) = n$

(iii) more than one solution if and only if $\rho(A) = \rho([A; b]) < n$.

(b) Suppose that G is a generalized inverse of A. Find a general solution of $Ax = b$ in terms of A, G and b.

(c) Suppose $m = n$ and by a finite sequence of elementary row operations A is reduced to a matrix H which is in HCF and by the same sequence of operations b is changed to d. Then find the general solution of $Ax = b$ in terms of H and d.

6+5+7=18

(4) Let $T : V_1 \to V_2$ be a linear map between two finite dimensional vector spaces V_1 and V_2. Show that

$$\dim \ker T + \dim T(V_1) = \dim V_1.$$

7

(5) Let V be a vector space over $\mathbb{C}$ and $T : V \to V$ be a $\mathbb{C}$-linear transformation. Define the real and imaginary parts of T. Denote them by u and v respectively, i.e.

$$T(x) = u(x) + iv(x) \ \forall x \in V.$$

Show that T can be defined using only the real part of T. 8

(6) Let $\mathcal{P}_n$ be the vector space of all polynomials in one variable with real coefficients, of degree less than n. Fix distinct points $t_1, t_2, \ldots, t_n \in \mathbb{R}$.

(a) Find polynomials $h_i \in \mathcal{P}_n$ such that $h_i(t_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta.

(b) Show that $\{h_1, h_2, \ldots, h_n\}$ is a basis of $\mathcal{P}_n$.

(c) Find the matrix which changes the natural basis of $\mathcal{P}_n$ (i.e. $\{1, x, x^2, \ldots, x^{n-1}\}$) to $\{h_1, h_2, \ldots, h_n\}$.

6+6+5=17

(7) Let V be an n-dimensional real vector space. Let $T : V \to V$ be nilpotent, i.e $T^p x = 0$ for all $x \in V$ for some $p \in \mathbb{N}$. Suppose that for $\xi \in V$, $T^{p-1}\xi \neq 0$. Show that Span $\{\xi, T\xi, T^2\xi, T^{p-1}\xi\}$ is a p-dimensional subspace of V which is invariant under T.

7

INDIAN STATISTICAL INSTITUTE

Analysis I : B. Stat 1st year
Back Paper Examination: 2012-13
Full Marks 100
24 ~~November~~ , 2012.
December

Maximum Marks 45 Maximum Time 3 hrs.

(1) Describe all polynomials $P : \mathbb{R} \to \mathbb{R}$ such that $P(x + 2\pi) = p(x)$, for all $x \in \mathbb{R}$. [4]

(2) If A and B are bounded subsets of $\mathbb{R}$ then prove that Sup $\{a + b \mid a \in A, b \in B\} =$ Sup $A +$ Sup B. [8]

(3) Let $f : [0, \infty) \to [0, \infty)$ be a bounded continuous function. Prove that there exists $c \in [0, \infty)$ such that $f(c) = c$. [8]

(4) Solve the equation $3^x + 4^x = 5^x$ for $x \in \mathbb{R}$. [8]

(5) Let $f : [0, 1] \to \mathbb{R}$ be such that f'' is continuous on $[0, 1]$ and $f^{(3)}$ exists in $(0, 1)$. Prove without using Taylor's theorem that there exists $c \in (0, 1)$ such that $f(1) = f(0) + f'(0) + \frac{1}{2}f''(0) + \frac{1}{6}f^{(3)}(c)$. [8]

(6) (a) Let $\{a_n\}$ be a monotonically decreasing sequence of positive real numbers. If $\sum_{n=1}^{\infty} a_n$ converges then prove that $\lim_{n\to\infty} n a_n = 0$.

(b) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ diverges if $0 < \alpha \leq 1$. [6+4=10]

(7) Draw the graph of the function $f(x) = x^3 - 6x^2 + 9x + 1$ for all $x \in [0, 4]$ indicating maxima, minima, convexity and concavity. [10]

(8) Let $f; [0, 1] \to \mathbb{R}$ be continuous and differentiable on $(0, 1)$. Suppose that $f(0) = f(1) = 0$ and there exists $x_0 \in [0, 1]$ such that $f(x_0) = 1$. Prove that $|f'(c)| > 2$ for some $c \in (0, 1)$. [10]

(9) Define $f(x) = \frac{e^x - e^{-x}}{2}$ for $x \in \mathbb{R}$. Prove that there exist positive constants c_1 and c_2 (independent of x) such that

$$c_1 \frac{xe^x}{1+x} \leq f(x) \leq c_2 \frac{xe^x}{1+x},$$

for all $x \geq 0$. [10]

(10) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function which is differentiable in $(0, \infty)$. Also assume that $\lim_{x\to\infty} f'(x) = b \in \mathbb{R}$.

(a) If $\lim_{x\to\infty} f(x) = a \in \mathbb{R}$ then prove that $b = 0$.

(b) If f is bounded then prove that $\lim\limits_{x\to\infty} \frac{f(x)}{x} = b$. [6+6=12]

(11) Prove that every sequence of real numbers has a monotonic subsequence. [12]

INDIAN STATISTICAL INSTITUTE

Analysis 2 : B. Stat 1st year
Mid Semester Examination: 2012-13
19 February , 2013.

Maximum Marks 40 Maximum Time 2 hrs.

(1) If $f \in R[a.b]$ and $F'(x) = f(x)$ for all $x \in [a,b]$ the prove that $\int_a^b f(x)dx = F(b) - F(a)$. [7]

(2) Let
$$A = \left\{ f : [0,1] \to \mathbb{R} \mid \text{f is differentiable, } f(0) = 0 \text{ and } \int_0^1 f'(x)^2 dx \leq 1 \right\}.$$
Prove that the set $\left\{ \int_0^1 f(x)dx \mid f \in A \right\}$ is a bounded set. [6]

(3) Without using Dirichlet's theorem prove that the improper integral $\int_\pi^\infty \frac{\sin x}{x} dx$ converges. Also prove that the above integral does not converge absolutely. [6]

(4) Discuss convergence/divergence of the improper integral
$$\int_1^\infty \frac{dx}{x[(\log x)^{1/2} + (\log x)^2]}.$$
[7]

(5) Let $A = \mathbb{Q} \cap [0,1]$ and $\{f_n\}$ be a sequence of real valued continuous functions on $[0,1]$ which converges uniformly on A. Prove that $\{f_n\}$ converges uniformly on $[0,1]$. [6]

(6) If $f : [0,1] \to \mathbb{R}$ is a differentiable function with non negative derivative then evaluate $\lim_{n\to\infty} \int_0^{1/2} f(1-x^n)dx$. [6]

(7) If $f : [0,1] \to \mathbb{R}$ is a continuous function then evaluate the limit
$$\lim_{n\to\infty} n \int_0^1 x^n f(x)dx.$$
[6]

INDIAN STATISTICAL INSTITUTE

B. Stat 1st year: Mid Semester Examination: 2012-13
Vectors and Matrices II : February 22, 2013.

Maximum Marks 40 Maximum Time 2 hrs.

Answer all questions.

Notation: For a square matrix A, $|A|$ is its determinant. In (2) $A^\star$ is the adjugate matrix of A (made of co-factors of A), and in (3) $A^\star$ is the adjoint of A, i.e. $\langle Ax, y\rangle = \langle x, A^\star y\rangle$.

(1) Let V be a vector space with a pseudo norm ρ.
(a) Find a subspace W such that $X = V/W$ is a normed space with a norm $\|.\|$.
(b) Find a linear map $T : V \to X$ such that $\rho(x) = \|Tx\|$. 3+3

(2) Let A be an $n \times n$ matrix with determinant 1. Let $A^\star$ is the adjugate matrix of A.
(a) Using the formula for $|A|$ expanding it in k-th row and using minors, show that $AA^\star = I$. Use this to show that $(A^\star)^\star = A$. (Do not use the formula $A^{-1} = A^\star/|A|$.)
(b) Given two vectors x, y of unit length in $\mathbb{R}^n$ show that, there exists a $B \in SO(n)$ such that $Bx = y$. 9+4=13

(3) Let A be an $n \times n$ matrix and $AA^\star = A^\star A$. Prove that $kerA = kerA^\star$. Using this and induction show that $kerA^k = kerA$ for any $k \in \mathbb{N}$. 3+9

(4) Let T be a triangle in $\mathbb{R}^2$ with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. Show that its area is $\frac{1}{2}|A|$ where

$$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}.$$

(Hint: Use (a) Area is invariant under translation, (b) a triangle is half of a parallelogram, (c) formula of area of parallelogram.) 5

(5) Let B be a fixed $n \times n$ matrix with real entries. Let $\mathbb{R}^{n^2} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be the n-fold product of $\mathbb{R}^n$. Note that $\mathbb{R}^{n^2}$ can be identified with $M_n(\mathbb{R})$ which is the set of $n \times n$ real matrices. Show that the map $f : \mathbb{R}^{n^2} \to \mathbb{R}$ defined by

$$f(A) = |AB|$$

is an alternating, multilinear map. Use this to prove that $|AB| = |A||B|$. 9

Probability Theory II
B. Stat. Ist Year Semester 2
Indian Statistical Institute

Mid-semestral Examination
Time: 2 Hours Full Marks: 35
Date: February 25, 2013

1. If X is a standard normal random variable, find the density of the random variable $|X|$. [4]
2. Find the probability that the first digit of the square root of a uniform (0,1) random variable is 3. [4]
3. Let $\{X_n\}$ be an i.i.d. sequence of Cauchy random variables. Define $M_n = \max\{X_1, \ldots, X_n\}$. Show that $\mathrm{P}[M_n/n \leq x]$ converges for every real number x. Call the limit $F(x)$. Show that it is a distribution function. [4+2=6]
4. If X is a lognormal random variable, find $\mathrm{E}[X^k]$, where k is a natural number. Show that, for any $\lambda > 0$, $\mathrm{E}[\exp(\lambda X)]$ does not exist. [4+4=8]
5. Consider a sample space Ω and a collection $\mathfrak{F}$ of subsets of Ω satisfying the following two conditions:
   (a) $\Omega \in \mathfrak{F}$.
   (b) If $A, B \in \mathfrak{F}$, then $A \setminus B \in \mathfrak{F}$.
   Show that:
   (a) If $A \in \mathfrak{F}$, then $A^c \in \mathfrak{F}$.
   (b) If $A, B \in \mathfrak{F}$, then $A \cup B \in \mathfrak{F}$. [3+5=8]
6. For a monotone increasing right continuous function U, let $U^{\leftarrow}$ be its left continuous inverse. Show that $U(x) < u$ holds if and only if $U^{\leftarrow}(u) > x$. [5]

Indian Statistical Institute
Statistical Methods II
B-I, Midsem

Date: Feb 27, 2013 Duration: 2hrs.

Attempt all questions. The maximum you can score is 40. Justify all your steps. This is a closed book, closed notes examination. You may use your own calculator. No need to perform more than three steps of any iterative method.

If copying is detected in the solution for any problem, all the students involved in the copying will get 0 for that problem. Also an additional penalty of 5 will be subtracted from the overall aggregate of each of these students.

1. A box contains 5 white balls and 2 black balls. A coin with unknown $P(Head) = p$ is tossed. A white ball is added to the box if the outcome is head; otherwise a black ball is added. Then a ball is drawn at random from the box, and the colour is recorded. If the colour is "white", then find the mle of p. [10]

2. Consider the following random experiment. A coin with unknown $P(Head) = p \in (0,1)$. is tossed. If it shows head, then a random variable is generated with distribution $Uniform(0,p)$, otherwise X is generated from $Uniform(p,1)$. This random experiment is performed 10 times independently (each time starting with a fresh coin toss) to produce data $X_1, ..., X_{10}$. Find a suitable estimator (mle, mme or something else) of p based on this data. Comment on the your choice of the estimator. [10]

3. Find mle and mme of θ based on a random sample $X_1, ..., X_{10}$ from the distribution $Uniform(-\theta, \theta)$. Suggest how you can compare the performances of the two estimators. [15]

4. Consider the discrete uniform distribution over the set
$$\{\theta - 2, \ \theta - 1, \ \theta, \ \theta + 1, \ \theta + 2\},$$
where $\theta \in \mathbb{R}$ is an unknown parameter. Assume that the following is a random sample from this distribution:
$$8, 6, 8, 10, 9.$$
Find mle of θ based on this data. Justify your steps. [5]

5. Based on two iid observations 2.3 and 4.1 from the Cauchy distribution with density
$$f_\theta(x) = \frac{1}{\pi \cdot (1 + (x-\theta)^2)}, \ \ x \in (-\infty, \infty),$$
find mle of θ. [10]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: 2012-13

Course Name : B. STAT. I YEAR

Subject Name : Numerical Analysis

Date: 01.03.13 **Maximum Marks: 50,** **Duration: $2\frac{1}{2}$ hrs.**

Note : Ordinary calculator is allowed in the Examination Hall.

Answer all the questions:

1. What is interpolation? Establish Newton's forward interpolation formula without error term. 1+5
2. Prove that
   (i) $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$,
   (ii) $\left(\frac{\Delta^2}{E}\right) e^x \frac{E e^x}{\Delta^2 e^x} = e^x$ where $\Delta f(x) = f(x+h) - f(x)$, $Ef(x) = f(x+h)$. 3+3
3. Define "degree of precision" of a numerical integration formula. Find the error term in Simpson's one-third rule for numerical integration. 6
4. Use Euler's method to evaluate $y(2)$, from $\frac{dy}{dx} = \frac{1}{2}(x+y)$ with $y(0) = 2$, taking $h = 0.5$. Also find the relative percentage error. 4+2
5. State Euler's method and modified Euler's method for solving the differential equation $\frac{dy}{dx} = f(x,y)$ given $y(x_n) = y_n$. Using Taylor's series find the truncation error of the modified Euler's method. 1+1+4
6. Explain the principle of numerical differentiation. Deduce Lagrange's numerical differentiation formula and give in particular the formula for computing the derivative at an interpolating point. 1+3+2
7. Let $f(x)$ be a real-valued function defined on $[a,b]$ and $n+1$ times differentiable on (a,b). If $p_n(x)$ is the polynomial of degree $\leq n$ which interpolates $f(x)$ at the $n+1$ distinct points $x_0, x_1, \cdots, x_n$ in $[a,b]$, then for all $\bar{x} \in [a,b]$, there exists $\xi = \xi(x) \in (a,b)$ such that $e_n(\bar{x}) = f(\bar{x}) - p_n(\bar{x}) = \frac{f^{n+1}(\xi)}{(n+1)!} \prod_{j=0}^{n} (\bar{x} - x_j)$. 6
8. Solve by Predictor-corrector method of $\frac{dy}{dx} = x - \frac{1}{y}$, $y(0) = 1, h = 0.1$ for $x = 0$ to $x = 0.1$. 6
9. Solve by Forth Order Runge-Kutta method for $x = 0$ to $x = 0.1$ from $\frac{dy}{dx} = x + y$ with $x_0 = 0$, $y_0 = 1, h = 0.1$. Also find the relative error. 4+2
10. What do you mean by round off errors in numerical data? Show how these errors are propagated in a difference table and explain how this affected the computations. 2+2+2
11. Establish Newton-Cotes numerical integration formula. 6

INDIAN STATISTICAL INSTITUTE

B. Stat 1st year: Semester Examination: 2012-13
Vectors and Matrices II : Date April 30, 2013.

Maximum Marks 60 Maximum Time 3 hrs.

Answer all questions.

All vector spaces in the questions below are finite dimensional. If the underlying field is not mentioned, then both the real and complex cases have to be considered.

(1) Let V be the vector space of all polynomials in one variable with real coefficients of degree at most three. Equip V with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$. Use Gram-Schmidt process to transform the basis $\{1, x, x^2, x^3\}$ of V to an o.n.b. 10

(2) Find the possible signatures of the bilinear forms ϕ and ψ described below.

(a) The bilinear form $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$\phi(x, y) = x^T p(A) y$$

where A is a symmetric $n \times n$ real matrix and the polynomial $p(x) = x^2 + bx + c$ has no real roots.

(b) The symmetric bilinear form $\psi : V \times V \to \mathbb{R}$ is defined on a real vector space V with $\dim V = 5$ and for every $v \in V$, there exists a $v' \in V$ such that $\psi(v, v') \neq 0$.

10 + 5

(3) (a) Let A be a $n \times n$ complex matrix. Prove that A is diagonalizable if and only if minimal polynomial of A has no repeated roots.

(b) Let A_1, A_2 be two commuting $n \times n$ complex matrices, each of which is diagonalizable. Show that they are simultaneously diagonalizable. 8+10

(4) Let V be a real inner product space and $A : V \to V$ be a nonnegative definite (n.n.d) self adjoint linear operator. If $B : V \to V$ is n.n.d. self adjoint and $B^2 = A$, then show that $V_\lambda^A = V_{\sqrt{\lambda}}^B$ for any eigenvalue λ of A. [Here V_λ^A and $V_{\sqrt{\lambda}}^B$ are the eigenspaces of A with eigenvalue λ and of B with eigenvalue $\sqrt{\lambda}$ respectively.] 7

P.T.O.

(5) Let V be a complex inner product space and $A : V \to V$ be a self adjoint linear operator. For nonzero vectors $v \in V$, define $\rho(v)$ by

$$\rho(v) = \frac{\langle Av, v\rangle}{\langle v, v\rangle}.$$

Show that $\rho(v) \leq \lambda_k$, for all $v \in V$, where λ_k is the largest eigenvalue of A. Does $\rho(v)$ necessarily attain the value λ_k? 5

(6) Let V be a complex vector space and $A : V \to V$ be a linear map which has an eigenvalue λ. Let $v_1, \dots, v_k$ be a finite sequence of nonzero vectors which satisfy

$$Av_1 = \lambda v_1, Av_2 = v_1 + \lambda v_2, \dots, Av_k = v_{k-1} + \lambda v_k.$$

(a) Show that $(A - \lambda I)^j v_j = 0$ for $j = 1, \dots, k$. (Hint:Use induction.)
(b) Show that the set $\{v_1, \dots, v_k\}$ is linearly independent.
(c) Let $W = span\{v_1, \dots, v_k\}$. Then show that W is A invariant and find the matrix of $A|_W$ with respect to the basis $\{v_1, \dots, v_k\}$. 5+5+5=15

INDIAN STATISTICAL INSTITUTE

Analysis II : B. Stat 1st year
Second Semester Examination: 2012-13
May , 2013.

Full Marks 70 Maximum Time 3 hrs.

Answer all the questions but maximum you can score is 60.

(1) Give an example of a sequence of functions $\{f_n\}$ defined on $[0,1]$ such that each f_n is discontinuous at every point of $[0,1]$ but converges uniformly to a continuous function in $[0,1]$. [2]

(2) If $f:\mathbb{R}\to\mathbb{C}$ is 2π-periodic and C^1 then prove that $f(t)=\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{int}$ for all $t\in[-\pi,\pi]$. [4]

(3) Let $\{f_n\}$, f be real valued functions on $\mathbb{R}$ such that $\{f_n(x_n)\}\to f(x)$ whenever $\{x_n\}\to x$. Prove that for every subsequence $\{f_{n_k}\}$ of $\{f_n\}$ we have $\{f_{n_k}(x_n)\}\to f(x)$ whenever $\{x_n\}\to x$. [4]

(4) Prove or disprove: There exists a real valued continuous function g on $[0,1]$ with $g(x)\neq x$ for all $x\in(0,1)$ such that given any $\epsilon>0$ and any real valued continuous function f on $[0,1]$ there exist real numbers $a_1,a_2,\ldots,a_n$ (depending only on f, g and ϵ) such that

$$|f(x)-\sum_{k=0}^{n}a_k(g(x))^k|<\epsilon,$$

for all $x\in[0,1]$. [4]

(5) Prove or disprove: There exists a sequence of non negative, continuous functions $\{f_n\}$ on $[0,1]$ such that $\sum_{n=1}^{\infty}f_n(x)$ converges uniformly on $[0,1]$ but $\sum_{n=1}^{\infty}\sup_{x\in[0,1]}f_n(x)=\infty$. [4]

(6) If $k\in\mathbb{R}\setminus\{0\}$ then prove that the equation $y''-k^2y=R(x)$ has a particular solution y_1 given by $y_1(x)=\frac{1}{k}\int_0^x R(t)\sinh(x-t)dt$. Hence find the general solution of the equation $y''-k^2y=e^{\pi x}$. [8]

(7) (a) If f, g are C^1, 2π periodic functions with $\int_{-\pi}^{\pi} f(t)dt = 0$ then prove that

$$\left|\int_0^{2\pi} \overline{f(t)}g(t)dt\right|^2 \leq \int_0^{2\pi} |f(t)|^2 dt \int_0^{2\pi} |g'(t)|^2 dt.$$

(b) If $f : [0,\pi] \to \mathbb{C}$ is a C^1 function with $f(0) = f(\pi)$ then prove that

$$\int_0^{\pi} |f(t)|^2 dt \leq \int_0^{\pi} |f'(t)|^2 dt.$$

Also discuss the case of equality. [4+4=8]

(8) For $n \geq 1$ define

$$a_n = \int_0^{\pi/2} \sin 2nx \cot x dx, \qquad b_n = \int_0^{\pi/2} \frac{\sin 2nx}{x} dx.$$

(a) Prove that $a_{n+1} = a_n$ for all $n \geq 1$.
(b) Prove that $\lim_{n\to\infty} a_n - b_n = 0$.
(c) Using the above results prove that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$. [8]

(9) If f and g are continuous, 2π-periodic functions then prove that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)g(nt)dt = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)dt\right).$$

[8]

(10) For $\alpha \in \mathbb{R}$ define

$$f_\alpha(x) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n.$$

(a) Prove that the above series converges for all $x \in (-1,1)$. [2]
(b) Differentiating the series or otherwise prove that $f_\alpha(x) = (1+x)^\alpha$ for all $x \in (-1,1)$. [8]

(11) (a) Describe when a sequence of 2π-periodic continuous functions $\{K_n\}$ is called a family of good kernels. [3]
(b) For $k \in \mathbb{N}$ and $t \in [-\pi,\pi]$ define

$$Q_k(t) = c_k \left(\frac{1+\cos t}{2}\right)^k,$$

where c_k is such that $\frac{1}{2\pi}\int_{-\pi}^{\pi} Q_k(t)dt = 1$. Prove that $c_k \leq \pi(k+1)/2$. [1]
(c) If f is a continuous 2π-periodic function then prove that the sequence of functions $f * Q_k(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)Q_k(t)dt$, $x \in [-\pi,\pi]$ converges uniformly to f. [6]

Probability Theory II
B. Stat. Ist Year Semester 2
Indian Statistical Institute

Semestral Examination
Time: 3 Hours Full Marks: 50
Date: May 6, 2013

1. If $\{X_n\}$ is a sequence of bounded random variables, show that $\limsup X_n$ is also a random variable. [6]
2. If X is a Cauchy random variable, what is the distribution of $1/X$? [4]
3. Define m as a median of a random variable X if we have $P[X \le m] \ge 0.5$ as well as $P[X \ge m] \ge 0.5$. Show that it always exists, but need not be unique. Describe the set of possible values of median of X in terms of its distribution function F. [3+2+3=8]
4. For a nonnegative random variable X, define
$$X_n^* = \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbb{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})}(X).$$
Show that $E[X_n^*]$ form a decreasing sequence and converge to $E[X]$. [2+4=6]
Hint: For the convergence, consider two cases separately depending on the finiteness of $E[X]$.
5. Consider the function
$$F(x,y) = \begin{cases} 0, & \text{for } x+y<1, \\ 1, & \text{for } x+y \ge 1. \end{cases}$$
Is it a bivariate distribution function? [5]
6. Pick two points A and B at random independently from the circumference of the unit circle. (You do this by choosing two angles independently and uniformly from $[0, 2\pi)$ and choosing two points with corresponding angles in the polar representation.) If D is the perpendicular distance of AB from the origin and Θ is the angle AB makes with the x-axis, show that the joint density of (D, Θ) is
$$f(d,\theta) = \frac{1}{\pi^2\sqrt{1-d^2}}, \quad 0 \le d \le 1, 0 \le \theta < 2\pi.$$
[7]
7. Let $X_1, \dots, X_n$ be i.i.d. exponential random variables with parameter 1.
Find the density of $V_n = \max_{1\le k\le n} X_k$. Find its moment generating function.
Hence or otherwise show that V_n and W_n have same distribution, where $W_n = \sum_{k=1}^{n} \frac{1}{k} X_k$.
[2+3+3=8]
8. If (X, Y) have joint density $\lambda^2 e^{-\lambda y}$ for $0 \le x \le y < \infty$, find the conditional density of Y given X. [6]

Indian Statistical Institute
Statistical Methods II
B-I, Second Semestral Examination

Date: May 08, 2013 Duration: 3 hrs.

This paper carries 70 marks. Attempt all questions. The maximum you can score is 60. Justify all your steps. This is a closed book, closed notes examination. You may use your own calculator. No need to perform more than three steps of any iterative method.

If copying is detected in the solution for any problem, all the students involved in the copying will get 0 for that problem. Also an additional penalty of 5 will be subtracted from the overall aggregate of each of these students.

1. Let $X_1, ..., X_n$ be iid from the distribution $Unif[-2\theta, \theta]$ where $\theta > 0$ is an unknown parameter. Derive the MLE of θ. Is it unbiased? Justify your answer. [10+5]

2. State true or false. Justify your answers with proofs or counter examples as appropriate.

   (a) It is possible to have three variables X_1, X_2, X_3 such that $r_{12} = 1$ but partial correlation $r_{12\bullet 3} = -1$. [5]

   (b) If X_1 and X_2 are two variables, then the multiple correlation $r_{1\bullet 2}$ is the same as the product-moment correlation r_{12}. [5]

3. The sample covariance matrix for X_1, X_2, X_3, X_4 based on 100 cases is

$$S = \begin{bmatrix} 54 & 22 & 11 & 20 \\ 22 & 108 & 108 & 2 \\ 11 & 108 & 174 & -45 \\ 20 & 2 & -45 & 78 \end{bmatrix}$$

   What is the partial regression coefficient $b_{23\bullet 14}$? Also find the partial correlation coefficient $r_{14\bullet 2}$. [5+5]

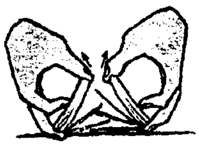
4. We have a data set consisting of n cases of $p+2$ variables (not necessarily centred) $X_1, ..., X_p, Y_1, Y_2$. Y_1 and Y_2 are linearly regressed on $X_1, ..., X_p$ (plus intercept term) using OLS to produce residuals Z_1 and Z_2, respectively. Is it always true that

$$cov(Y_1, Z_2) = cov(Z_1. Z_2)?$$

   Justify your answer with proof or counterexample as appropriate. [10]

[More excitement overleaf!]

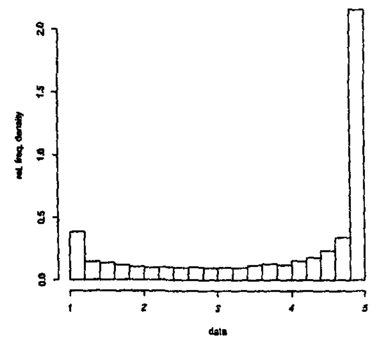
5. A rifle shuffle is the following way to shuffle a deck of 52 cards.



First the deck is cut into two equal halves, and held in two hands. Then the cards are dropped one by one from the two hands randomly according to the following probability distribution: If there are L cards in the left hand and R cards in the right hand then the chance that next card comes from the left hand is $\frac{L}{L+R}$. Assuming that the deck was initially labelled as 1,...,52, write down an algorithm to simulate a rifle shuffle. The output should be the permutation of the shuffled deck. You can only generate iid random numbers from $Unif(0,1)$ distribution. [15]

6. You are given just a single coin with unknown probability $p \in (0,1)$ of head. Suggest (with proof) a method by which you can use this coin to simulate a random variable X with $Bernoulli(\frac{1}{3})$ distribution. [5]

7. It is known that a continuous random variable can take only values between 1 and 5. If the histogram of a random sample looks as follows, then suggest how you can use the standard distributions to model this data. Assuming that you have a software to maximise any given function of one or more variables, suggest how you can fit the distribution to the random sample.



[2+3]

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: 2012-13

Course Name : B. STAT. I YEAR

Subject Name : Numerical Analysis

Date: 10.05.2013 **Maximum Marks: 50,** **Duration: $2\frac{1}{2}$ hrs.**

Answer all the questions:

1. Define n[th] order divided difference of $f(x)$. Show that $f(x_0, x_1, x_2, \cdots, x_n) = \frac{\Delta^n f(x_0)}{n!\, h^n}$ for the case of equally spaced interpolating points $x_r = x_0 + rh, r = 0, 1, 2, \cdots, n$. 1+4
2. Explain the method of fixed point iteration for numerical solution of the equation of the form $x = \phi(x)$. Derive the condition of convergence. 2+3
3. Let y_n be an approximate solution of $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ generated by Euler's method. If the exact solution $y(x)$ has a continuous second order derivative on the interval $[x_0, b]$ and if on this interval the inequalities $|f_y(x, y)| \leq L$, $|y''(x)| < M$ are satisfied for fixed positive constants L and M, show that the error $e_n = y(x_n) - y_n$ of Euler's method at a point $x_n = x_0 + nh$ is bounded as follows
$$|e_n| \leq \frac{hM}{2L}\left(e^{(x_n - x_0)L} - 1\right).$$ 5
4. Calculate the polynomial of degree at most 3 that best approximates e^x over the interval [-1, 1] in the least-square sense. 5
5. Describe the Regula-Falsi method for computing a simple real root of the equation $f(x) = 0$ and discuss the geometrical significance of the method. When does the method fail? 3+1+1
6. Prove that, if X and Y are vectors with the same norm, there exists a Householder matrix P such that $Y = PX$, where $P = I - 2\, W\, W^T$, $W = \frac{X-Y}{\|X-Y\|_2}$ and I is an identity matrix. Hence show that $P^{-1} = P$. 3+2
7. Describe power method to calculate numerically greatest eigenvalue of a real square matrix of order n. 5
8. Transform the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$ to tri-diagonal form by Given's method. Hence obtain the Sturm sequence of the corresponding characteristic equation. 3+2
9. Solve the following system $x + 3y + 2z = 5$, $2x - y + z = -1, x + 2y + 3z = 2$ by matrix factorization method. 5

P.T.O.

10. Let A be a rectangular matrix with m columns and $n > m$ rows and let A have rank m. Then show that for each vector b, the overdetermined linear system $AX = b$ has a unique solution in the sense of the method of least squares and the solution is identical with the solution of the non-singular system $A^T AX = A^T b$. 5

11. Using Jacobi's method find all eigenvalues and corresponding eigenvectors of the matrix, $\begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$. 3+2

12. Write down the recursion relation for Newton-Raphson method used to derive a real root of an algebraic equation $f(x) = 0$. Compute to 2 decimal places the real root of the equation $x^2 + 4\sin(x) = 0$. 2+3

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PROBABILITY THEORY II
B. STAT. IST YEAR SEMESTER 2
INDIAN STATISTICAL INSTITUTE

Backpaper Examination

Time: 3 Hours   Full Marks: 100

Date: June 26, 2013

1. A collection $\mathcal{A}$ of subsets of the sample space Ω is called a Dynkin class, if it satisfies:
   (a) $\Omega \in \mathcal{A}$.
   (b) If $A, B \in \mathcal{A}$, $B \subseteq A$, then $A \setminus B \in \mathcal{A}$.
   (c) If $A_n \in \mathcal{A}$ and $A_n \uparrow$ for $n \geq 1$, then $\cup_1^\infty A_n \in \mathcal{A}$.
   A collection $\mathcal{A}$ of subsets of the sample space Ω is called a λ-system, if it satisfies:
   (a) $\Omega \in \mathcal{A}$.
   (b) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.
   (c) If $A_n \in \mathcal{A}$ for $n \geq 1$ and $A_m A_n = \emptyset$ for $m \neq n$, then $\cup_1^\infty A_n \in \mathcal{A}$.
   Show that $\mathcal{A}$ is a Dynkin class iff it is a λ-system. [16]

2. Show that
$$\frac{1}{x} - \frac{1}{x^3} < \frac{1 - \Phi(x)}{\phi(x)} < \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5},$$
   where ϕ and Φ are standard normal density and distribution function respectively. [14]

3. For a random variable X, denote $\mu_k = \mathrm{E}\left[(X - \mathrm{E}[X])^k\right]$ and assume that they are defined for $k = 1, 2, 3, 4$. Show that the matrix
$$\begin{pmatrix} 1 & 0 & \mu_2 \\ 0 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix}$$
   is nonnegative definite. [12]

4. Two points are chosen independently and uniformly on a rod of unit length and the rod is broken at those two points giving three pieces. What is the probability that the three pieces will form a triangle? [14]

5. Write down the bivariate normal density with mean vector zero, common variance 1 and correlation ρ. Let (X, Y) be a random variable with such a density. Find the joint density of (X, Z) where $Z = (Y - \rho X)/\sqrt{1 - \rho^2}$. [12]

6. If X and Y are i.i.d. exponential random variables, identify the distribution of X/Y. (Do not do any calculations.) [8]

7. If (X, Y) have joint density $x(y - x)e^{-y}$ for $0 \leq x \leq y < \infty$, find the conditional density of X given Y. [12]

8. Let X be a gamma random variable with shape parameter α and scale parameter λ. Show that
$$\mathrm{P}[X \geq 2\alpha/\lambda] \leq (2/e)^\alpha.$$
   [12]

INDIAN STATISTICAL INSTITUTE

B. Stat 1st year: Back paper Examination: 2012-13
Vectors and Matrices II : Date 24/06......, 2013.

Maximum Marks 45 Maximum Time 3 hrs.

Answer all questions.

All vector spaces in the questions are finite dimensional. If the underlying field is not mentioned, then both the real and complex cases have to be considered.

(1) Let A, B be both $n \times n$ real symmetric matrices and B is positive definite. Show that there exists an invertible matrix P such that both $P^T AP$ and $P^T BP$ are diagonal matrices. 5

(2) Let V be an inner product space, $\|\cdot\|$ be the norm induced by the inner product and $T : V \to V$ be a linear map. Suppose that $\{x_1, \dots x_k\}$ is a basis of V.

(a) Is T invertible if $Tx_i \neq 0$ for $i = 1, \dots, k$?

(b) Is T invertible if $x_i \perp KerT$ for $i = 1, \dots k$?

(c) Suppose for two elements $x, y \in V$, $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Is $x \perp y$? 3+7+7=17

(3) Let A be a $m \times n$ matrix with real entries and $b \in \mathbb{R}^m$. Show that only one of the following system of equations can have a solution:

(a) $Ax = b$,

(b) $A^T x = 0$ and $x^T b \neq 0$. 8

(4) Let $\mathbb{R}^{n^2} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (n times product of $\mathbb{R}^n$). Suppose that $f : \mathbb{R}^{n^2} \to \mathbb{R}$ is a multilinear map. Show that the conditions below are all equivalent.

(a) $f(x_1, \dots, x_n) = 0$ if two adjacent elements x_i, x_{i+1} are same.

(b) $f(x_1, \dots, x_n) = 0$ if two elements x_i, x_j are same.

(c) $f(x_1, \dots, x_i, x_{i+1} \dots, x_n) = -f(x_1, \dots, x_{i+1}, x_i \dots, x_n)$ 10

(5) Consider $\mathbb{R}^3$ with the bilinear form $\phi((x, y, z), (x', y', z')) = xx' + 2yy' + 3zz'$. Justify that it defines an inner product on $\mathbb{R}^3$. Use Gram-Schmidt process to transform the set of vectors $\{(1, 1, 1), (1, 0, 1), (0, 1, 2)\}$ to an orthonormal set in $\mathbb{R}^3$ with respect to this inner product. 10

P.T.O.

(6) Let P be the orthogonal projection on a subspace W of an inner product space V with inner product $\langle \cdot \cdot \rangle$. Show that $\langle Px, x\rangle \geq 0$ for all x in V. 10

(7) Suppose T is a nonzero self adjoint linear operator on an inner product space V. Show that T cannot be nilpotent. 8

(8) Show that given $x, y \in S^{n-1} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$, there exists a $k \in SO(n)$ such that $kx = y$ 7

(9) Let A be an $m \times n$ complex matrix and $A = USV^*$, where U and V are unitary matrices of size $m \times m$ and $n \times n$ respectively and S be an $m \times n$ matrix of the form:

$$S = \begin{bmatrix} D & O \\ O & O \end{bmatrix},$$

with D a $r \times r$ diagonal matrix with positive entries $\gamma_1, \ldots, \gamma_r$ and O are zero blocks of appropriate sizes.

(a) Show that the first r columns of V are eigenvectors of A^*A with eigenvalues γ_i^2 and the first r columns of U are eigenvectors of AA^* with eigenvalues γ_i^2.

(b) Using the decomposition $A = USV^*$ and U, V, S, D find a generalized inverse of A. 5+5

(10) Let $\phi : V \times V \to \mathbb{R}$ be a symmetric bilinear form for a real vector space V. Suppose for every $v \in V$, there exists a $v' \in V$ such that $\phi(v, v') \neq 0$.

(a) Show that $V = V^+ \oplus V^-$ where $\phi|_{V^+ \times V^+}$ is positive definite and $\phi|_{V^- \times V^-}$ is negative definite.

(b) If V admits two such decompositions $V = V^+ \oplus V^-$ and $V = W^+ \oplus W^-$ with the same property, then show that

$$\dim V^+ = \dim W^+ \text{ and } \dim V^- = \dim W^-.$$

7+8