MSTAT I - Measure Theoretic Probability

Midsem. Exam. / Semester I 2010-11

Time - 2 hours/ Maximum Score - 30

<u>NOTE</u> : **SHOW ALL YOUR WORK TO GET THE FULL CREDIT. RESULTS USED MUST BE CLEARLY STATED.**

1. (4+4+4+4=16 marks) Write TRUE or FALSE and justify clearly.

   (a) Set of point of discontinuities of a right continuous function (from $\mathcal{R}$ to $\mathcal{R}$) must be measurable.

   (b) Let $\{f_n\}$ be a sequence of real-valued measurable function on $\mathcal{R}$, such that, $f_n \uparrow f$. Assume, $\int_{\mathcal{R}} f_n d\mu$ exists for all n and also $\int_{\mathcal{R}} f d\mu$ exists, where μ is a probability. Then $\int_{\mathcal{R}} f_n d\mu \to \int_{\mathcal{R}} f d\mu$.

   (c) Let $f : (\Omega, \mathcal{F}) \to (\mathcal{R}, \mathcal{B}(\mathcal{R}))$ be a Borel-measurable function and μ be a σ-finite measure on $(\Omega, \mathcal{F})$. If $\int_{\Omega} f d\mu$ exists and finite then, for every sequence of disjoint sets $\{B_n\}$ in $\mathcal{F}$, $\sum_{n=1}^{\infty} \int_{\Omega} f_n d\mu$ exists and finite, where $f_n = f I_{B_n}$

   (d) Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let X be a random variable on this space with finite expectation. If for some reals a, b, and for all $A \in \mathcal{A}$, $a \leq \int_A X dP \leq b$ then $a \leq X \leq b$, almost surely $[P]$.

2. (4+4=8 marks)

   (a) Show that for any Borel measurable set $A \subset \mathcal{R}^2$, the map, $x \mapsto \lambda_1(A^x)$ is a Borel-measurable function. where λ_1 is the one-dimensional Lebesgue measure and $A^x = \{y : (x, y) \in A\}$.

   (b) Suppose $A \subset \mathcal{R}^2$, is a Lebesgue measurable would the map $x \mapsto \lambda_1(A^x)$ be a Lebesuge-measurable function? Justify your answer.

3. (3+2+4=9 marks)

   (a) Let $\{f_n\}$, g be real-valued measurable functions on $\mathcal{R}$, such that, $-2g^+ \leq f_n \leq 3g^-$, where g is integrable with respect to Lebesuge measure λ. If $f_n \to f$, a.e. $[\lambda]$, as $n \to \infty$, then show that f is integrable with respect to Lebesuge measure and $\lim_{\{n \to \infty\}} \int f_n d\lambda = \int f d\lambda$.

   (b) With same set up as in (a), show that $\lim_{\{n \to \infty\}} \int |f_n - f| d\lambda = 0$.

   (c) Let P be a probability on $(\mathcal{R}, \mathcal{B}(\mathcal{R}))$. For sequence of non-negative measurable function $\{f_n\}$ on $\mathcal{R}$, with $\int f_n dP = 1$, for all n. Show that, $f_n \to 1$ a.e. $[P]$ implies $\lim_{\{n \to \infty\}} \int |f_n - 1| dP = 0$. [Hint: You may use, $\int_{\mathcal{R}} (f_n - 1)^+ dP = \int_{\mathcal{R}} (f_n - 1)^- dP$.]

All the best.

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2010-11

M. Stat. I Year (NB Stream)
Real Analysis

Date: 30.08.10 **Maximum Marks: 40** **Duration: $1\frac{1}{2}$ 1Hours**

The Paper contains questions of 46 marks. You may attempt all questions.
Maximum marks you can score is 40.

1. a) Show that for $A \subseteq R$, boundary of A = boundary of A^c

   b) Show that if $A \subseteq R$ and A is open or closed, then the interior of the boundary set of A is empty.

   c) Give example of an open set (in R) whose boundary set is uncountable. Justify your answer.

   [3+4+5]

2. a) Show that the set of all algebraic numbers is countable.

   b) Show that the set of all prime numbers is infinite.

   [5+5]

3. Check if the following sequences are Cauchy or not. Give reasons.

   a) $s_n = 1 + 1/2 + 1/3 + \ldots\ldots + 1/n$

   b) $s_n = a_1 - a_2 + a_3 - \ldots\ldots + (-1)^{n-1} a_n$
   where $\{ a_n \}$ is a decreasing sequence with limit 0 .

   [5+5]

4. Give examples (with justifications) :

   a) Two subsets A , B of R for which $\text{Int}(A) \cup \text{Int}(B) \neq \text{Int}(A \cup B)$

   b) $(S')' \neq S'$ where S is a subset of R and S′ denotes derived set of S.

   c) $f(A \cap B) \neq f(A) \cap f(B)$ where f is from S to T and A , B are two subsets of S.

   d) An infinite subset of R with no accumulation point.

   [2+2+2+1]

5. State and prove Bolzano- Weierstrass Theorem.

   [7]

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Midterm Examination
Large Sample Statistical Methods
First Semester
2010-2011 Academic Year
M.Stat. First Year (B-Stream and NB-Stream combined)
Date : 03.09.10 Maximum Marks: 75 Duration :- 3 hours

Group A
Answer each question

1. Let $\{x_n\}$ be a sequence of reals. Suppose δ_{x_n} converges in distribution to some distribution F. Show that $F = \delta_c$ for some real c. [6]

2. Let $X_n \sim \text{Bin}(n, \frac{1}{2})$ for $n \geq 1$ and $\Phi(.)$ denotes the distribution function of a $N(0,1)$ distribution. Does $P(X_n \leq \frac{n+n^{3/4}}{2}) - \Phi(n^{1/4})$ converge to a limit as $n \to \infty$? Prove your assertion. [4]

3. Suppose X_n converges in probability to both X and Y. Show that $P(X = Y) = 1$. [5]

4. Suppose $\frac{X_n - \mu}{\sigma_n}$ converges in distribution to $N(0,1)$, where $\sigma_n > 0$, $\forall n$. Show that X_n converges in probability to μ if and only if $\sigma_n \to 0$ as $n \to \infty$. [5]

5. Suppose $X_1, X_2, \ldots$ are iid with finite mean μ. Prove that $(1 + \frac{1}{n^2})^{S_n}$ converges to 1 in probability, where $S_n = \sum_{i=1}^{n} X_i$. [5]

6. Let $X_1, X_2, \ldots$ be a sequence of random variables defined on a common probability space. Suppose $E(X_n^2) \leq 1$ for each n. Does the sequence $\frac{X_n}{n}$ converge almost surely ? Prove or give a counterexample. [5]

7. Suppose $X_1, X_2, \ldots, X_{100}$ are iid observations from an unknown continuous distribution F. Can you suggest any method of finding a confidence band of approximate confidence probability 95% for the unknown distribution function using the above observations ? [4]

8. (a) State with assumptions, the asymptotic representation theorem of a sample quantile due to J.K. Ghosh. Briefly argue how one can show asymptotic normality of sample quantiles starting from this representation theorem. [2+2]

   (b) Let $X_1, X_2, \ldots X_n$ be iid observations from a $N(\mu, 1)$ distribution with μ unknown. Find the joint asymptotic distribution of (properly centred and scaled) sample mean and sample median. Suppose now a random sample of size 50 has been drawn from a $N(1,1)$ distribution and you are only told that the sample median is 1.1. Based on this information, can you give an estimate of the sample mean ? Justify your answer. [6+4=10]

Continued to next page

Group B

Answer as many questions as you can. The maximum you can score in this group is 27

9. Suppose a sequence $\{X_n\}$ of random variables is stochastically bounded. Does this mean that $P(\sup_n |X_n| < \infty) = 1$? Prove or give a counterexample. [6]

10. Let $X_1, X_2, \ldots$ be independent random variables with

$$\begin{aligned} X_k &= -ke^k \text{ with probability } e^{-2k}, \\ &= +ke^k \text{ with probability } e^{-2k}, \\ &= -k \quad \text{with probability } \frac{1}{2} - e^{-2k}, \\ &= +k \quad \text{with probability } \frac{1}{2} - e^{-2k}. \end{aligned}$$

Let $S_n = X_1 + \ldots + X_n$. Can you find constants $a_n > 0$ such that $a_n S_n$ converges to a non-degenerate distribution ? Prove your assertion. [10]

11. Let P and Q be two probability measures on $(\Omega, \mathcal{A})$. Then show that statement (a) below implies statement (b). [8]

    (a) For each $A \in \mathcal{A}$, $\{P(A) = 0\}$ implies $\{Q(A) = 0\}$.

    (b) For each $\epsilon > 0$, there exists $h_\epsilon > 0$ such that $\{A \in \mathcal{A}$ and $P(A) < h_\epsilon\}$ implies $\{Q(A) < \epsilon\}$.

12. Suppose $\sum_{i=1}^{\infty} P(|X_i| > i) = \infty$ where $X_1, X_2, \ldots$ are independent random variables. Show that

$$P\left(\lim_{n\to\infty} \frac{S_n}{n} \text{ exists in } (-\infty, \infty)\right) = 0,$$

where $S_n = \sum_{i=1}^{n} X_i$. [8]

(Hint: Express $\frac{S_n}{n} - \frac{S_{n+1}}{n+1}$ in terms of S_n and X_n.)

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2010-11

M. Stat. I Year
Sample Surveys and Design of Experiments

Date: 6. 9. 10 **Maximum Marks: 20** **Duration:** $1\frac{1}{2}$ **1Hours**

No Books, Tables, Calculating machines or any other tools & Equipments are allowed.

No special arrangements are needed.
Answer the 3 questions each carrying an equal value.
Terms & Notations are as usual.

Use separate answer-scripts for Group A and Group B

GROUP – A

1. Explain if the Ratio estimator for a finite population total is a homogeneous linear estimator. Can it be exactly unbiased for a total ? Give reasons.

   OR

   Explain and illustrate, giving necessary proofs, contributions of Basu, Godambe, Hanurav towards UMV estimation of a finite population total.

2. Given raw survey data, derive, giving requisite proofs, a minimal sufficient statistic.

   OR

   Derive, giving proofs, an exact expression for the variance of a ratio estimator, unbiased for a total, in terms of the parameters of a design on which it is based. How will you unbiasedly estimate this variance ?

3. Show how may you rank the Hansen-Hurwitz, Des Raj & Symmetrized Des Raj estimators for a finite population total in terms of relative efficiencies in comparable circumstances ?

   OR

   For a PPSWR sampling in n draws how many distinct units may you expect to realize ? For this scheme work out the inclusion probabilities of the first two orders and hence give a formula for the variance of the number of distinct sample units

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INDIAN STATISTICAL INSTITUTE
KOLKATA
Midsemestral Examination : M. Stat. I – Sem I of 2010-11
Group B
Design of Experiments

Date : September 6, 2010 **Full Marks: 20** **Time: 1.5 hours**

Note : Answer any TWO questions. Marks allotted to a question are indicated in brackets [ ] at the end.

1. (a) Give the two definitions of connectedness of a block design, and prove their equivalence.
(b) State and prove a necessary and sufficient condition for a connected block design to be orthogonal.
(c) Give an example of a block design that is orthogonal and disconnected, with proof of your claim.

[5+3+2 = 10]

2. (a) Given the following three MOLS of order four L_1, L_2 and L_3, construct an SBIBD with parameters $v=21$, $k=5$, $\lambda=1$:
Rows of L_1 = [A B C D, B A D C, C D A B, D C B A],
Rows of L_2 = [A B C D, C D A B, D C B A, B A D C],
Rows of L_3 = [A B C D, D C B A, B A D C, C D A B].
(b) Develop the ANOVA of a BIBD (v, b, r, k, λ).

[5+5 = 10]

3. (a) State and prove a necessary and sufficient condition for the variance balance of a connected block design.
(b) Define an Youden design with an example. Develop the ANOVA of an Youden design with parameters v, k, λ.

[3+(2+5) = 10]

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination : (2010-2011)

M.Stat. 1st Year

STATISTICAL INFERENCE I

Date: 8 September, 2010 Max. Marks: 80 Duration: 3 Hours

1. (a) Define admissible and extended Bayes rules.

(b) Let δ_0 be an equalizer rule. Show that it is minimax if any one of the following is true.
(i) δ_0 is admissible
(ii) δ_0 is extended Bayes. [4+4=8]

2. Consider a decision problem with a finite parameter space. Show that any admissible rule is Bayes with respect to some prior distribution. [7]

3. Let the parameter space Θ be the real line R and assume that the risk $R(\theta, \delta)$ is a continuous function of θ for all decision rules δ. Suppose that δ_0 is a Bayes rule with respect to a prior distribution. Under appropriate conditions on the prior, show that δ_0 is admissible. [7]

4. Let the action space $\mathcal{A}$ be a convex subset of R^k. Discuss on essential completeness of the class of nonrandomized decision rules in the class of behavioural decision rules under appropriate conditions (prove the assertions you make). [8]

5. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two independent random samples from two normal distributions $N(\theta_1, 1)$ and $N(\theta_2, 1)$ respectively. Find a minimax estimator of $\theta = \theta_1 - \theta_2$ for the squared error loss. [12]

6. Consider a decision problem with finite parameter space for which the risk set is bounded from below and closed from below. Show, **without** using the

Separating Hyperplane Theorem, that there exists an admissible minimax decision rule. [16]

7. (a) let X be a random observable following a distribution with a density involving a parameter θ. Our problem is to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ where θ_0 and θ_1 are two specified parameter values. Consider the usual 0-1 loss function and find the risk of a randomized test given by a test function ϕ. Show that the risk set S is a convex subset of R^2.

(b) Consider the set up of Question 7(a) with $X \sim U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$, $\theta_0 = \frac{1}{2}$ and $\theta_1 = 0$. It is given that an MP test of level α is given by the test function $\phi(x)$ which takes the value 1, if $-\frac{1}{2} < x \leq 0$, the value 0, if $\frac{1}{2} \leq x < 1$ and the value 2α, if $0 < x < \frac{1}{2}$. Sketch the risk set S for this testing problem (hints: use the fact that S is convex, contains the point (1,0) and is symmetric about the point $(\frac{1}{2}, \frac{1}{2})$).

Also find the minimax decision rule and the least favourable prior distribution.

[(4+4)+(7+3+4)=22]

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination : Semester I (2010-2011)

M. Stat Ist Year

Linear Models and Markov Chain

Date: 10. 09. 10 Maximum marks: 40 Time: 2 hours.

Note: Answer all questions. This question paper carries 45 points. Maximum you can score is 40. You may use any results proved in the class by stating the results clearly. If you use other results not discussed in class, they need to be proved.

1. (a) Show that $X(X'X)^-$ is a g-inverse of X'. [5 points]

   (b) Suppose $\beta^* = X^- P_X Y$ where P_X is the projection matrix onto the column space of X and X^- is a g-inverse of X. Show that $P'\beta^*$ is BLUE for every estimable function $P'\beta$. [5 points]

2. Suppose $H = (X'X)^-(X'X)$ for some g-inverse $(X'X)^-$ of $(X'X)$. In the Gauss-Markoff setup, show that each of the following two conditions is necessary and sufficient for $A\beta$ to be estimable for some matrix A.

   (a) $AH = A$. [5 points]

   (b) $AX^-X = A$, for some g-inverse X^- of X. [5 points]

3. Prove that a linear function of the response is the BLUE of its expectation iff it is uncorrelated with every linear zero function. [8 points]

4. Let y_1, y_2, y_3, y_4 be uncorrelated observations with common variance σ^2 and expectations given by:

$$E(y_1) = E(y_2) = \beta_1 + \beta_2 + \beta_3, \qquad E(y_3) = E(y_4) = \beta_1 - \beta_2.$$

   (a) Show that $p'\beta$ is estimable iff $p_1 + p_2 = 2p_3$. [3 points]

   (b) Show that $\theta = 3\beta_1 + \beta_2 + 2\beta_3$ is estimable. Find its BLUE and the variance of the BLUE. [6 points]

   (c) Evaluate the residual sum of squares, and an unbiased estimate of σ^2. [4 points]

   (d) Assuming that the observations follow normal distribution, evaluate an appropriate test-statistic for testing $\theta = \theta_0$. [4 points]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination : Semester I (2010-11)

M. Stat. I Year

Applied Stochastic Processes

Date: 10.9.2010 Maximum marks: 40 Time: 2 hours

The total mark is 44

1. Customers arrive at a service counter according to a Poisson process with rate λ till time t_0 (known) after which, as the reputation of the service grows or worsens, the rate changes to $\lambda + \alpha$. Based on continuous monitoring data on the arrival of customers during the period $(0, \tau]$ with $\tau > t_0$, obtain maximum likelihood estimates of λ and α. If t_0 is also an unknown parameter, how do you propose to estimate t_0 along with λ and α? [6+4=10]

2. Describe linear birth and death process. Write down the differential equation for the corresponding probability generating function (pgf) and then solve for it. [2+4+4+2=12]

3. A telephone switchboard has s lines. Telephone calls arrive according to a Poisson process with rate λ and the call holding times are independent exponential random variables with mean $1/\mu$. If all the s lines are busy, no further call is entertained.

   Write this as a suitable birth and death process, specifying the birth and death rates. and write down the corresponding equilibrium probabilities. [2+3=5]

4. In case of some population, one can observe the number of deaths until time t in addition to the current population size. Suppose the population grows according to a linear birth and death process with rates λ and μ, respectively, starting with one individual at time 0. Find the expected number of deaths until time t. Find the distribution of the time of first death as explicitly as possible. [6+2=8]

5. (a) Obtain the expected time of extinction for a linear death process starting with N individuals at time 0.

   (b) Consider a two-state Markov process with α and β being the rates of $0 \to 1$ and $1 \to 0$ transitions, respectively. Derive the probability that the process is in state 1 at time t starting in state 0 at time 0.

   (c) Describe an algorithm for simulating a homogeneous planar Poisson process.

   [$3 \times 3 = 9$]

INDIAN STATISTICAL INSTITUTE
Mid-Semester (Supplementary) Examination: 2010-11

M. Stat. I Year
Sample Surveys and Design of Experiments

Date: 24.09.10 **Maximum Marks: 20** **Duration: $1\frac{1}{2}$ 1Hours**

GROUP - A

No Books/external equipments allowed

Answer any 2 questions bearing 10 marks each.

1. Discuss why every unit of a population must be assigned a positive inclusion probability in deriving an unbiased estimator for the population total.

2. Derive Hartley and Ross's unbiased ratio-type estimator for a finite population total based on a simple Random sample selected without replacement. How will you find an unbiased variance estimator for it ?

3. Explain the scheme of systematic sample selection with probability proportional to measures of size. How will you unbiasedly estimate a population total on choosing such a sample ? Can you always unbiasedly estimate the variance of this estimator ? If so, how ?

4. Derive Des Raj's estimator for a finite population total available in a context to be explained by you. Show how you may derive an estimator better than it from the same sample.

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INDIAN STATISTICAL INSTITUTE
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Supplementary Mid-Sem Examination, Semester I, 20010 - 11

M Stat I

Sample Surveys & Design of Experiments

Date : September 24, 2010 **Time : 1 and half hours** **Full Marks : 20**

GROUP B
(Design of Experiments)

Note : Answer both the Qusetions.

1. (a) Develop the analysis of variance of a general block design, indicating clearly how the various sums of squares are to be computed.
   (b) Define variance balance of a connected block design and show that the constancy of the off-diagonal elements of its C matrix is a necessary and sufficient condition for the variance balance to hold
   (c) Give an example of a block design that is (i) disconnected, and orthogonal; and (ii) connected and variance balanced.

   [6 + 4 + 2 =12]

2. (a) Give a method of construction of the BIBD's having the parameters $v=s^2+s+1=b$, $r=s+1=k$, $\lambda=1$, showing clearly that the method actually leads to the parameters shown.
   (b) Show that for a BIBD(v,b,r,k,λ), $b \geq v$, equality holding if and only if every pair of blocks intersects in λ common treatments.

   [4 + 4 = 8]

M. Stat First Year Question Paper

II Semester Supplementary Examination (2009-10)

Measure Theoretic Probability

Date : 28/09/2010 Maximum Score : 50 Time : 2 hrs.

1. Ω is a non-empty set and £ is a collection of subsets of Ω. Consider the following two sets of conditions on £:

(I) (1) $\Omega \in$ £, (2) A, B $\in$ £, $A \subset B$ imply $B - A \in$ £ and, (3) $A_n \in$ £, $A_n \subset A_{n+1}$ for each n $\Rightarrow \bigcup_{n\geq 1} A_n \in$ £

(II) (1) $\Omega \in$ £, (2) A $\in$ £ $\Rightarrow A^c \in$ £ and, (3) For each n, $A_n \in$ £, $A_n \bigcap A_m = \emptyset$ for $m \neq n$ imply $\bigcup_{n\geq 1} A_n \in$ £.

Show that the two sets of conditions are equivalent (i.e., (I) $\Rightarrow$ (II) & (II) $\Rightarrow$ (I)).

[12 + 13 = 25]

2. Consider the set O of all open intervals of reals, bounded or unbounded. Is it a monotone class? Justify your answer.

Prove or disprove the following statements.
(a) A semi-field which is also a monotone class is a field.
(b) A semi-field which is also a λ-system is a field.

[3 x 5 = 15]

3. Let S be a fixed non-empty proper subset of Ω. A subset $A \subset \Omega$ is said to <u>split</u> S if the sets $A \bigcap S$ and $A^c \bigcap S$ are both non-empty. Clearly, a set A does not split S if and only if A either contains S or is disjoint from S.

Show that if $\mathcal{G}$ is a class of subsets of Ω, such that none of the sets in $\mathcal{G}$ *splits S*, then none of the sets in $\sigma(\mathcal{G})$, the σ-field generated by $\mathcal{G}$, can split S either. (Hint : Use 'Good-Set' principle).

[10]

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INDIAN STATISTICAL INSTITUTE

M. Stat. I yr. (NB Stream)

Suppl. Mid-Semester Examination

Subject : Real Analysis   Date : 28/09/10

Full marks : 40   Time : 1 hr. 30 mins.

Let $f: R \to R$. Show that $\lim_{x \to a} f(x) = l$ iff $\forall$ sequences $\{x_n\}$ with $x_n \to a$ (and $x_n \neq a, \forall$ we have $f(x_n) \to l$. [7]

Find Supremum, infimum, closure and interior of the following sets.

a) $(-1, 1]$ b) Set of all integers

c) Set of all rational numbers in $(0,1)$ d) $(2, \infty)$

[2 × 4]

a) Show that set of all rational numbers is countable

b) $A \subset R$. Show that set of all isolated points of A is a countable set.

c) For $A \subset R$, show that $\partial A = \partial A^c$.

d) Let $a_{n+2} = (a_{n+1} + a_n)/2$ for all $n \geq 1$. Show that $a_n \longrightarrow (a_1 + 2a_2)/3$

e) If $0 < x_1 < 1$ and if $x_{n+1} = 1 - \sqrt{1 - x_n}$ for all $n \geq 1$, prove that limit of x_n exists and equals 0.

[5 × 5]

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INDIAN STATISTICAL INSTITUTE
First Semester Examination: 2010-11
M. Stat. I Year (NB – Stream)
Real Analysis

Date: 19.11.2010 Maximum Marks: 100 Duration: 3 Hours

The paper contains questions of 125 marks. You may attempt all questions. Maximum marks you can score is 100.

1.(a) Let $f:[a,b]\to R$ be bounded. Show that $\int_a^b f(x)dx$ exists iff $S(U,P_n,f)-S(L,P_n,f)\to 0$ as $n\to\infty$ where P_n is a partition of $[a,b]$ given by $P_n=\left\{a,\ a+\frac{b-a}{n},\ a+2\frac{b-a}{n},\cdots\cdots,\ a+(n-1)\frac{b-a}{n},\ b\right\}$

(b) Using the above in question (a) show that $\int_0^{\pi/2}\sin x\,dx$ exists.

[12+5]

2. Show that $\int_a^b f(x)dx$ exists if f is a continuous function defined on $[a,b]$.

[10]

3.(a) Let A be a compact subset of $[0,\infty)$. Define $B=\left\{\frac{4+x}{u}: x\in A \text{ and } u\geq 1\right\}$. Is B compact? Justify your answer.

(b) Let $C=\left\{x: x \text{ is irrational and } x\in\left[-\sqrt{2},+\sqrt{2}\right]\right\}$. Is C compact? Justify your answer.

[5+5]

4.(a) State and prove Taylor's Theorem.

(b) Derive infinite power series expansion of $\cos x$ around $x=0$, using Taylor's Theorem.

[15+5]

5. Let $g(\underline{x})=x_1^3+x_2^3$ and $f(\underline{x})=x_1+x_2$, $\forall\underline{x}=(x_1,x_2)\in R^2$. Let $\{\underline{x}_n\}$ be a sequence in R^2. Show that $g(\underline{x}_n)\to 0$ implies $f(\underline{x}_n)\to 0$.

[8]

6. Answer any <u>three</u> questions.

(a) Let $0<x_1<1$ and $x_{n+1}=1-\sqrt{1-x_n}$ for all $n\geq 1$. Show that x_n is a decreasing sequence with limit 0. Also show that $\frac{x_{n+1}}{x_n}\to\frac{1}{2}$.

[P.T.O.]

(b) Let $a_{n+2} = (a_{n+1} + a_n)/2$ for all $n \geq 1$. Show that $a_n \to (a_1 + 2a_2)/3$.

(c) Let f be a continuous function on $[a,b]$ and $f(x) = 0$ when x is rational. Show that $f(x) = 0$ for all x in $[a,b]$.

(d) Find lim sup and lim inf of the following sequences.

(i) $\dfrac{n+(\log n)^2}{1+(-1)^n n\sqrt{n}}$ (ii) $(-1)^n n^{-\frac{1}{n}}$

[6×3]

7.(a) Consider the series $a_0 + a_1 + a_2 + a_3 + \cdots\cdots$. Show that it is convergent if $\limsup |a_n|^{\frac{1}{n}} < 1$ and it is divergent if $\limsup\ |a_n|^{\frac{1}{n}} > 1$.

(b) Use the above in question (a) to derive the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$.

[8+4]

8.(a) Let $f(\underline{x}) = \exp(1 + x_1 + 2x_2)$ where $\underline{x} = (x_1, x_2) \in R^2$. Find the total derivative of f at $\underline{x} = (0,0)$.

(b) In the above in question (a), find the directional derivative of f at $(0,0)$ in the direction (p,q).

[5+5]

9.(a) Let $A \subseteq R$.
Show that the set of all isolated points of A is a countable set and the derived set of A is a closed set.

(b) Show that any interval (a,b) in R contains infinitely many algebraic numbers.

(c) Give example of a bounded function on $[0,1]$ with countably infinite number of discontinuities. Can f be Riemann integrable?

[10+5+5]

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MSTAT I - Measure Theoretic Probability

Final Exam. / Semester I, 2010-11

Time - 3 hours/ Maximum Score - 50

19/11/10

NOTE : SHOW ALL YOUR WORK TO GET FULL CREDIT. RESULTS USED MUST BE CLEARLY STATED.

1. (6+6=12 marks)

   Suppose f be a Borel-measurable function and μ be a σ-finite measure.

   (a) Let $\int_\Omega f d\mu$ exists and let $\{B_n\}$ be a sequence of disjoint sets. Show that $\sum_{n=1}^{\infty} \int_{B_n} f d\mu$ is either an absolutely convergent series or it diverges to $+\infty$ (or $-\infty$).

   (b) Let f be defined on reals and μ be the Lebesgue measure. If f is Riemann integrable then show that the set of continuity points of f is Borel measurable and its complement has measure zero.

2. (5+5=10 marks)

   (a) Show that for any Borel measurable set $A \subset \mathcal{R}^2$, the map, $x \mapsto \lambda_1(A^x)$ is a Borel-measurable function. where λ_1 is the one-dimensional Lebesgue measure and $A^x = \{y : (x, y) \in A\}$.

   (b) If $A \subset \mathcal{R}^2$, is a Lebesgue measurable would the map $x \mapsto \lambda_1(A^x)$ be a Lebesuge-measurable function? Justify your answer.

3. (4+4+4=12 marks)

   Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables and let $S_n = X_1 + \cdots + X_n$. Decide for the following cases whether (i) S_n converge almost surely, as $n \to \infty$;
   (ii) $(S_n - E(S_n))/n$ converge almost surely, as $n \to \infty$ whenever $E(S_n)$ is finite for each n;
   (iii) $(S_n - E(S_n))/\sqrt{Var(S_n)}$ converge to $N(0,1)$ in distribution, as $n \to \infty$, whenever $E(S_n)$ and $Var(S_n)$ are finite for each n.

   (a) Let $P(X_n = n) = \frac{1}{2n^2}(1 - \frac{1}{a}) = P(X_n = -n)$, $P(X_n = 0) = 1 - \frac{1}{n^2}(1 - \frac{1}{a})$ for some constant $a > 1$.

   (b) Let $\{X_n\}$ be a Poisson random variable with parameter $\lambda_n > 0$, such that, $\sum_{n\geq 1} \lambda_n < \infty$.

(c) X_n has the probability density function

$$f_n(x) = \begin{cases} \frac{1}{2c_n} & \text{if } x \in (-c_n, c_n), \\ 0 & \text{otherwise,} \end{cases}$$

where c_n's are nonzero and uniformly bounded constants with $\sum_{n\geq 1} c_n^2 = \infty$.

4. (4+4+4+4+4=20 marks)

Write TRUE or FALSE and justify by proving or disproving it.

(a) Let $\mathcal{A}$ be a σ-field which is countably generated. Then there is a countable field $\mathcal{F}$ which generates $\mathcal{A}$.

(b) Let $\{X_n\}$ be an i.i.d. sequence of Cauchy random variable with median zero. Let $S_n = |X_1| + \cdots + |X_n|$. Then $S_n/n \to \infty$ with probability one, as $n \to \infty$.

(c) Let $\{X_n\}$ be a sequence of random variables such that $X_n \to X$ in probability, as $n \to \infty$. Assume further that, $\sup_n E(X_n^2) < \infty$. Then, as $n \to \infty$, $X_n \to X$ in L_1.

(d) Let $\{f_n\}_{n\geq 1}$, f be Borel measurable functions on the real line and μ be the Lebesgue measure. $f_n \to f$ in measure and also $f_n \to f$ almost everywhere $[\mu]$, as $n \to \infty$. Then $f_n \to f$ almost uniformly, with respect to μ as $n \to \infty$.

(e) Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let X be a random variable on this space with finite expectation. If for some reals a, b, and for all $A \in \mathcal{A}$, $a \leq \int_A X dP \leq b$ then $a \leq X \leq b$, almost surely $[P]$.

All the best.

INDIAN STATISTICAL INSTITUTE

First- Semester Examination: 2010-2011
M. Stat. I Year
Sample Surveys and Design of Experiments

Date: 23.11.2010 **Duration: 3 hours**

Answer Part A and Part B in Separate Answer Books.

Part A
(Sample Surveys)

Maximum Marks: 20 Duration: 1.30 hours

No external equipments will be allowed or provided.

Answer any 2 questions. Each question carries 10 marks.

1. Suppose n villages out of N villages in a district are selected with replacement with probabilities proportional to their known numbers of households. Every time a village is selected it is independently surveyed on taking a circular systematic sample of 10 households contained in it determining the numbers of graduate household members in each household surveyed.

   How will you unbiasedly estimate (i) the total current number of people in the district graduated and (ii) Variance of your estimator in (i)?

2. A sample of 15 people is taken from 50 persons assembled in a big hall following Rao, Hartley and Cochran's scheme of selection without replacement using their known ages as size measures. How will you unbiasedly estimate the proportion of those 50 people addicted to gambling on asking them questions by 2 suitable randomized response (RR) techniques ? How will you compare their respective levels of accuracy? Derive specific results for estimation.

P.T.O.

3. Check if for Lahiri-Sen-Midzuno Scheme of sampling the unbiased estimator, by Yates & Grundy's method, of the variance of the Horvitz & Thompson's estimator of a finite population total is uniformly non-negative. Apply usual notations.

4. What are post-strata? Describe how to derive an unbiased estimator for a finite population mean by post stratified sampling. Give an alternative estimation procedure. Indicate how you may judge their respective merits and demerits, on deriving appropriate results.

N.B. Assignment Records to be submitted on the date of Semestral Examination carry 10 marks.

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INDIAN STATISTICAL INSTITUTE
203 B. T. ROAD
KOLKATA 700108
First Semestral Examination, 2010 - 11
M Stat I
Sample Surveys & Design of Experiments

Date : Time : 1 and 1/2 hours Full Marks : 30

GROUP B
(Design of Experiments)

Note : Answer Question No. 1, and any ONE from the rest. Marks allotted to a question are indicated in [ ] at its end.

1. (a) Give the rank and the structural definitions of connectedness of a block design, and prove their equivalence.

(b) Suppose D is BIBD with $\lambda = 1$, and D* is another block design obtained from D by deleting one of its blocks. Examine if D* is variance balanced or not. Justify your answer, stating clearly the results that you need to use.

(c) Construct a one-fourth fraction of a 2^6 factorial experiment, so that only four-factor or higher order interactions are taken in the identity group of interactions defining the fraction. Examine if this is an orthogonal array of strength three or not. Prove your claim.

[4+5+(3+3)=15]

2. (a) Prove that the constancy of all the off-diagonal elements of the C matrix of a connected block design is a necessary and sufficient condition for its variance balance.

(b) Show that for a BIBD (v,b,r,k,λ), $\lambda(v-1) = r(k-1)$, and that $b \geq v$, equality holding if and only if every pair of blocks intersects in λ common treatments.

(c) Give the ANOVA of a one-fourth fraction of a 2^5 factorial experiment with I=ABD=ACE=BCDE as the identity group of interactions, clearly indicating how the various sums of squares are to be computed, and stating clearly the assumptions that you make about the negligible effects.

[4+5+6=15]

3. (a) Define the main effects and interactions in an $(1/s^f)$ fraction of an s^n factorial experiment, where s is a prime or a prime power. Show how two or more interactions in such experiments may become "aliases" of one another.

(b) Show that the $(s^{n-f} - 1)$ d.f. associated with an $(1/s^f)$ fraction of an s^n factorial experiment, where s is a prime or a prime power, are represented by $(s^{n-f} - 1)/(s-1)$ distinct alias sets, each containing s^f distinct factorial effects of the s^n factorial experiment.

(c) Construct an orthogonal array for three symbols having 13 factors, 27 runs and strength two. (You may give only the independent treatment combinations of the array and indicate how the remaining treatment combinations are to be generated.)

[4 + 6 + 5 = 15]

INDIAN STATISTICAL INSTITUTE
First Semester Examination: 2010-11
M. Stat. I Year
Statistical Inference I

Date: 26.11.2010 Maximum Marks: 60 Duration: $3\frac{1}{2}$ Hours

Group – A
Answer all questions.

1. Consider a statistical decision problem where the parameter space and the action space are both equal to $\mathbb{R}$ and the loss function is proportional to squared error. Prove that the nonrandomized decision rules based on a given sufficient statistic T form an essentially complete class.

[8]

2. Let $X_1,\ldots\ldots,X_n$ be a random sample from a two-parameter exponential family of distributions with a joint density of the form

$$f\left(\underset{\sim}{x}\middle|\mu,\sigma\right)=C(\mu,\sigma)h\left(\underset{\sim}{x}\right)\exp\left\{\mu T\left(\underset{\sim}{x}\right)+\sigma\, S\left(\underset{\sim}{x}\right)\right\}$$

where μ and σ are unknown real parameters. Consider the problem of testing $H_0:\mu\le\mu_0$ against $H_1:\mu>\mu_0$ where μ_0 is some specified value of μ.

(a) Define similar tests and tests having Neyman structure in the context of the above problem.

(b) Show that a UMP α-similar test of level α is also UMP unbiased of level α.

(c) Using appropriate results for one-parameter exponential family of distributions find a UMP unbiased test of level $\alpha\,(0<\alpha<1)$ for this problem.

[2+3+9=14]

3.(a) Consider observations from a location-scale family of densities with a location parameter $\theta\in\mathbb{R}$ and a scale parameter $\sigma>0$. Describe how one can find the minimum risk equivariant (MRE) estimator of $\sigma^r, r>0$, under the usual location-scale transformations for an appropriate loss function. Use the results on MRE estimators for the location family and scale family of densities.

P.T.O.

(b) Let $X_1, \ldots\ldots, X_n$ be i.i.d. observations with a common $N\left(\mu, \sigma^2\right)$ distribution where μ and σ^2 are both unknown. Find the MRE estimator of σ^2 with loss function $L(\sigma, t) = \left(t - \sigma^2\right)^2 / \sigma^4$.

(c) Consider a location parameter family of densities. Show that if a UMVU estimator of the location parameter exists and is equivariant, it is also MRE for squared error loss.

[6+4+4=14]

4. Discuss on the differences between the Bayesian paradigm and classical inference with respect to evaluation of performance of a decision rule.

[4]

Group – B

Answer as many questions as you can.
Maximum you can score is 20.

5. Let $\underset{\sim}{X}$ follow $N_p\left(\underset{\sim}{\theta}, I\right)$. Extending the idea from the univariate case $(p=1)$, show that $\underset{\sim}{X}$ is a minimax estimator of $\underset{\sim}{\theta}$ for the loss function $L\left(\underset{\sim}{\theta}, a\right) = \sum_{i=1}^{p} (\theta_i - a_i)^2$.

[9]

6. Suppose that for a statistical decision problem, the upper value is equal to the lower value, i.e. (with usual notations)

$$\sup_{\pi} \inf_{\delta} r(\pi, \delta) = \inf_{\delta} \sup_{\pi} r(\pi, \delta).$$

Show that any minimax rule is extended Bayes.

[3]

7. Let X_1 and X_2 be independent with a common density of the form $f(x-\theta)$ where $f(\cdot)$ is a known density symmetric about zero and θ is an unknown (location) parameter. Show that $(X_1 + X_2)/2$ is MRE estimator of θ for squared error loss.

[5]

8. Let $X_1, \ldots\ldots, X_n$ be a random sample from a uniform distribution $U(0,\theta), \theta > 0$. Consider the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. Show that the test that rejects H_0 when $\max(X_1, \ldots\ldots, X_n) > \theta_0$ or $\max(X_1, \ldots\ldots, X_n) \leq \theta_0 \alpha^{1/n}$, and accepts H_0 otherwise, is UMP level α $(0 < \alpha < 1)$.

[8]

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Indian Statistical Institute
First Semestral Examination
2010-2011 Academic Year
M.Stat. First Year (B-Stream and NB-Stream combined)
Large Sample Statistical Methods

Maximum Marks: 60 Date: 30.11.10 Duration :- 3 hours

Answer as many questions as you can. The maximum you can score is 60

1. (a) Suppose $X_1, \ldots, X_n$ are iid N(0,1). Find the asymptotic distribution of

$$\frac{n(X_1X_2 + X_3X_4 + \ldots + X_{2n-1}X_{2n})^2}{(\sum_{i=1}^{2n} X_i^2)^2}.$$

[6]

(b) Show that for any sequence $\{X_n\}_{n\geq 1}$ of random variables on a common probability space, there exists a sequence $\{c_n\}$ of strictly positive real numbers such that $c_nX_n \to 0$ almost surely. [6]

2. (a) Suppose $X_1, X_2, \ldots$ are iid Poisson(λ), $\lambda > 0$. Find suitable constants a_n and b_n (allowed to be dependent on λ also) such that $a_n(Y_n - b_n)$ converges to a non-degenerate limit where $Y_n = (1 - \frac{1}{n})^{n\bar{X}_n}$ and $\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}$, for $n \geq 1$. Prove your answer. [6]

(b) Let $\{X_n\}$ be a sequence of iid observations from a population having finite fourth moment. Show that after appropriate renormalization the sample mean and sample variance are asymptotically independent if the population skewness is zero. [7]

3. (a) Let $X_1, \ldots, X_n$ be iid normal with unknown mean μ and variance 1, where μ can take only integer values. Find the maximum likelihood estimator $\hat{\mu}_n$ of μ. Prove that $P_\mu(\hat{\mu}_n = \mu$ for all sufficiently large $n) = 1$. [5]

(b) Give an example of an inconsistent maximum likelihood estimator and prove that it is indeed inconsistent. [5]

(c) Let $X_1, \ldots, X_n$ be iid with common density $f(x, \theta)$ where $\theta \in \Theta$, Θ being an open interval of the real line. Stating appropriate regularity assumptions, show that if there is a unique solution of the likelihood equation for all n and all sample values $(x_1, \ldots, x_n)$, then with probability tending to one (under the true value of the parameter) as $n \to \infty$, this unique solution is the global maximum of the likelihood function. [5]

(d) Suppose $X_1, \ldots, X_n$ are iid with common density $f(x, \theta)$, where $\theta \in \Theta$, Θ consisting of only finitely many real numbers. Assume also that density under each θ has the same support and that the distributions under different θ's are different. If θ_0 is the true value of θ, prove that with probability tending to 1 (under θ_0) as $n \to \infty$, the likelihood will be maximized at the

P.T.O.

value $\theta = \theta_0$. [7]

4. (a) Listing appropriate assumptions, state Hoeffding's result on asymptotic normality of one-sample U-statistic. [2]

(b) Suppose $X_1, \ldots, X_n$ are iid having a continuous distribution which is symmetric about its (unknown) median θ. Derive the asymptotic null distribution of the Wilcoxon Signed-Rank test statistic for testing $H_0 : \theta = 0$ against $H_1 : \theta > 0$. [7]

5. Suppose $X_1, \ldots, X_n$ are iid having density $f(x, \theta)$, where $\theta \in \mathbf{R}$. Stating appropriate regularity assumptions, derive the asymptotic null distribution of the likelihood ratio test statistic for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, where θ_0 is a given constant. [10]

INDIAN STATISTICAL INSTITUTE

First Semestral Examination : (2010-2011)

M. Stat Ist Year (NB - stream)

Markov Chain and Linear Models

Date: 3.12. 10 — Maximum marks: 100 — Time: 3 hours.

Note: Answer all questions. This paper carries 110 marks. Maximum you can score is 100.
Answer Group A and Group B questions in separate answerscripts.

Group A: Markov Chain

1. Consider a discrete time Markov Chain with state space $S = \{0,1,2,3,4,5\}$, and the transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

   (a) Determine which states are recurrent and which states are transient. [5]

   (b) Find the absorption probabilities to different classes for each of the transient states. [6]

   (c) Find the mean time spent in each of the transient states. [6]

   (d) Find the stationary distribution concentrated on each of the irreducible closed sets. [8]

2. Consider an irreducible birth and death chain on the non-negative integers.

   (a) Find a necessary and sufficient condition for the chain to be transient. [10]

   (b) Find a necessary and sufficient condition for the chain to be positive recurrent. [12]

   (c) Find a necessary and sufficient condition for the chain to be null recurrent. [3]

3. (a) For a branching chain, let $p_j, j = 0, 1, \ldots$ be the probability of j offsprings in the following generation where $p_0 < 1$. Let ρ be the probability of extinction. Show that ρ satisfies the equation (in s)

$$s = \sum_{j=0}^{\infty} s^j p_j,$$

   and ρ is the smallest positive number to satisfy this equation. [10]

   (b) Consider a branching chain with $p_0 = p_3 = \frac{1}{2}$. Find ρ. [10]

[P.T.O]

Group B: Linear Models

4. Let y_1, y_2, y_3, y_4 be uncorrelated observations with common variance σ^2 and expectations given by:

$$E(y_1) = \beta_1 + \beta_2 + \beta_3 + \beta_4,$$

$$E(y_2) = \beta_1 + \beta_2 - \beta_3 + \beta_4,$$

$$E(y_3) = \beta_1 - \beta_2 + \beta_3,$$

$$E(y_4) = \beta_1 - \beta_2 - \beta_3.$$

(a) Show that $p'\beta$ is estimable iff $p_1 + p_2 = 2p_4$, where $p = (p_1, p_2, p_3, p_4)'$. [2]

(b) Show that $\theta = 2\beta_1 + \beta_3 + \beta_4$ is estimable. Find its BLUE and the variance of the BLUE. [2 + 4 + 4]

(c) Find the residual sum of squares and an unbiased estimate of σ^2 based on it. [4]

(d) Assuming that the observations follow normal distribution, find an appropriate test-statistic for testing the hypothesis $\theta = \theta_0$. [4]

5. (a) In the Gauss-Markoff setup, prove that a parametric function $p'\beta$ is estimable if and only if $p'\hat{\beta}$ is unique where $\hat{\beta}$ is a solution of the least squares equations. [10]

(b) Let β^* be such that for every estimable $p'\beta$, the BLUE is $p'\beta^*$. Then show that β^* must be a least square estimate of β. [10]

INDIAN STATISTICAL INSTITUTE

Semestral Examination : Semester I (2010-11)

M. STAT. I Year

Applied Stochastic Processes

Date: 03.12.2010 Maximum marks: 100 Time: $3\frac{1}{2}$ hours

This examination is open notes. Books cannot be used and notes cannot be exchanged. Refer to your notes properly, but do not reproduce derivations from there. Answer as many as you can. Total mark is 105.

1. Consider a linear death process $X(t)$ with death rate μ per individual, starting with N individuals at time 0.

   (a) Obtain the differential equation for the probability generating function of $X(t)$ and solve for it. Hence, or otherwise, find the distributions of $X(t)$ (for a fixed t) and T, the time of extinction. Also, derive $E[X(t)]$ and $E[T]$.

   (b) Prove or disprove if the times between successive deaths form a renewal process.

   (c) If immigration with rate λ is allowed to this process, then obtain the limiting distribution of the resulting process.

   (d) Suppose this process with immigration is observed from time 0 to τ. Describe the data and obtain maximum likelihood estimates of λ and μ.

   [(2+3+3+3+1+1)+4+4+(2+4)=27]

2. Consider a system with M machines, R repairmen ($R < M$) and S spare machines. The machines operate independently, each with *exponential*(λ) failure time. The repairmen act independently, each taking *exponential*(μ) repair time. The repair times and the failure times of the machines are independent. After failure, a machine goes for repair and the repair work starts immediately if a repairman is free, otherwise waits for repair. A spare, if available, replaces the failed machine immediately. After completion of repair, a machine joins operation immediately, if needed, otherwise becomes a spare. Let $X(t)$ denote the number of non-operative machines, under or awaiting repair, at time t. Write this as a birth and death process by specifying the birth and death rates. [6]

3. (a) Explain why, in the context of a renewal process, the total life time is not the same as renewal time.

   (b) Find the limiting distribution of the total life at time t for a renewal process with renewal distribution F. Prove that the mean of this limiting distribution is greater than or equal to the mean renewal time.

   (c) Suppose the life time of an item is uniformly distributed over the interval $(0, 10]$ and it is subject to age replacement at age $T \in (0, 10]$. Given that the cost of a failure replacement is twice the cost of a planned replacement, obtain the optimum replacement time T by minimizing the long run mean replacement cost.

   [3+(10+2)+10=25]

P.T.O.

4. Let $\{N(A) : A \subset \mathcal{R}\}$ be a Poisson process with intensity $\lambda(x) = \Lambda/2$, for $-1 \leq x \leq 1$, and 0, otherwise, where Λ is a constant. A second point process $\{M(B) : B \subset \mathcal{R}\}$ is created by randomly and independently translating each point x of N to a point y of M as $y = x + \epsilon$, where ϵ is a standard normal variate. Obtain the distribution for the number of points of M in a set B. Out of n input points, what is the probability that there are m points of M in the set B? [5+3=8]

5. Consider a computer system during peak load when a large unlimited number of jobs are waiting to be processed. Assume that the required processing time of a job is exponentially distributed with mean $1/\mu$. The processing times for different jobs are independent. Further assume that an adjustment time per job is required (before the job starts) by the system to perform overhead functions to be ready for the processing. The adjustment times for different jobs are independent (and also independent of the respective job processing time) and are exponentially distributed with mean $1/\lambda$. At time 0, an adjustment time starts to take up the first job-processing. Consider the point process defined by the point events of successive jobs being over. Write this as a self-exciting point process by specifying the intensity process. For fixed and unknown adjustment time τ per job and based on an observed path until time T, find maximum likelihood estimates of μ and τ. [4+6=10]

6. (a) A rare mutation takes place in one normal cell with probability $\nu(t)\Delta t + o(\Delta t)$ in the time interval $(t, t + \Delta t)$ during division (that is, during mutation a normal cell divides into one normal cell and one mutant cell). Probability of two or more mutations in $(t, t + \Delta t)$ is $o(\Delta t)$. Normal cells do not divide otherwise and they act independently. Start with a body of S normal cells at time 0. Each mutant cell born in this way grows according to a linear birth and death process with rates λ and μ, respectively, per cell, thereby forming a mutant clone. Let $Y(t)$ denote the number of non-extinct mutant clones at time t. Derive the distribution of $Y(t)$.

   (b) Let S, as in (a) above, follows a $Gamma(\alpha, \beta)$ distribution. Let $N(t)$ denote the number of mutations by time t. Find $P[N(t) = n]$.

[6+6=12]

7. (a) Give an argument for the Little's formula $E(W) = E(Q)/\mu$, where μ is the mean service time for a customer, W is the waiting time for an arriving customer before entering into service and Q is the system size in equilibrium.

   (b) Consider an $M/M/1$ queue with arrival rate 10/hour and the service time with mean 4 minutes. Answer the following questions assuming the system to be in equilibrium.

      i. What is the probability of having a queue?

      ii. What is the average queue length?

      iii. What is the average time a customer spends in the system?

   (c) In a tandem queue system with K servers in series, Poisson arrivals and exponential service times, give an expression for the expected system size in equilibrium.

   (d) For a $M^{[X]}/M/1$ system with bulk arrivals in equilibrium, prove that the probability that the system is busy is equal to the corresponding traffic intensity.

[3+(2+3+3)+3+3=17]

INDIAN STATISTICAL INSTITUTE

Back Paper Examination : Semester I (2010-2011)

M. Stat Ist Year (NB - stream)

Markov Chain and Linear Models

Date: 4.1.11 Maximum marks: 100 Time: 3 hours.

Answer Group A and Group B questions in separate answerscripts.

Group A: Markov Chain

1. Consider a discrete time Markov Chain. Let i be a recurrent state and suppose that i leads to j. Let f_{ij} be the probability that starting from the state i the chain ever visits the state j. Show that,

   (a) $f_{ji} = 1$, and hence, j leads to i. [5]

   (b) j is recurrent. [5]

2. Consider a discrete time Markov Chain with state space S, transition matrix P and stationary distribution π.

   (a) Show that if $\pi_i > 0$ and i leads to j, then $\pi_j > 0$. [5]

   (b) Suppose that i and j are two states such that $P_{ki} = cP_{kj}$, for all $k \in S$. Show that, $\pi_i = c\pi_j$. [5]

3. Consider a Markov Chain on non-negative integers such that starting from $i, i \geq 0$, the chain goes to $i+1$ with probability p_i and goes to 0 with probability $1 - p_i$.

   Find necessary and sufficient conditions for

   (a) the chain to be irreducible. [3]

   (b) the chain to be recurrent. [7]

4. Consider the two-state Markov Chain with states 0 and 1. Let, $p_{01} = a, p_{10} = b, 0 < a, b < 1$.

   (a) Is 0 a recurrent state? Justify your answer. [5]

   (b) Show that, $(\pi_0, \pi_1) = (\frac{b}{a+b}, \frac{a}{a+b})$ is the unique stationary distribution. [5]

   (c) Calculate, $f_{00}^n, n \geq 1$. [4]

   (d) Calculate f_{00} to see whether 0 is a recurrent state. [2]

   (e) Find $m_0 = \sum_{n=1}^{\infty} n f_{00}^n$ and verify that $\pi_0 = \frac{1}{m_0}$. [4]

[P.T.O]

Group B: Linear Models

5. (a) If A is of full column rank, then show that G is a g-inverse of A if and only if G is a left-inverse of A. [5]

   (b) Show that $X(X'X)^{-}$ is a g-inverse of X'. [5]

6. Let y_1, y_2, y_3, y_4 be uncorrelated observations with common variance σ^2 and expectations given by:

$$E(y_1) = E(y_2) = \beta_1 + \beta_2 + \beta_3, \quad E(y_3) = E(y_4) = \beta_1 - \beta_2.$$

   (a) Show that $p'\beta$ is estimable iff $p_1 + p_2 = 2p_3$. [3]

   (b) Show that $\theta = 3\beta_1 + \beta_2 + 2\beta_3$ is estimable. Find its BLUE and the variance of the BLUE. [2 + 4 + 4]

   (c) Find the residual sum of squares and an unbiased estimate of σ^2 based on it. [3]

   (d) Assuming that the observations follow normal distribution, find an appropriate test-statistic for testing $\theta = \theta_0$. [4]

7. (a) In the Gauss-Markoff setup, prove that a linear function of the response is the BLUE of its expectation iff it is uncorrelated with every linear zero function. [10]

   (b) Suppose $\beta^* = X^{-}P_XY$ where P_X is the projection matrix onto the column space of X and X^{-} is a g-inverse of X. Show that $P'\beta^*$ is BLUE for every estimable function $P'\beta$. [10]

INDIAN STATISTICAL INSTITUTE

Semestral Examination (Backpaper) : Semester I (2010-11)

M. STAT: I Year

Applied Stochastic Processes

04·01·11

Date: ~~??.??.20~~11 Maximum marks: 100 Time: 3 hours

This examination is open notes. Books cannot be used and notes cannot be exchanged. Refer to your notes properly, but do not reproduce derivations from there.

1. State if the following statements are true or false giving suitable reasons.

   (a) A renewal process is Markovian.

   (b) For a renewal process with renewal distribution $F(\cdot)$, the limiting probability that there are odd number of renewals up to a time t is $1/2$.

   (c) If the intensity process of a point process $X(t)$ is $\lambda E[Y(t)]$, where $Y(t)$ is another independent linear birth and death process and $\lambda > 0$, then $X(t)$ is a doubly stochastic Poisson process.

   (d) A delayed renewal process with the first renewal time having density $\frac{x}{50}$, $0 \leq x \leq 10$, and the subsequent renewal times having density $\frac{1}{10}$, $0 \leq x \leq 10$, has the expected number of renewals $M_D(t)$, by time t (≤ 10), as $\frac{t}{5}$.

   [4 × 4 = 16]

2. Let $X(t)$ be a non-homogeneous Poisson process with rate $\lambda(t)$, $t > 0$.

   (a) Find the conditional distribution of $X(s)$, given $X(t)$, for $s < t$.

   (b) Prove that the inter-occurrence times do not form a renewal process.

   (c) Derive the conditional distribution of the occurrence times in $(0, T]$, given $X(T) = n > 0$.

   (d) Describe how to simulate this process from 0 to T, for a given $\lambda(t)$.

   [3+3+3+4=13]

3. Consider a Markov linear growth process with two sexes in which a female gives birth to either a male or a female offspring with rates λ_1 and λ_2, respectively, and dies with rate μ_1; males do not give birth but die with rate μ_2 per individual. Assume the individuals to act independently. Starting with a female at time 0, derive

   (a) the expected number of females at time t and the expected number of males ever born up to time t, and

   (b) the distribution of the time of first birth of a male child by specifying the corresponding hazard rate as explicitly as possible.

   [(4+6)+5=15]

P.T.O.

4. (a) For a Poisson process with rate λ, derive $P[\beta_t > z]$, where β_t is the total life at time t. Hence, prove that the mean total life time at time t is $\frac{1}{\lambda}(2 - e^{-\lambda t})$.

   (b) Find the limiting distribution of β_t, the total life at time t, for a renewal process with renewal distribution F. Prove that the mean of this limiting distribution is greater than or equal to the mean renewal time.

[(9+5)+((9+5)=28]

5. Consider the mutation model in which the normal cells grow according to a deterministic curve $X(t)$. Mutations take place from the pool of normal cells with rate $\mu(t)$ per normal cell to generate one primary mutant cell. Each primary mutant cell grows according to a non-homogeneous linear birth and death process, starting from the time of its creation, with rates $\alpha(t)$ and $\beta(t)$, respectively, to generate a mutant clone (collection of mutant cells originated from one primary mutant cell). Let $Y(t)$ denote the number of non-extinct mutant clones at time t. Identify the process $Y(t)$ and, hence, find its moment generating function and mean. [2+6+2=10]

6. (a) An $M/M/s$ system has all the servers busy and additional n more customers waiting in the queue. If the system shuts off the arrival stream at this time, what is the mean time to empty the system?

   (b) Consider an $M^{[X]}/M/1$ queue with bulk arrivals in which arrivals take place according to a Poisson process with rate λ but, at each arrival time, a random X number of customers arrive (i.e., bulk arrival). Let X have probability mass function $c_x = (1-p)p^{x-1}$, $0 < p < 1$, $x = 1, 2, \cdots$. The service rate is μ. Derive the probability generating function $\Pi(u)$ of the system size in equilibrium (Q) as a power series in u. Hence, obtain $\pi_n = P[Q = n]$.

[5+(10+3)=18]

Indian Statistical Institute
Backpaper
First Semester
2010-2011 Academic Year
M.Stat. First Year (B-Stream and NB-Stream combined)
Large Sample Statistical Methods

07/01/11

Maximum Marks: 100 Duration :- 4 hours

Answer all questions

1. State and prove the Skorohod Representation Theorem. [25]

2. State and prove the asymptotic representation theorem (due to J.K.Ghosh) of sample quantiles under iid sampling from a common distribution. [25]

3. Stating appropriate assumptions, derive in the context of iid sampling from a multinomial population with k classes, the asymptotic null distribution of the chi-square test statistic for the composite hypothesis testing problem $H_0 : \pi_i = \pi_i(\theta),\ i = 1, \ldots, k$, where $\pi_i(\theta),\ i = 1, \ldots, k$ are specified functions of θ, an unknown vector of dimension q, where $q < k - 1$. [30]

4. Stating appropriate assumptions, prove asymptotic normality of one-sample U-statistic based on iid observations from a common distribution. [20]

INDIAN STATISTICAL INSTITUTE
First Semester Supplementary Examination: 2010-11

M. Stat. I Year
Statistical Inference I

Date: 12.01.2011 **Maximum Marks : 100** **Duration: 3 Hours**

Answer all questions

1. Consider a decision problem with finite parameter space. Show that under suitable conditions on the prior distribution, a Bayes rule is admissible. Also show that any admissible rule is Bayes with respect to some prior distribution.

[6+8 =14]

2. Show that if a minimal complete class of decision rules exists, it consists exactly of all admissible rules.

[14]

3. Let $X_1,\ldots.X_n$ be a random sample from an $N\ (\theta,1)$ distribution. Consider the problem of estimating θ with squared error loss. Assuming that the risk function of every nonrandomized rule is continuous in θ, prove that the sample mean $\overline{X}$ is admissible.

[14]

4. a) Let X have a density of the form $C(\theta)h(x)e^{\theta T(x)}$ where $\theta \in \Theta$, an open interval in R. Fix some $\theta_0 \in \Theta$ and consider the problem of testing $H:\theta=\theta_0$ vs $K:\theta \neq \theta_0$. Suppose that there exists a test ϕ_0 of the form.

$$\begin{aligned}\phi_0(x)&=1, \text{ if } \quad T(x)<C_1 \text{ or } T(x)>C_2\\ &=0, \text{ if } \quad C_1<T(x)<C_2\end{aligned}$$

where $C_1<C_2$ are such that

$$E_{\theta_0}\phi_0(X)=\alpha \text{ and } E_{\theta_0}\left[T(X)\phi(X)\right]=\alpha\ E_{\theta_0}T(X),\ o<\alpha<1.$$

Prove that ϕ_0 is UMP among all unbiased tests of level α.

b) Let $X_1,\ldots,X_n$ be i.i.d observations with a common $N(\theta,1)$distribution. Find the UMP unbiased test of level α for testing $H:\theta=0$ Vs $K:\theta\neq 0$. Show that the power function of this test is an increasing function of $|\theta|$.

[15+(7+8)=30]

5. Let $X_1,\ldots,X_n$ be observations with a joint density of the form $f(x_1-\theta,\ldots,x_n-\theta)$ where f is a known density and θ is an unknown real parameter.

c) Describe (with proof of the results you state) how one can find the minimum risk equivariant (MRE) estimator of θ.

d) What is Pitman estimator of θ ? Find the Pitman estimator when the observations are i.i.d. with a common uniform distribution $U\left(\theta-\frac{1}{2},\theta+\frac{1}{2}\right)$

[18+10=28]

3. (10+10+5=25 marks)

Let $\{h_n\}$, h be a sequence of Borel-measurable functions on $(\mathcal{R}, \mathcal{B}(\mathcal{R}))$, such that, $h_n \to h$ in measure (with respect to a finite measure μ), as $n \to \infty$.

(a) If $\{h_n\}$ is uniformly integrable, show that $h_n \to h$, in L_1, as $n \to \infty$.

(b) If $\sup_n \int_{\mathcal{R}} |h_n|^2 d\mu < \infty$ then show that $h_n \to h$, in L_1, as $n \to \infty$.

(c) Give an example in (a) where L_1 convergence fails in absence of uniform integrability.

4. (5+5+5+5+5=25 marks)

Write TRUE or FALSE and justify by proving or disproving it.

(a) A right continuous function on the real line must have at most countably many discontinuities.

(b) Let Ω be an uncountable set and μ be a measure on the σ-field that consists of all countable and co-countable subsets of Ω, with $\mu(\Omega) = \infty$. Then μ cannot be σ-finite.

(c) Let $\{X_n\}$ be an i.i.d. sequence of random variable, distributed as student-t distribution with 1 degree of freedom. Let $S_n = X_1 + \cdots + X_n$. Then S_n/n converges with probability one, as $n \to \infty$.

(d) Let $\{X_n\}$ be a sequence of independent random variables with $sup_n E|X_n|^{2+c} < \infty$, for some constant $c > 0$. Then, for all $\epsilon > 0$, $\max_{1 \le k \le n} P(|X_k| \ge \epsilon) \to 0$ as $n \to \infty$.

(e) Let $\{f_n\}$, f be a sequence of Borel function on the line. Let $f_n \to f$ almost everywhere (with respect to the Lebesgue measure μ), as $n \to \infty$. Then $f_n \to f$ almost uniformly, as $n \to \infty$. Also, $f_n \to f$ in measure, as $n \to \infty$.

All the best.

INDIAN STATISTICAL INSTITUTE
First Semester Backpaper Examination: 2010-11

M. Stat. I Year
Sample Surveys & Design of Experiments

Date: 20.01.11 **Maximum Marks 50** **Duration: $1\frac{1}{2}$ hrs.**

Group – A

(Sample Surveys)

Answer to Group A in a separate book

No external equipments permitted.
Answer any 2 questions each carrying 25 marks.

1.(a) Explain the uses in details of Politz-Simmons's techniques giving results in details.

(b) How will you unbiasedly estimate the total number of gamblers in a community eliciting randomized responses by Warner's technique from a sample taken by Rao, Hartley and Cochran's scheme providing a measure of its accuracy ?

[12+13=25]

2. Explain situations relevant respectively to Double Sampling and Post-stratified Sampling. How will you unbiasedly estimate a population mean in these two cases along with measures of its accuracy ?

3. How will you unbiasedly estimate the variance of Basu's generalized Difference estimator for a finite population total from a sample if the sampling scheme admits samples of varying sizes?

P.T.O.

INDIAN STATISTICAL INSTITUTE
203 B. T. ROAD
KOLKATA 700108

Backpaper Examination, Semester I, 2010 – 11

M Stat I

Sample Surveys & Design of Experiments

Date : 20.1.11 Time : 1 and 1/2 hours Full Marks : 50

GROUP B

(Design of Experiments)

Note : Answer all the questions. Marks allotted to a question are indicated in [ ] at the end.

1. (a) Define variance balance of a block design, and, state and prove a necessary and sufficient condition for a connected block design to be variance balanced.

(b) Give a method of construction of the BIBD's having the parameters $v=s^2+s+1=b$, $r=s+1=k$, $\lambda=1$, showing clearly that the method actually leads to the parameters shown.

(c) Show that for a BIBD(v,b,r,k,λ), $b \geq v$, equality holding if and only if every pair of blocks intersects in λ common treatments.

[6 + 10 + 8 = 24]

2. (a) Consider a block design D consisting of the following five blocks : $B_1 = (1, 2, 3, 4)$, $B_2 = (1, 2, 3, 5)$, $B_3 = (1, 2, 4, 5)$, $B_4 = (1, 3, 4, 5)$, $B_5 = (2, 3, 4, 5)$, where the integers 1,2,...,5 represent the treatments of the design. What kind of a block design is D? Obtain its parameters. Is it connected? Why? Is it variance balanced? Why? State precisely the results that you may be using while answering these whys.

(b) Develop the ANOVA of D, indicating clearly how the various sums of squares are to be computed.

[(4+4+4)+14=26]

Discrete Mathematics
M.Stat. I

21 February 2011

Max. Marks : 80 Time : 3 hrs.

Answer all questions

1. Find the number of ways to assign 6 jobs to 4 workers so that each job gets a worker and each worker gets at least 1 job. (10)

2. Find the number of integer solutions to the following equation (10)

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = 15, \qquad 0 \le b_i \le 3 \text{ for all } i$$

3. Find the number of code words of length 3 from an alphabet {a, b, c, d, e, f} if a, c and e occur an odd number of times. (10)

4. Find the number of onto function from a set of 8 elements to a set with 4 elements. (10)

5. Find the number of distinct ways to 2-color a 4 X 4 array that can rotate by 0^0 or 180^0 (15)

6. Provide a "good" upper bound for the Ramsey number R (3, 3, 3). (10)

7. Find a BCH code with n =15 that can correct upto 2 errors. (15)

5

Discrete Mathematics
M.Stat. I

21 February 2011

Max. Marks : 80 Time : 3 hrs.

Answer all questions

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5

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2010-11

M. Stat. I Year (NB Stream)
Measure Theoretic Probability

Date: 21.02.11 **Maximum Marks: 40** **Duration: $1\frac{1}{2}$ Hours**

The Paper contains questions of 46 marks.
Maximum marks you can score is 40.

1. a) Let Sample Space $= \{1,2,3,4,5,6\}$, $A = \{1,3,4\}$, $B = \{4,6\}$. Derive the cardinality of $\sigma(\{A, B\})$.

   b) Show that there does not exist any σ-field which is countably infinite.

   c) Let $\mathcal{J} = \{(a,b] : 0 \le a \le b \le 1\}$ and $\mathcal{B} = \sigma(\mathcal{J})$. Show that $\mathcal{B} = \sigma$-field generated by the collection of all open subsets of $(0,1]$.

[4+8+8]

2. a) Let P denote Lebesgue measure on $(0,1]$ with Borel σ-field defined on it. Give an example of Borel measurable subsets $\{A_n\}$ such that
$0 < P(\liminf A_n) < \liminf P(A_n) < \limsup P(A_n) < P(\limsup A_n) < 1$

   b) Let two probability measures P_1 and P_2 be defined on a σ-field $\sigma(\mathcal{F})$, where $\mathcal{F}$ is a field. Using $\pi - \lambda$ theorem show that if P_1, P_2 agree on $\mathcal{F}$, they must agree on $\sigma(\mathcal{F})$.

   c) Let P be a probability measure defined on Ω with a field $\mathcal{F}$ and let P* be its outer measure. Show that $\{A : P^*(A) + P^*(A^C) = 1,\ A \subset \Omega\}$
$= \{A : A \subset \Omega$ and $P^*(A \cap E) + P^*(A^C \cap E) = P^*(E)$, for any subset E of $\Omega\}$.

[6+8+12]

*****

2502·11

M. Stat First Year, Second Semester, 2010-11

Time : 2 Hours | Mid-Semester Examination | Full Marks 50

Regression Techniques

1. Consider a usual multiple linear regression problem with model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ containing an intercept. Assume that the $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$ for some unknown $\sigma^2 > 0$, and define $\widehat{\boldsymbol{\beta}}$ as the usual least squares estimator of $\boldsymbol{\beta}$ based on n observations. Now, prove or disprove the following statements.

   (a) If $\widehat{\boldsymbol{\beta}}^{(0)}$ is the least squares estimator of $\boldsymbol{\beta}$ based on first $n-1$ observations, for any $\mathbf{m} \in R^{p+1}$, $\text{Var}(\mathbf{m}'\widehat{\boldsymbol{\beta}})$ cannot exceed $\text{Var}(\mathbf{m}'\widehat{\boldsymbol{\beta}}^{(0)})$. [2]

   (b) The rows and columns of $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}$ all add to 1. [3]

   (c) No diagonal element of $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}$ can be smaller than $1/n$. [2]

   (d) MSE is the UMVUE for σ^2. (if required, assume $X_1, X_2, \ldots, X_p$ are all non-stochastic) [3]

   (e) If the usual $100(1-\alpha)\%$ confidence interval of β_p contains the origin, the null hypothesis $H_0 : \beta_p = 0$ cannot be rejected (when $H_1 : \beta_p \neq 0$) by the usual test of level α. [2]

   (f) In the case of $p = 1$, $\widehat{\beta}_1$, the least squares estimate of β_1 can be expressed as a weighted average of slopes between pairs of observations. [3]

   (g) Consider the class models based on all possible subsets of $\{X_1, X_2, \ldots, X_p\}$. If the models selected by AIC and BIC contain k_1 and k_2 parameters, respectively, k_2 cannot be larger than k_1. [2]

   (h) The Durbin-Watson statistic reaches its minimum value only if $R^2 = 1$. [2]

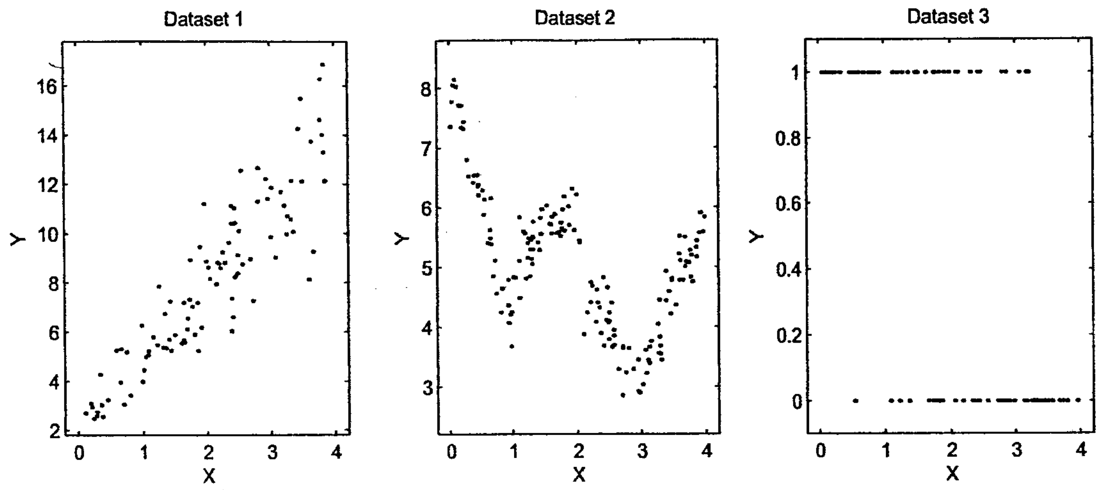
   (i) If we regress the residual on the response variable, the regression co-efficient will be $1 - R^2$. [2]

   (j) Instead of $N(0, \sigma^2\mathbf{I})$, if $\boldsymbol{\epsilon} \sim N(0, \sigma^2\boldsymbol{\Sigma})$, the generalized least squares estimator of $\boldsymbol{\beta}$ matches with the maximum likelihood estimator. [3]

2. Consider a multiple linear regression problem with two covariates X_1 and X_2, where Y and X_1 are real valued, but X_2 has 3 categories. Suppose that we have n observations on (Y, X_1, X_2). Write down a suitable linear regression model and describe how you will test whether X_2 has any statistically significant effect on Y. [5]

[Turn over]

3. Suggest a suitable regression model for each of the following three data sets and briefly describe a method for estimating the associated model parameters. [3 × 3 =9]



4. (a) Show that the LMS estimator is regression and affine equivariants. [2]

(b) Show that in the case of one-dimensional location model, the LTS estimator will coincide with the MCD location estimator when both of them have 50% breakdown point. [3]

(c) Suppose that a random variable Y follows a one parameter location model with parameter θ. If 15, 7, 24, 8, 10, 2, 11, 9, 28 and 5 are ten observed values of Y, find the LTS estimate (of θ) with 50% breakdown point. [4]

(d) Consider the ten observed values of Y given in (c). Find a value of β that minimizes $\psi(\beta) = 2\sum_{i=1}^{10}(y_i - \beta) + 3\sum_{i=1}^{10}|y_i - \beta|$. Check whether this value if unique. [3]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: (2010 – 2011)

M. Stat (NB-Stream) 1st Year

Programming and Data Structures

Date: 22.02.2011 Maximum Marks: 40 Duration: 2 hours

Answer all questions in brief.

1. Write a program in C to rearrange a list of strings alphabetically. [7]

2. Write the function *strindex (s, t)*, which returns the position of the rightmost occurrence of character *t* in string *s*, or -1 if there is none. [5]

3. What is the difference between (explain with suitable example) [3 x 2 = 6]

   a) int *ptr[10] and int (*ptr)[10]

   b) const int *ptr and const int *const ptr

4. Answer the following: [3 + 3 + 2 + 2 = 10]

   a) Explain with suitable example the difference between int **data and int *(*data)( ).

   b) Illustrate the main purpose of passing variables by pointers, with the help of a function that swaps two integers.

   c) What will be the output of the following C code?

```
#include<stdio.h>
int main( )
{
    struct node
    {
            int value;
            struct node *ptr;
    };
    struct node *p, q;
    printf("\n%d\t%d\n",sizeof(p),sizeof(q));
    return 0;
}
```

d) Explain the error (compile or run-time), if any, in the following C code:

```
#include<stdio.h>
int main( )
{
    int x = 10;
    int *ptr;
    float y;
    *ptr = 15;
    y = (float)x/*ptr;
    printf("The value of y = %f\n",y);
    return 0;
}
```

5. Define a macro *swap(t, x, y)* that interchanges two variables *x* and *y* of type *t*. [5]

6. Write a procedure to transpose an *m x n* sparse matrix in *O(n + t)* time complexity, where *t* represents the number of non-zero elements. [7]

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination – Semester II : 2010-2011
M. Stat. I Year
Metric Topology and Complex Analysis

Date : 28.02.11 | Maximum Score : 80 | Time : $2\frac{1}{2}$ Hours

Note : This paper carries questions worth a **total** of **93 MARKS**. Answer as much as you can. The **MAXIMUM** you can score is **80**. While writing definitions, be **precise and to the point**. For proofs, **clearly state the results**, if any, that you are using.

1. Let ρ be a metric on a non-empty set X.
(a) When is a subset V called an **open set** in the topology givem by ρ?
(b) Show that ρ satisfies $|\rho(x,y) - \rho(z,y)| \leq \rho(x,z)$ for every $x, y, z \in X$.
(b) Define $\overline{\rho}(x,y) = \rho(x,y)/[1 + \rho(x,y)]$. Show that $\overline{\rho}$ defines a metric on X. Show also that ρ and $\overline{\rho}$ give the same topology on X (that is, the same open sets).

2+4+(7+7)=[20]

2. (a) Define what is meant by **uniform continuity** of a function $f : X \to Y$, where X and Y are metric spaces.
(b) Let (X, ρ) be a metric space. For a given non-empty $A \subset X$, define $\rho(x, A) = \inf\{\rho(x,y) : y \in A\}$ for $x \in X$. Show that $\rho(x, A) = 0$ if and only if $x \in \overline{A}$. Show also that the function $f(x) = \rho(x, A)$, $x \in X$ is uniformly continuous on X [May use 1(b)].
(c) Show that if F and G are two disjoint closed subsets of X, then they can be separated by open sets, that is, there exist two disjoint open sets V_1 and V_2, each containing one of the given closed sets.

2+(6+7)+7=[22]

3. (a) Define what is meant by a subset of a metric space being **totally bounded.**
(b) Show that if a set is totally bounded, then it is bounded. Show also that if a set is totally bounded, then its closure is also totally bounded.
(c) Let A be a subset of a complete metric space. Show that $\overline{A}$ is compact if and only if A is totally bounded.

2+(6+6)+6=[20]

4. (a) Let A be a subset of a metric space X. Define what is meant by a **lebesgue number** for an open cover $\{V_\alpha, \alpha \in \Lambda\}$.
(b) When is a subset of a metric space said to be **sequentially compact**? Show **directly from the definition**, that if a subset of a metric space is sequentially compact, then every open cover of that subset has a lebesgue number.

3+(3+9)=[15]

5. Let X be a compact metric space. Let $f : X \to X$ be a contraction, that is, there is a number $\alpha \in (0,1)$ such that $\rho(f(x), f(y)) \leq \alpha\rho(x,y)$ for all $x, y \in X$. Show that f has a unique fixed point, that is, there is a unique $x \in X$ such that $f(x) = x$. Show also that, for any $y \in X$, the sequence $\{x_n\}$, defined as $x_1 = f(y)$ & $x_n = f(x_{n-1}), n \geq 2$, converges to x.

(8+8)=[16]

TIME SERIES ANALYSIS

M Stat 1st Year (2010-11)

Mid Semester Examination

Date: 01.03.2011 — Time: 2 hours

Total Marks 40

(The question paper carries **45** marks. Maximum you can score is **40**)

1. The production figures for four quarters for 5 years are given in the following table. The last column indicates the trend values for annual production.

Year / Quarter	I	II	III	IV	Annual Trend
2000	75	60	54	59	217.6
2001	86	65	63	80	257.3
2002	90	72	66	85	297.0
2003	100	78	72	93	336.7
2004	109	87	80	101	376.4

(a) Calculate the seasonal indices for each quarter.
(b) Suppose the value corresponding to quarter III in 2001 is mistakenly printed as 63 instead of 93. Assuming the true value to be 93, modify your indices.

(10+8=18)

2. Suppose that $X_t = Z_t + \theta Z_{t-1},\ t = 1,2,\ldots$ where $Z_0, Z_1, Z_2, \ldots$ are independent random variables each with moment generating function $E(\exp[\lambda Z_i]) = m(\lambda)$.
(a) Express the joint moment generating function $E(\exp[\sum_{i=1}^{n} \lambda_i Z_i])$ in terms of $m(.)$.
(b) Deduce from (a) that $\{X_t\}$ is strictly stationary. **(4+5=9)**

3. Let $Z_t,\ t = 0, \pm 1, \pm 2, \ldots,$ be independent normal random variables each with mean 0 and variance σ^2 and let a, b and c are constants. Which, if any, of the following processes are stationary? For each stationary process, specify the mean and autocovariance function.
(a) $X_t = a + bZ_t + cZ_{t-1}$ (b) $X_t = a + bZ_0$ (c) $X_t = Z_t Z_{t-1}$ **(3+3+3=9)**

4. Let $\{S_t,\ t = 0,1,2,\ldots\}$ be the random walk with constant drift μ, defined by $S_0 = 0$ and

$$S_t = \mu + S_{t-1} + X_t,\ t = 1,2,\ldots$$

where $X_1, X_2, \ldots$ are independently and identically distributed random variables with mean 0 and variance σ^2. Compute the mean of S_t and the autocovariance function of the process $\{S_t\}$. Show that $\{\nabla S_t\}$ is stationary and compute its mean and autocovariance function. **(4+5=9)**

MULTIVARIATE ANALYSIS
M. Stat. I year
Mid-semester Examination

Time : 3 hours | **Maximum marks 100** | **[illegible] March 2011**

Answer all questions

To justify your steps, clearly state any result that you use. You may assume results for univariate distributions.

1. Denote by $\underset{-}{X} = (X_1, X_2, X_3, \ldots\ldots\ldots, X_N)$, where X_α's are mutually independent $N_p[\mu, \Sigma]$, $\alpha = 1, 2, \ldots N$.

1.1 Show that, when $\mu = 0$,

$$\underset{-}{X}\,\mathrm{H}\,\underset{-}{X}' \sim W_p\,[r, \Sigma] \text{ iff } l'\underset{-}{X}\,\mathrm{H}\,\underset{-}{X}'l \sim \sigma_l^2\,\chi^2(r)$$

for any non- null l, where $\sigma_l^2 = l'\Sigma l$ and rank(H) = r.

1.2 How would the result be affected when $\mu \neq 0$? **[16+4=20]**

2. Let W be $W_p\,[r, \Sigma]$ and $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, $r \geq p$,

where W_{11} is (p-q) X (p-q).

2.1 Show that both W_{11} and $W_{22.1}$ have Wishart distribution and that they are independently distributed.

2.2 Consider a standard linear multiple regression set-up of Y on $X_1, X_2, \ldots, X_p$, based on N observations on Y and each of X's. Find the distribution of the residual sum of squares of Y after regressing on $X_1, X_2, \ldots, X_p$ by the method of least-squares.

2.3 Show that the distribution of $|W|$ can be expressed as a product of mutually independent Chi- square distributions. **[(3+18+4)+ 20+15 = 60]**

3. 3.1 What is Union – intersection principle for testing?

3.2 Using the Union-Intersection principle, derive a level $\alpha = 0.05$ test procedure for testing the mean $\mu = 0$ of a p = 4 variate normal population with unknown covariance matrix based on a random sample of size N = 17.

[5+15 =20]

MULTIVARIATE ANALYSIS
M. Stat. I year
Mid-semester Examination

Time : 3 hours **Maximum marks 100** **4 March 2011**

Answer all questions

To justify your steps, clearly state any result that you use. You may assume results for univariate distributions.

1. Denote by $\underset{\sim}{X} = (X_1, X_2, X_3, \ldots\ldots\ldots, X_N)$, where X_α's are mutually independent $N_p[\mu, \Sigma]$, $\alpha = 1, 2, \ldots N$.

   1.1 Show that, when $\mu = 0$,
   $$\underset{\sim}{X} \mathrm{H} \underset{\sim}{X}' \sim W_p\,[r, \Sigma] \text{ iff } l' \underset{\sim}{X} \mathrm{H} \underset{\sim}{X}' l \sim \sigma_l^2 \chi^2(r)$$
   for any non- null l, where $\sigma_l^2 = l'\Sigma l$ and rank(H) = r.

   1.2 How would the result be affected when $\mu \neq 0$? [16+4=20]

2. Let W be $W_p\,[r, \Sigma]$ and $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, $r \geq p$,

   where W_{11} is (p-q) X (p-q).

   2.1 Show that both W_{11} and $W_{22.1}$ have Wishart distribution and that they are independently distributed.

   2.2 Consider a standard linear multiple regression set-up of Y on $X_1, X_2, \ldots, X_p$, based on N observations on Y and each of X's. Find the distribution of the residual sum of squares of Y after regressing on $X_1, X_2, \ldots, X_p$ by the method of least-squares.

   2.3 Show that the distribution of $|W|$ can be expressed as a product of mutually independent Chi- square distributions. [(3+18+4)+ 20+15 = 60]

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   [5+15 =20]

INDIAN STATISTICAL INSTITUTE

M. STAT. FIRST YEAR

Optimisation Techniques

Date:4. 3. 11 Mid.Semestral Examination Time : 2 hrs.

This paper carries 60 marks.The maximum you can score is 40.

1. Consider the following problem:

   $P_1, P_2, \cdots P_m$ are plants producing an equal amount α of a certain good each. $M_1, M_2, \cdots M_n$ are markets each having a demand β for the good. Suppose α_{ij} is the cost per unit of transporting the good from P_i to M_j. Find amounts η_{ij} that should be sent from P_i to M_j so that the demands are met from the productions at minimum cost.

   (a) Formulate this as a linear programming problem. [3]

   (b)Give,with justification, neccessary and sufficient conditions for an optimal solution to exist. [12].

2. Solve the following problem and its dual:

   Find $\eta_1, \eta_2, \eta_3 \geq 0$ such that $8\eta_1 + 19\eta_2 + 7\eta_3$ is maximum subject to

   $$3\eta_1 + 4\eta_2 + \eta_3 \leq 25$$

   $$\eta_1 + 3\eta_2 + 3\eta_3 \leq 50.$$

   [15]

3. Without using simplex method, prove that the following problem has an optimal solution and find the solution for the problem and its dual:

   Find $\eta_1, \eta_2, \eta_3 \geq 0$ such that $2\eta_1 + 3\eta_2$ is maximum subject to

   $$4\eta_1 + 2\eta_2 + \eta_3 = 4$$

   $$\eta_1 + 3\eta_2 = 5.$$

   [15]

4. Does there exist $\eta_1, \eta_2 \geq 0$ such that

   $$5\eta_1 - 4\eta_2 \leq 7$$

   $$-3\eta_1 + 3\eta_2 \leq -5?$$

   Justify your answer. [15]

Indian Statistical Institute

Second Midsemestral Examination, 2010-11

M. Stat. I year
Multivariate Analysis

Date: March 23, 2011 Maximum Marks: 15 Time: 2 hours

Note that the paper contains 20 marks

1. Let $S^{n-1} = \{(u_1, u_2, \ldots, u_n) : u_1^2 + u_2^2 + \ldots + u_n^2 = 1\}$ denote the unit sphere in $\mathbb{R}^n$. Correspondingly, let λ_n denotes the Lebesgue measure defined on S^{n-1} for $n \geq 2$. Find a suitable density function $g(x, u)$ for $x \in [-1, 1]$ and $u \in S^{n-1}$ such that
$$\int_{S^n} f(z)\, d\lambda_{n+1}(z) = \int\int_{[-1,1]\times S^{n-1}} f(\sqrt{1-x^2}u, x)\, g(x,u)\, d(\lambda \times \lambda_n)(x,u),$$
for any integrable function f and under the transformation of variables $T : [-1,1] \times S^{n-1} \longrightarrow S^n$, given by $T(x, u_1, u_2, \ldots, u_n) = (\sqrt{1-x^2}(u_1, u_2, \ldots, u_n), x)$. Here λ denotes the standard Lesbesgue measure on $[-1, 1]$. Hence obtain the value of the integral when $f(z) = z_1^2 z_2^2 \ldots z_{n+1}^2$ using this recursion formula. [7]

2. (a) Given a symmetric and nonnegative definite kernel $K : T \times T \longrightarrow \mathbb{R}$, describe the notion of the reproducing kernel Hilbert space of functions $\mathscr{H} \subseteq \{f : T \longrightarrow \mathbb{R}\}$ generated by the kernel K.
(b) Assuming that K represents the covariance kernel of a zero-mean gaussian process $\{X_t : t \in T\}$, discuss how would you interpret the identity
$$\text{Cov}\,(X_{s_1}, X_{s_2}) = \text{Cov}\,(X_{s_1}, X_{s_2}|Y) + \text{Cov}\,(E(X_{s_1}|Y), E(X_{s_2}|Y)),$$
for $s_1, s_2 \in S \subset T$ and $Y = \{X_t : t \notin S\}$. Assume that T is finite if necessary.
[5 · 8 =13]

INDIAN STATISTICAL INSTITUTE, KOLKATA

M.Stat I year, Second Semestral Examination

Time 3 hours | Regression Techniques | Date : 02.05.2011

Answer as many as you can. The maximum you can score is 100.

1. Consider a simple linear regression model $y_i = \alpha + \beta x_i + \epsilon_i$, $i = 1, 2, \ldots, n$, where the x_i's are non-stochastic and the ϵ_i's are independent and identically distributed with mean 0 and variance σ^2.

   (a) Assume that no two x_i's are equal, and for all $i < j$, define $\widehat{\beta}_{ij} = (y_i - y_j)/(x_i - x_j)$ as the slope between the pair of observations (x_i, y_i) and (x_j, y_j). Now, consider two estimators of β, namely, $\widehat{\beta}_1 = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \widehat{\beta}_{ij}$ and $\widehat{\beta}_2 = \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_{ij}\widehat{\beta}_{ij}$, where $\sum_{i=1}^{n} \sum_{j=i+1}^{n} w_{ij} = 1$ and $w_{ij} \propto (x_i - x_j)^2$. Check whether $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are unbiased. Which of these two estimators will you use for estimating β ? Justify your answer. [2+4]

   (b) Suppose that the x_i's take values in the interval $[0, 1]$, and we have 10 observations $(x_1, y_1), (x_2, y_2), \ldots, (x_{10}, y_{10})$ from this model, where at least two x_i's are distinct. Assume that the mean of these ten x-values $x_1, x_2, \ldots, x_{10}$ is 0.57. If you are asked to choose the 11th value x in $[0, 1]$, which value of x will you choose to minimize the variance of the least squares estimator of β ? Justify your answer. [4]

2. Consider a multiple linear regression model $Y = \mathbf{X}\beta + \epsilon$, where $\epsilon \sim N(\mathbf{0}, \sigma^2\mathbf{I})$. Also consider the partitions $\mathbf{X}_{n\times p} = [\mathbf{X^1}_{n\times p_1} : \mathbf{X^2}_{n\times p_2}]$ and $\beta' = \left[\beta_1' : \beta_2'\right]$, where β_1 and β_2 are of dimensions p_1 and p_2, respectively.

   (a) Suppose that instead of the full model, a person considers a subset model $Y = \mathbf{X}^1\beta_1 + \epsilon$ and finds an estimate $\widehat{\beta}_1$ of β_1 using the method of least squares. If $\beta_2 \neq \mathbf{0}$, can $\widehat{\beta}_1$ ever be unbiased for β_1 ? Justify your answer. [2]

   (b) Let $\widehat{\widehat{\beta}}_1$ be the estimate of β_1 obtained from the full model. If $\mathbf{D_1}$ and $\mathbf{D_2}$ denote the dispersion matrices of $\widehat{\beta}_1$ and $\widehat{\widehat{\beta}}_1$, respectively, show that $\mathbf{D_2} - \mathbf{D_1}$ is positive semi-definite. [3]

   (c) Describe how you will test $H_0 : \beta_2 = 0$ against $H_1 : \beta_2 \neq 0$. Find a 95% confidence interval for β_2 associated with this test. [3+2]

3. Consider a multiple linear regression model $\mathbf{Y} = \mathbf{X}\beta + \epsilon$, where $\epsilon \sim N(\mathbf{0}, \sigma^2\mathbf{I})$ and $\text{rank}(\mathbf{X}_{n\times p}) < p$.

   (a) Check whether β is unbiasedly estimable. [2]

   (b) Define $\widehat{\beta}_0 = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Y}$, where A^{-} denotes a generalized inverse of $\mathbf{A}$. If $\mathbf{m}'\beta$ is unbiasedly estimable, show that $\mathbf{m}'\widehat{\beta}_0$ is an unbiased estimator of $\mathbf{m}'\beta$ and also check whether $\mathbf{m}'\widehat{\beta}_0$ depends on the choice of the generalized inverse of $\mathbf{X}'\mathbf{X}$. [2+2]

   (c) If $\|\mathbf{m}\| = 1$, show that no linear unbiased estimator of $\mathbf{m}'\beta$ can have variance smaller than σ^2/λ, where λ is the largest eigenvalue of $\mathbf{X}'\mathbf{X}$. [4]

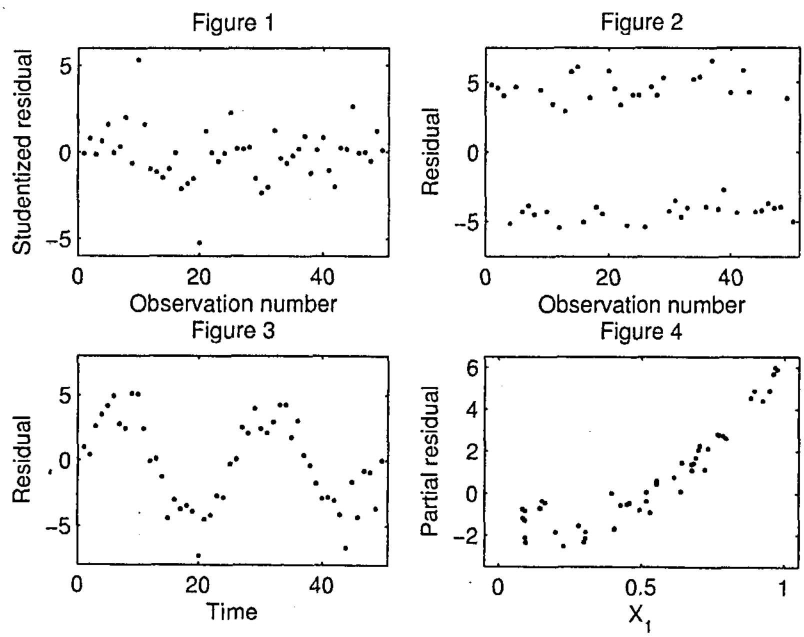
P.T.O

4. (a) Consider a logistic regression model, where the response variable takes only two values 0 and 1. Suppose that we have a data set $(x_1, y_1), \ldots, (x_n, y_n)$ of size n. If there exist $\beta_0 \in \mathbb{R}$ and $\beta \in \mathbb{R}^p$ that satisfy $(2y_i - 1)(\beta_0 + \mathrm{x}_i'\beta) > 0$ for all $i = 1, 2, \ldots, n$, then show that the maximum likelihood estimate of the parameters of the logistic regression model will not exist. [4]

(b) Check whether this logistic regression model can be expressed as a single layer perceptron model with an appropriate transformation function. [2]

(c) How will you generalize this logistic regression model and estimate the model parameters if the response has more than two categories. [4]

5. In a linear regression problem, after fitting a multiple linear regression model with p ($p > 1$) regressors, if you find one of the following plots, how will you modify your estimates ? Give details with proper explanations. [2+2+3+3]



[In Figure 3, assume that the observations are collected in accordance with the time of occurrence. So, time can be considered as the observation number.]

6. Consider a linear regression model $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \epsilon_i \;\; i = 1, 2, \ldots, n$, where the ϵ_i's are independent and identically distributed as $N(0, \sigma^2)$ variables.

(a) Show that LASSO regression and ridge regression methods can be viewed as Bayesian estimation techniques for suitable choices of priors on regression parameters. [3]

(b) Show that these two regression methods can also be viewed as penalized least squares methods for suitable choices of the penalty function. [3]

(c) Give geometric interpretation of the estimates obtained by LASSO regression and ridge regression. Also explain how LASSO regression helps in selecting a parsimonious regression model. [4]

7. (a) If $y_1, y_2, \ldots, y_n$ are n observed values of a random variable Y, for any $p \in (0,1)$, show that $\theta_p = \inf\{y : \sum_{i=1}^n I\{y_i \leq y\} \geq np\}$ is a minimizer of $\{\sum_{i=1}^n |y_i - \theta| + (2p-1)\sum_{i=1}^n (y_i - \theta)\}$. [5]

(b) Consider a simple linear regression model $y_i = \alpha + \beta x_i + \epsilon_i$, $i = 1, 2, \ldots, n$, where the ϵ_i's are independent and identically distributed with p-th quantile 0. Describe how you will estimate the p-th quantile of the response variable Y for a given x. [5]

8. (a) A model is called generalized linear model if for given any $\mathbf{x}$, the conditional p.d.f. (or p.m.f.) of Y is of the form $f(y, \mu_{\mathbf{x}}, \phi) = \exp\{[yg(\mu_{\mathbf{x}}) - \psi(\mu_{\mathbf{x}})]/a_\phi + c(y, \phi)\}$, where $\mu_{\mathbf{x}} = E(Y \mid \mathbf{x})$, $g(\mu_{\mathbf{x}}) = \mathbf{x}'\beta$, a linear predictor, and a_ϕ is a constant that depends on the scale parameter ϕ. Show that the logistic regression model can be expressed as a generalized linear model. Also find the functions g and ψ for this model. [5]

(b) Assume that $Y_1, \ldots, Y_n$ are independent random variables, where $\mu_i = E(Y_i) = \mathbf{x}_i'\beta$ for given $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})'$ and unknown parameter β. If Y_i $(i = 1, 2, \ldots, n)$ is uniformly distributed over the range $\mu_i \pm \sigma$ (σ unknown), then show that the maximum likelihood estimate of β is obtained by minimizing $\max_i |y_i - \mathbf{x}_i'\beta|$. [5]

9. (a) Show that in a location model, the LMS estimator coincides with the MVE estimator when both of them have 50% asymptotic breakdown point. [2]

(b) "The LAD regression estimator has 50% asymptotic breakdown point" - justify the validity of this statement. [3]

(c) Suppose that tha heights (in inches) of 11 students of a class are recorded as 55, 62, 75, 67, 63, 70, 72, 74, 65, 66 and 73. Find out the MCD and the MVE estimates of the average height of the students in that class. [5]

10. Consider a data cloud consisting of m observations $\{(y_i, x_{1i}, x_{2i}),\ i = 1, 2, \ldots, m\}$ and a linear fit $f(x_1, x_2)$ that does not pass through any data point.

(a) What is the minimum value of m for which the fit f may have positive regression depth? Justify your answer. Give an example of such a data set and the corresponding linear fit. [3+3]

(b) Given a data set of $m \geq 6$ observations, show that there exists a linear fit with regression depth greater than or equal to $m/3$. [4]

11. (a) What is a cubic spline ? Describe how will you fit an additive regression model to a data set using cubic splines ? [4]

(b) Give an example of a regression problem, where additive regression is not appropriate, but projection pursuit regression can be helpful in properly estimating the regression surface. Give justification to your answer. [3]

(c) Briefly describe an algorithm for fitting a projection pursuit regression model. [3]

P.T.O

2. (a) Let $\hat{f}_h$ be the Nadaraya-Watson estimate of the regression function f based on n observations when a Gaussian kernel $K(\mathbf{t}) = (2\pi)^{-p/2}e^{-\frac{1}{2}\mathbf{t}'\mathbf{t}}$ and a bandwidth h are used. Study the behavior of $\hat{f}_h$ when (i) h tends to infinity and (ii) h shrinks to 0. [3+4]

(b) Describe how you will find a local linear estimate of the regression function f at $\mathbf{x}$. [3]

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INDIAN STATISTICAL INSTITUTE

SECOND SEMESTER EXAMINATION: (2010 - 2011)

M. STAT, I YEAR (NB-Stream)
Subject: Measure Theoretic Probability

Date: May 4, 2011 **Maximum Marks: 100** **Duration: 3 hrs**

Note: This paper contains questions of 120 marks. You may attempt all questions. You can score a maximum of 100 marks.

Answer the following questions

1. Let us consider $(0,1]$.
F_1 =Borel $\sigma-$ field on $(0,1]$
$F_2 = \sigma-$field of countable and co-countable sets in $(0,1]$. Show that
i) $F_2 \subset F_1$
ii) F_1 is countably generated.
iii) F_2 is not countbly generated

[4+8+8]

2. a) Let X be a random variable and $\phi(t)$ denote the characteristic function of X. If X has a density $f(x)$ on $\mathbb{R}$, show that
$\phi(t) \to 0$ as $|t| \to \infty$.

b) Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables with mean c and variaance σ^2. Show that

$$\frac{(\sum_{i=1}^{n} X_i - nc)}{\sigma\sqrt{n}} \to Normal(0,1) \text{ (weakly)}$$

[10+15]

3. a) Let F_1 be the $\sigma-$ field generated by the collection of all open sets in $\mathbb{R}^2$ and F_2 be given by $\sigma(\{A \times B : A \in R, B \in R\})$ where R is the Borel $\sigma-$ filed on $\mathbb{R}$. Show that $F_1 = F_2$.

b) Let $f : \Omega \to \Omega'$ and C be a class of subsets of Ω'. Show that $\sigma\left(f^{-1}(C)\right) = f^{-1}(\sigma(C))$, where $f^{-1}(C) = \left\{f^{-1}(A) : A \in C\right\}$.

[12+13]

4. a) Let (Ω, F, μ) be a measure space, where μ is $\sigma-$ finite. Show that there can not exist uncountably many disjoint $F-$sets, each with positive $\mu-$ measure.

b) Let X be a random variable with distribution function F on $\mathbb{R}$. Show that F, as a function from $\mathbb{R} \to \mathbb{R}$, is measurable.

[P.T.O]

c) Give an example of a sequence of measurable functions $f_n : \mathbb{R} \to \mathbb{R}$, with $f_1 \leq f_2 \leq f_3 \leq \ldots$ and $lim_n f_n(x) = f(x)$ such that

$$\int_{\mathbb{R}} f_n d\mu \quad \text{does not converge to} \quad \int_{\mathbb{R}} f d\mu$$

.where $\mu$is Lebesgue measure on $\mathbb{R}$.

[12+8+10]

5) a) Let $f : \Omega \to [0, \infty)$ be measurable. Then show that there exists a sequence of simple functions $\{s_n(w)\}$ such that

$$s_n(w) \uparrow f(w) \text{ for each } w \in \Omega.$$

b) Let $f : \Omega \to [0, \infty)$ be measurable and μ be a measure defined on Ω.Then show that
i) If $\mu(\{w : f(w) > 0\}) > 0$, then $\int_\Omega f d\mu > 0$.
ii) If $\int_\Omega f d\mu < \infty$, then $\mu(\{w : f(w) = \infty\}) = 0$.

[10+10]

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Indian Statistical Institute

Second Semester Examinations, (2010-2011)

M. Stat, First Year

Discrete Mathematics

Date : May 05, 2011 | Maximum Marks : 80 | Duration : 3 Hours

Attempt all questions. The paper carries a total of 92 marks. Maximum you can score is 80. Figures in the right margin indicate the marks on the different parts of a question.

1. a) Prove that in every tree T, any two longest paths intersect (have a common vertex).
   b) Given two spanning trees S and T of a connected graph G and $s \in E(S) \setminus E(T)$, prove that there exists an edge $t \in E(T) \setminus E(S)$ such that both $S + t - s$ and $T + s - t$ are spanning trees of G. [5+10=15]

2. Show that the number of spanning trees of a complete graph K_n which do not contain a fixed edge e of K_n is $(n-2)\, n^{n-3}$. [12]

3. An n-cube is a cube in n dimensions. A cube in one dimension is a line segment; in two dimensions, it's a square, in three, a normal cube, and in general, to go to the next dimension, a copy of the cube is made and all corresponding vertices are connected. What is the expected distance between two randomly chosen vertices? Show that every n-cube ($n > 1$) has a Hamiltonian cycle. [10+10=20]

4. One deals out a deck of 52 cards, faced up, into a 4×13 array. Then one tries to select 13 cards, one from each column, in such a way as to get one card of each denomination (but not necessarily of the same suit). What is the probability that such a selection is possible? Justify your answer. [15]

5. A digraph is called a tournament if its underlying graph is complete. A vertex v in a digraph is called a leader if every other vertex can be reached from v by a path of length at most 2. Prove that every tournament has a leader. [10]

6. State and prove Perron-Frobenius Theorem for non-negative primitive matrices. [20]

Indian Statistical Institute
M. Stat - I: 2010-11
Semester Examination
Time Series Analysis

Date: 06.05.2011 Duration: 3 hours

Marks: 60

(The question paper carries 70 marks. Maximum you can score is 60)

1. Let Y_t be a stationary time series. Compare the limiting value of the coefficient obtained in the regression of $Y_t - \hat{r}(1)Y_{t-1}$ on $Y_{t-1} - \hat{r}(1)Y_{t-2}$ with the limiting value of the regression coefficient of Y_{t-2} in the multiple regression of Y_t on Y_{t-1} and Y_{t-2}. (11)

2. Determine whether the process $X_t + 0.6X_{t-2} = Z_t + 1.2Z_{t-1}$ is causal and / or invertible. (8)

3. Show that in order for an AR(2) process with autoregressive polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$ to be causal, the parameters (ϕ_1, ϕ_2) must lie in the triangular region: $\{(\phi_1,\phi_2): \phi_2 + \phi_1 < 1,\ \phi_2 - \phi_1 < 1,\ |\phi_2| < 1\}$. (7)

4. Find the autocorrelation function $\rho(h)$ of the ARMA(2,1) process,

$$(1 - 0.5B + 0.4B^2)X_t = (1 + 0.25B)Z_t,\ \{Z_t\} \sim WN(0,\sigma^2).$$

Plot $\rho(h)$ for $h = 1,2,\ldots,10$ and comment. (11)

5. The estimated autocorrelations for a sample of 100 observations on the time series $\{X_t\}$ are $\hat{r}(1) = 0.8$, $\hat{r}(2) = 0.5$ and $\hat{r}(3) = 0.4$.

(a) Assuming the times series $\{X_t\}$ is defined by
$$X_t = \beta_1 X_{t-1} + \beta_2 X_{t-2} + e_t,$$
where the e_t are independent $N(0,\sigma^2)$ variables, estimate β_1 and β_2.

(b) Test the hypothesis that the order of the autoregressive process is two against the alternative that the order is three. (17)

6. Let the time series X_t be defined by $X_t = \sum_{j=0}^{\infty} \alpha_j e_{t-j}$, where e_t are independent $N(0,\sigma^2)$ variables with fourth moment $\eta\sigma^4$ and $\sum_{j=0}^{\infty} j^{1/2} |\alpha_j| < \infty$. Then show that,

$$Cov\{I_n(\omega_j), I_n(\omega_k)\} = \begin{cases} 2(4\pi)^2 f^2(0) + o(1), & \omega_j = \omega_k = 0 \\ (4\pi)^2 f^2(\omega_k) + o(1), & \omega_j = \omega_k,\ \omega_k \neq 0,\pi \\ O(n^{-1}), & \omega_j \neq \omega_k \end{cases}$$

where $f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-i\omega h}$. (16)

INDIAN STATISTICAL INSTITUTE

Semestral Examination – Semester II : 2010-2011

M. Stat. I Year

Metric Topology and Complex Analysis

Date : 09.05.11 | **Maximum Score : 120** | Time : 3 Hours

Note : This paper carries questions worth a total of **138 MARKS**. Answer as much as you can. The **MAXIMUM** you can score is **120**. While writing definitions, be **precise and to the point**. For proofs, **clearly state the results**, if any, that you are using.

1. Let D be a non-empty open subset of the complex plane $\mathbb{C}$. Fix $z_0 \in D$ and consider the set $A = \{z \in D :$ there is a polygonal path lying in D and joining z_0 to $z\}$. Show that A and $D \setminus A$ are both open in D.

12+12=[24]

2. (a) Define what is meant by an analytic function on some open subset of $\mathbb{C}$.
(b) Show that an analytic function on an open set must satisfy the Cauchy-Riemann equations.
(c) Let D be a connected open subset of the complex plane and let $f : D \to \mathbb{C}$ be analytic. Show that if $|f(z)| = 4$ for all $z \in D$, then f must be constant.

4+10+10=[24]

3. (a) State Liouville's Theorem.
(b) Let f be an entire function satisfying $|f(z)|/|z|^3 \to 0$ as $|z| \to \infty$. Show that f must be a polynomial of degree at most 2.

4+12=[16]

4. Let f be an analytic function on and open set D. Define $g : D \times D \to \mathbb{C}$ by $g(w, z) = (f(w) - f(z))/(w - z)$ for $w \neq z$ and $g(w, z) = f'(z)$ for $w = z$. Show that g is continuous on $D \times D$ and that for every $w \in D$, the function $z \mapsto g(w, z)$ is analytic on D.

12+12=[24]

5. (a) State Morera's Theorem.
(b) Let $f_n, n \geq 1$, be a sequence of analytic functions on an open set D. Show that if the sequence f_n converges to a function f on D and if the convergence is uniform on every compact subset of D, then f is analytic on D.

4+10=[14]

6. Let f be analytic on all of $\mathbb{C}$ except for an isolated singularity at $z = 0$.
(a) Show that, for $z \in \mathbb{C}$, the integral $\int_{\sigma_r} \frac{f(w)}{(w - z)^2}\, dw$, where $\sigma_r(t) = re^{it}, 0 \leq t \leq 2\pi$ and $r > |z|$, does not depend on r.
(b) Denoting the integral in (a) by $g(z)$, show that g is an entire function.
(c) If the singularity of f at $z = 0$ is a removable singularity, then show that for all $z \neq 0$, $2\pi i\, f'(z) = g(z)$.

12+12+12=[36]

INDIAN STATISTICAL INSTITUTE

M. STAT. FIRST YEAR
Semester Examination-2011

Subject: Optimization Techniques

Date: 11.5.11 Time : 3 hrs.

This paper carries 70 marks.The maximum you can score is 60.

1. (a) Solve the following problem:

Find $\eta_1, \eta_2, \eta_3 \geq 0$ such that $\eta_1 + 2\eta_2 + \eta_3$ is maximum subject to

$$\eta_1 + \eta_2 + \eta_3 \leq 16$$

$$\eta_1 - \eta_2 + 3\eta_3 \leq 12$$

$$\eta_1 + \eta_2 \leq 4$$

(b) Solve (a) with the additional condition that η_1, η_2, η_3 are integers. [15+15]

2. Solve the optimal assignment problem with 4 individuals and 4 jobs where the efficiency ratings are given by the matrix:

	J_1	J_2	J_3	J_4
I_1	4	4	9	3
I_2	3	5	8	8
I_3	3	5	7	9
I_4	2	6	5	7

[15]

3. Let $f(x, y) = |x - 1|^3 + |x|^3 + |y - 1|^3 + |y - 2|^3$.

What is the minimum value of $f(x, y)$? [15]

4. (a) State an additive type model for dynamic programming problem.

(b) The sales manager for a publisher of college text books has four sales persons to assign to three different regions of the country. The following table gives the estimated increase in sales in each region. Use dynamic programming method to determine how many salespersons should be assigned to the respective regions in order to maximize sale.

Salesman	Estimated increase in sales (in appropriate unit) for three regions		
	1	2	3
1	35	21	28
2	48	42	41
3	70	56	63
4	89	70	75

[10]

INDIAN STATISTICAL INSTITUTE

Second-Semester Examination: (2010 – 2011)

M. Stat. (NB-Stream) 1st Year

Programming and Data Structures

Date: 11.05.2011 Maximum Marks: 100 Duration: 3 hours

Answer all questions.

1. A concordance is an alphabetical list of all the distinct words in a text along with the number of occurrences of each word. Write a C program that makes a concordance from an input text file. While writing the program, you need to take care of the following:
   a) The names of the variables should reflect their functions.
   b) Use dynamic memory allocation.
   c) Proper documentation of the program should be provided. [20]

2. Write a C program to find maximum and minimum of a set of integer numbers using recursion. [10]

3. For any nonempty binary tree *T*, if N_E is the number of external nodes and N_I is the number of internal nodes, then prove that $N_E = N_I + 1$. Write a procedure to perform the in-order traversal of a binary tree using stack. [5+ 10 = 15]

4. Answer the following: [2 + 3 + 4 + 6 = 15]
   a) What is the speciality of void pointer?
   b) Explain with example the differences between malloc( ) and calloc( ) in terms of the functions they perform.
   c) Describe an efficient way of storing two $n \times n$ symmetric matrices in memory.
   d) Construct the Max-Heap for sorting the following list: [50, 44, 65, 31, 73, 67, 85, 69]

5. a) Write a procedure to evaluate a postfix expression using stack.
   b) Construct an AVL search tree by inserting the following elements in the order of their occurrence: [64, 10, 45, 33, 15, 117, 100, 85]. [10 + 10 = 20]

6. Explain with suitable examples the differences between (*any two*) [5 x 2 = 10]
   a) int **data and int *(*data)( )
   b) stack and queue
   c) const int *ptr and const int *const ptr
   d) fprintf( ) and fwrite( )

7. Describe *any two* of the following with suitable examples: [5 x 2 = 10]
   a) Priority queue
   b) B tree
   c) Heap

MULTIVARIATE ANALYSIS
M. Stat. I year [NB stream]

Semester Examination

Time : 3 hours **Maximum marks 100** **13 May 2011**

Answer any three of the questions 1 to 5, and an alternative set in question 6.

To justify your steps, clearly state any result that you use. You may assume results for univariate results for proving multivariate results.
Show steps that are essential for computations in question 6.

1.1 Let Y be $N_p[\,0\,,\Sigma\,]$, W be $W_p\,[r, \Sigma]$ and Y and W be independently distributed. Derive the distribution of $Y'W^{-1}Y$.

1.2 Let X_α's be mutually independent $N_p[\,\mu\,,\Sigma\,]$, $\alpha = 1, 2, \ldots N$.
Using a standard method, find a test procedure for testing $\mu = \mu_0$ when Σ is unknown. Construct a 100(1-α)% confidence region for μ. [8+6+6=20]

2 Let X_α's be mutually independent $N_p[\,\mu\,,\Sigma\,]$, $\alpha = 1, 2, \ldots N$. Derive a test procedure to test $\Delta^2_p = \Delta^2_{p-q}$ where $\Delta^2_p = \mu'\Sigma^{-1}\mu$ is based on all p components and Δ^2_{p-q} is based on the first p-q components. [20]

3. Consider data on a p x 1 response vector from an RBD with t treatments in b blocks.
3.1 Write a MANOVA model.
3.2 Write the MANOVA table.
3.3 Show that the treatment and the error SSP matrices are independently distributed.
3.4 Suggest a test statistics and a test procedure to test the equality of treatment effects. [3+5+8+4=20]

4. Consider $B = (X_1 - \overline{X}, X_2 - \overline{X}, X_3 - \overline{X}, \ldots\ldots\ldots, X_N - \overline{X})$ where X_α, $\alpha = 1, 2, \ldots N$ are observations on a p x 1 vector X and $\overline{X}$ is the mean vector of X_α's.
4.1 Define sample PCs Y of X and their use.
4.2 Construct data on Y from B.
4.3 Compute the sample means and the sample covariances of the components of Y.
4.4 What can you say about the expectations and the covariances of the first two components of Y [(3+3)+4+6+4 =20]

P.T.O

5.1 Write an m-factor orthogonal model for factor analysis of a p x 1 vector X and identify its differences from a p variate regression model.

5.2 How would m=2 and m=3 compare?

5.3 Discuss the principle component method of estimation of parameters in the model using data X_α, $\alpha = 1, 2, \ldots N$ on X. [7+6+7=20]

6. *Either* 6.1.1 and 6.1.2

6.1.1 The problem is to use one of X_1 and X_2, or to use one of the two principal components of X_1 and X_2, to predict Y using a linear regression equation.

Decide on your choice when the covariance matrix of $\begin{pmatrix} Y \\ X_1 \\ X_2 \end{pmatrix}$ is

$$\Sigma = \begin{pmatrix} 2 & 0.4 & 0.6 \\ 0.4 & 2 & -2 \\ 0.6 & -2 & 5 \end{pmatrix}.$$

6.1.2 Compute and compare the two multiple correlations between Y and X_1 and X_2, and between Y and the two principal components of X_1 and X_2,

[30+10=40]

Or 6.2.1 and 6.2.2

6.2.1 Let $\bar{x}_1 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$, $\bar{x}_2 = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $\bar{x}_3 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ be the sample means and $S = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$ be the pooled sample covariance matrix based on equal size samples from three populations.. Classify the unit into one of the three populations based on the observation $\begin{pmatrix} 4.5 \\ 4.5 \end{pmatrix}$ on the unit using Fisher's discriminant function.

6.2.2 How would the classification change with the assumption of normality of the three populations with different unknown mean vectors but the same unknown covariance matrix?

[30+10=40]

Indian Statistical Institute

Second Semestral Examination, 2010-11

M. Stat. I year (B-stream)
Multivariate Analysis

Date: May 13, 2011 | Maximum Marks: 60 | Time: 3 hours

Note that the paper contains 80 marks. Symbols used in the paper have their usual significance.

1. Given a non-negative definite covariance matrix K, describe the notion of reproducing kernel Hilbert space (RKHS) corresponding to K. Suppose that K is given by $K(i,j) = \min(i,j)$ for $i, j = 1, 2, \ldots, p$. Using induction find a suitable orthonormal basis of the RKHS of K explicitly for $p \geq 3$. [5 + 10 = 15]

2. Consider a multivariate linear model setup where $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ip})'$ are independent multivariate normal random vectors with $EY_{ij} = x_i'\beta_j$ and common non-singular covariance matrix K for some $p > 1$, $i = 1, 2, \ldots, n$ and a sequence of $r \times 1$ regression vectors x_i's. Assume that the design matrix is of full rank $r(< n)$. The vectors $\beta_1, \beta_2, \ldots, \beta_p$ are regression coefficients for p components respectively. Show that by virtue of a suitable linear transformation on the variables Y_{ij}'s it is possible to create another set of independent multivariate normal random vectors $Z_1, Z_2, \ldots, Z_n$ such that Z_i has multivariate normal with mean $B'T_i$ for suitable transformed regressors T_i's, coefficient matrix $B = [\beta_1, \beta_2, \ldots, \beta_p]$ and covariance matrix K respectively for $i = 1, 2, \ldots, r$, with $T = [T_1, T_2, \ldots, T_r]$ being non-singular $r \times r$ and Z_i is distributed $N_p(0, K)$ for $i = r+1, \ldots, n$. Hence show that the MLE of K under the above model has a $W_p(n-r, K)$ distribution upto a scale factor. [15]

3. Let $X_1, X_2, \ldots, X_n$ be iid $N_p(\mu, K)$ with K being unknown, nonsingular. Obtain a suitable version of the Likelihood ratio test statistic for testing $H_0 : \mu = 0$ and derive its distribution under the null. [15]

4. Let S_n denote the sample dispersion matrix of a set of iid $N_p(\mu, I_p)$ observations. Further let $S_n^{(p-1)}$ denote the upper left $(p-1) \times (p-1)$ submatrix of S_n (that is, the minor of $S_{n,pp}$). Find the conditional distribution of $|S_n|$ given $S_n^{(p-1)}$. [15]

5. Consider the problem of discriminating between two multivariate normal populations $N_p(0, K)$ and $N_p(\mu, \sigma^2 K)$ respectively. Describe the optimal classifier ϕ which minimizes the total misclassification error $P_{12}(\phi) + P_{21}(\phi)$ with standard notation. Suppose further that μ, K, σ^2 are all unknown. How would you estimate these parameters based on two training samples from the two populations? [10 + 10 = 20]

INDIAN STATISTICAL INSTITUTE

Second-Semester Examination: (2010 – 2011)
(Back Paper)
M. Stat. (NB-Stream) 1st Year

Programming and Data Structures

Date: 27.06.2011 Maximum Marks: 100 Duration: 3 hours

Answer all questions.

1. Answer the following: [3 + 4 + 5 + 3 = 15]
   a) How can one dynamically allocate an one-dimensional array?
   b) Explain with an example the differences between realloc( ) and calloc( ) in terms of the functions they perform.
   c) Explain the output of following C program:

```
#include<stdio.h>
struct number
{
    int x;
    float y;
};
int main()
{
    int m;
    float n;
    struct number p;
    FILE *fp;
    fp=fopen("input.txt","w");
    m=10;
    n=213.76;
    fwrite(&m,sizeof(int),1,fp);
    fwrite(&n,sizeof(float),1,fp);
    m=15;
    n=518.29;
    fwrite(&m,sizeof(int),1,fp);
    fwrite(&n,sizeof(float),1,fp);
    fclose(fp);
    fp=fopen("input.txt","r");
    fread(&p,sizeof(struct number),1,fp);
    fread(&p,sizeof(struct number),1,fp);
    printf("\n%d\t%.2f\n",p.x,p.y);
    printf("\n%d\t%.2f\n",p.x,p.y);
    fclose(fp);
    return 0;
}
```

P.T.O

d) What is the difference between *struct x1{........}* and *typedef struct {......}x2.*

2. What is an AVL search tree? Construct an AVL search tree by inserting the following elements in the order of their occurrences: [68, 13, 49, 32, 19, 126, 99, 81]. [6 + 9 = 15]

3. What is a Min-Heap? Construct the Min-Heap for sorting the following list: [50, 44, 65, 31, 44, 73, 67, 85, 69]. [6 + 9 = 15]

4. Write a program to read a matrix from an input file "Input.txt'' and find whether the given matrix is symmetric. While writing the code, you need to take care of the following:
   a) The names of the variables should reflect their functions.
   b) Proper documentation of the program is to be provided.
   c) Use dynamic memory allocation. [25]

5. Explain with suitable examples the differences between [5 x 4 = 20]
   a) int *data( ) and int *(*data)( )
   b) a string and an array of characters
   c) fscanf( ) and fread( )
   d) structure and union

6. Discuss the following with suitable examples: [5 x 2 = 10]
   a) Circular linked list
   b) Deque

MULTIVARIATE ANALYSIS
M. Stat. I year [NB stream]

Semester Backpaper Examination

Time : 3 hours **Maximum marks 100** 28.06.11 ~~XX-XX-2011~~

Answer all questions

To justify your steps, clearly state any result that you use. You may assume results for univariate distributions.

1. Denote by $\underline{X} = (X_1, X_2, X_3, \ldots\ldots\ldots, X_N)$, where X_α's are mutually independent $N_p[\,0\,,\Sigma\,]$, $\alpha = 1, 2, \ldots N$.
   Show that
   $$\underline{X}\,\mathrm{H}\,\underline{X}' \sim W_p\,[r, \Sigma] \text{ if } l'\underline{X}\,\mathrm{H}\,\underline{X}'l \sim \sigma_l^2\,\chi^2(r)$$
   for any non- null l, where $\sigma_l^2 = l'\Sigma l$. [12+8=20]

2. Let B and E be the between and the within SSP matrices in the MANOVA table for a CRD with r_i replications of the ith treatment, $i = 1, 2, \ldots., t$.
   Show that B and E are independently distributed.
   Express the distribution of $|B|/|B+E|$ in terms of univariate random variables [8+12=20]

3. Let Y be $N_p[\,0\,,\Sigma\,]$, W be $W_p\,[r, \Sigma]$ and Y and W be independently distributed.
   Derive the distribution of $l'\Sigma^{-1}l\,/\,l'W^{-1}l$, for any non - null l.
   Derive the distribution of $Y'\Sigma^{-1}Y$.
   Consider a standard linear multiple regression set-up of U on $X_1, X_2, \ldots, X_p$, based on N observations on U and each of X's.
   Find the distribution of the residual sum of squares of U after regressing U on $X_1, X_2, \ldots, X_p$ by the method of least-squares. [12+8+10=30]

4. Discuss essential steps to find Fisher's LDF for classification into one of g populations. [10]

5. **Consider $B = (X_1 - \overline{X}, X_2 - \overline{X}, X_3 - \overline{X}, \ldots\ldots\ldots, X_N - \overline{X})$ where X_α, $\alpha = 1, 2, \ldots N$ are observations on a p x 1 vector X and $\overline{X}$ is the mean.**
Let B be of rank $r<p$. Find the p x N matrix C of rank $s<r$ that minimizes the square of the Frobenious norm of B-C.
How were principle components used in finding the minimum of the norm?
[15+5=20]

INDIAN STATISTICAL INSTITUTE
Backpaper Examination – Semester II : 2010-2011
M. Stat. I Year
Metric Topology and Complex Analysis

28.06.11
Date : 00.00.00 **Maximum Score : 100** Time : 3 Hours

Note : This paper has **FIVE** questions, each carrying **20 MARKS**. In your proofs, you should **clearly state the results**, if any, that you are using.

1. (a) Let X, Y be metric spaces. Show that if $f : X \to Y$ is uniformly continuous, then for any Cauchy sequence $\{x_n\}$ in X, the sequence $\{f(x_n)\}$ is cauchy in Y.

   (b) Assuming that Y is complete, show that if D is a dense subset of X, then any function $g : D \to Y$ which is uniformly continuous, has a unique extension to a uniformly continuous function $f : X \to Y$.

   10+10=[20]

2. (a) Let X be a metric space. A collection of subsets of X is said to have the **finite intersection property (fip)**, if every finite subcollection has a non-empty intersection. Show that X is compact if and only if for every collection of subsets with the fip, the intersection of the closures of all the sets in the collection is non-empty.

   (b) Let X, Y be metric spaces and suppose that Y is compact. Show that a function $f : X \to Y$ is continuous if and only if the set $G_f = \{(x, f(x)) : x \in X\}$ is closed in the product space $X \times Y$.

   10+10=[20]

3. Let u and v be real-valued functions with continuous first order partial derivatives on a connected open subset $D \subset \mathbb{C}$. Show that the function $f : D \to \mathbb{C}$ defined as $f(z) = u(z) + iv(z)$ is analytic if and only if u and v satisfy the Cauchy-Riemann equations on D.

   [20]

4. (a) Show that for an analytic function f on a connected open subset $D \subset \mathbb{C}$, the following conditions are equivalent:
   (1) $f \equiv 0$;
   (2) there is a point $z_0 \in D$ such that $f^{(n)}(z_0) = 0$ for all $n \geq 0$;
   (3) the set $\{z \in D : f(z) = 0\}$ has a limit point in D.

   (b) Let f and g be analytic functions on a connected open subset $D \subset \mathbb{C}$. Show that if $fg \equiv 0$, then either $f \equiv 0$ or $g \equiv 0$.

   12+8=[20]

5. (a) Prove the following version of Cauchy Integral Formula:
   Let f be an analytic function on an open subset $D \subset \mathbb{C}$ and let γ be a piecewise smooth closed curve in D satisfying $n(\gamma; z) = 0$ for all $z \notin D$. Then, for all $z \in D \setminus \{\gamma\}$, one has
   $$n(\gamma; z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw.$$

   (b) State and prove Morera's Theorem.

   12+8=[20]

INDIAN STATISTICAL INSTITUTE
Second Semester Back Paper Examination: 2010-11
M. Stat. I Year
Measure Theoretic Probability

Date: 30.06.11 **Maximum Marks: 100** **Duration: 3 Hours**

1. (a) Let Ω be a non-empty set and $\mathcal{F}$ be a σ-field defined on Ω. Show that the cardinality of $\mathcal{F}$ can not be 30.

   (b) Give example of a σ-field on $(0,1]$ that is not countably generated. Justify your answer.

   (c) In some finitely additive probability measure space $(\Omega, \mathcal{F}, P)$, give example of a sequence of $\mathcal{F}$-sets $\{A_n\}$ such that

   (i) $A_n \uparrow A$, but $P(A_n) \nrightarrow P(A)$

   (ii) $A_n \downarrow A$, but $P(A_n) \nrightarrow P(A)$

   (d) Let $\mathcal{J} = \{(a,b] : 0 < a \le b \le 1\}$

   Show that $\sigma(\mathcal{J}) = \sigma$- field generated by the collection of all open sets in $(0,1]$

   (e) Give an example of an uncountable subset (of R) whose Lebesgue measure is Zero. Justify your answer.

   (f) Show that the characteristic function of standard Cauchy variable in $e^{-|t|}$, $t \in \mathrm{R}$.

   (g) Let $E \in \mathcal{R} \times \mathcal{R}$, where $\mathcal{R}$ is Borel σ-field on R, and $E_x = \{y : (x, y) \in E\}$. Show that for each fixed x, E_x is a measurable set.

   (h) In a probability space $(\Omega, \mathcal{F}, P)$, let $\{A_n\}$ be a sequence of independent $\mathcal{F}$-sets. Show that $P(\limsup_n A_n) = 0$ or, 1.

[9×8 = 72]

2. (a) State and Prove Dominated Convergence Theorem.

   (b) Give an example of a sequence of mesurable and bounded function $\{f_n\}$ with $f_n : R \to R$, such that $\lim_n f_n(x)$ exists and equals $f(x)$ for all x, but

   $$\lim_n \int_R f_n(x)\, d\lambda(x) \neq \int_R f(x)\, d\lambda(x)$$

   where λ = Lebesgue measure on R.

   (c) Let f and g be two measurable simple functions defined from Ω to $[0, \infty]$. Show that

   $$\int_\Omega (f+g)\, d\mu = \int_\Omega f\, d\mu + \int_\Omega g\, d\mu$$

   Where μ is some non-negative measure defined on Ω.

[14+6+8]

*****

Indian Statistical Institute
M. Stat - I: 2010-11
Back Paper
Time Series Analysis

Date: 01/07/11

Duration: 3 hours

Marks: 100

1. Suppose $X_t = (1+0.2B)(1-0.8B)Z_t$, $\{Z_t\} \sim WN(0,\sigma^2)$. Obtain $\{\pi_j\}$ in the representation $Z_t = \sum_{j=0}^{\infty}\pi_j X_{t-j}$. (10)

2. Consider the causal AR(1) process $X_t = \rho X_{t-1} + Z_t$ where $\{Z_t\}$ is a sequence of independent $N(0,\sigma^2)$ variables. Derive the conditional and unconditional maximum likelihood estimator of ρ. (10)

3. Show that there is no stationary solution of $X_t = \rho X_{t-1} + Z_t$ when $|\rho|=1$. (10)

4. Describe the spectral density of an MA(2) process. (10)

5. Suppose that a time series $\{X_t\}$ is represented by $X_t = \sqrt{2}\sum_{j=1}^{q}\sigma_j \cos(\lambda_j t - \gamma_j)$, where $\gamma_1, \gamma_2, \ldots, \gamma_q$ are independent random variables being uniformly distributed in the interval $[0, 2\pi]$ and σ_j and λ_j, $j = 1, 2, \ldots, q$ are constants. Show that $\{X_t\}$ is a weakly stationary process. (15)

6. Describe the Dickey-Fuller test for testing the presence of unit roots in a given time series. (10)

7. Consider the infinite-order MA process $\{X_t\}$ defined by

$$X_t = Z_t + \alpha(Z_{t-1} + Z_{t-2} + \ldots),\ Z_t \sim WN(0,\sigma^2),$$

where α is a constant. Show that this process is not weakly stationary. Also show that the series of first differences of $\{X_t\}$ i.e. $\{X_t - X_{t-1}\}$ is stationary. (15)

8. Let the time series X_t be defined by $X_t = e_t + 0.6e_{t-1}$, where $\{e_t\}$ is a sequence of independent $N(0,1)$ variables. Given a sample of 10,000 observations from such a time series, what is the approximate joint distribution of the periodogram ordinates associated with $\omega_{1500} = 2\pi(1500)/10000$ and $\omega_{2500} = 2\pi(2500)/10000$? (20)

Indian Statistical Institute
M. Stat – I: 2010-11
Back Paper
Time Series Analysis (Supplementary)

Date: 6.7.11 Duration: 3 hours

Marks: 100

1. Suppose $X_t = (1+0.2B)(1-0.8B)Z_t$, $\{Z_t\} \sim WN(0,\sigma^2)$. Obtain $\{\pi_j\}$ in the representation $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$. (10)

2. Consider the causal AR(1) process $X_t = \rho X_{t-1} + Z_t$ where $\{Z_t\}$ is a sequence of independent $N(0,\sigma^2)$ variables. Derive the conditional & unconditional maximum likelihood estimator of ρ. (10)

3. Show that there is no stationary solution of $X_t = \rho X_{t-1} + Z_t$ when $|\rho| = 1$. (10)

4. Describe the spectral density of an MA(2) process. (10)

5. Suppose that a time series $\{X_t\}$ is represented by $X_t = \sqrt{2}\sum_{j=1}^{q} \sigma_j \cos(\lambda_j t - \gamma_j)$, where $\gamma_1, \gamma_2, \ldots, \gamma_q$ are independent random variables being uniformly distributed in the interval $[0, 2\pi]$ and σ_j and λ_j, $j = 1, 2, \ldots, q$ are constants. Show that $\{X_t\}$ is a weakly stationary process. (15)

6. Describe the Dickey-Fuller test for testing the presence of unit roots in a given time series. (10)

7. Consider the infinite-order MA process $\{X_t\}$ defined by

$$X_t = Z_t + \alpha(Z_{t-1} + Z_{t-2} + \ldots),\ Z_t \sim WN(0,\sigma^2),$$

where α is a constant. Show that this process is ~~non-stationary~~ not weakly stationary. Also show that the series of first differences of $\{X_t\}$ i.e. $\{X_t - X_{t-1}\}$ is stationary. (15)

8. Let the time series X_t be defined by $X_t = e_t + 0.6e_{t-1}$, where $\{e_t\}$ is a sequence of independent $N(0,1)$ variables. Given a sample of 10,000 observations from such a time series, what is the approximate joint distribution of the periodogram ordinates associated with $\omega_{1500} = 2\pi(1500)/10000$ and $\omega_{2500} = 2\pi(2500)/10000$? (20)

(r) In the case of a d-dimensional spherically symmetric distribution, for any $\mathbf{x} \in \mathbb{R}^d$ and any orthogonal matrix $\mathbf{H}$, $S_r(\mathbf{Hx})=\mathbf{H}\, S_r(\mathbf{x})$, where $S_r(\eta)$ denotes the spatial rank of η. [3]

(s) If we have a data set consisting of three observations (x_1, y_1), (x_2, y_2) and (x_3, y_3), any linear fit that does not pass through any of these data points is a nonfit. [2]

(t) If there are 24 observations on (X, Y) and no three of them are collinear, no linear fit can have regression depth more than 13. [3]

3. Consider a multiple linear regression model $\mathbf{Y} = \mathbf{X}\beta + \epsilon$, where $\epsilon \sim N(0, \sigma^2\mathbf{I})$ and $\text{rank}(\mathbf{X}_{n\times p}) < p$.

(a) Check whether β is unbiasedly estimable. [2]

(b) Define $\widehat{\beta}_0 = (\mathbf{X}'\mathbf{X})^-\mathbf{X}Y$, where A^- denotes a generalized inverse of $\mathbf{A}$. If $\mathbf{m}'\beta$ is unbiasedly estimable, show that $\mathbf{m}'\widehat{\beta}_0$ is an unbiased estimator of $\mathbf{m}'\beta$. Check whether $\mathbf{m}'\widehat{\beta}_0$ is unique ? [2+2]

(c) If $||\mathbf{m}|| = 1$, show that no linear unbiased estimator of $\mathbf{m}'\beta$ can have variance smaller than σ^2/λ, where λ is the largest eigenvalue of $\mathbf{X}'\mathbf{X}$. [4]

4. (a) Consider a logistic regression model, where the response variable takes only two values 0 and 1. Suppose that we have a data set $(x_1, y_1), \ldots, (x_n, y_n)$ of size n. If there exist $\beta_0 \in \mathbb{R}$ and $\beta \in \mathbb{R}^p$ that satisfy $(2y_i - 1)(\beta_0 + \mathbf{x}_i'\beta) > 0$ for all $i = 1, 2, \ldots, n$, then show that the maximum likelihood estimate of the parameters of the logistic regression model will not exist. [4]

(b) Give an example of a regression problem, where additive regression is not appropriate, but projection pursuit regression can be helpful in properly estimating the regression surface. Give justification to your answer. [3]

(c) Briefly describe an algorithm for fitting a projection pursuit regression model. [3]

5. (a) Show that the local linear estimate of the regression function turns out to be usual linear regression estimate when the bandwidth h tends to infinity. [3]

(b) Show that k-nearest neighbor estimate of the regression function can also be viewed as a kernel estimate of regression function when a uniform kernel with an adaptive choice of h is used. [3]

(c) Show that the Nadaraya-Watson estimate (based on a Gaussian kernel) of the regression function converges to 1-nearest neighbor estimate as the bandwidth h shrinks to 0. [4]

6. Consider a linear regression model $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \epsilon_i \quad i = 1, 2, \ldots, n$, where the ϵ_i's are independent and identically distributed as $N(0, \sigma^2)$ variables.

(a) Show that LASSO regression and ridge regression methods can be viewed as Bayesian estimation techniques for suitable choices of priors on regression parameters. [3]

(b) Show that these two regression methods can also be viewed as penalized least squares methods for suitable choices of the penalty function. [3]

(c) Give geometric interpretation of the estimates obtained by LASSO regression and ridge regression. Also explain how LASSO regression helps in selecting a parsimonious regression model. [4]

7. (a) If $y_1, y_2, \ldots, y_n$ are n observed values of a random variable Y, for any $p \in (0, 1)$, show that $\theta_p = \inf\{y : \sum_{i=1}^n I\{y_i \leq y\} \geq np\}$ is a minimizer of $\{\sum_{i=1}^n |y_i - \theta| + (2p-1)\sum_{i=1}^n (y_i - \theta)\}$. [5]

(b) Assume that $Y_1, \ldots, Y_n$ are independent random variables, where $\mu_i = E(Y_i) = \mathbf{x}_i'\beta$ for given $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})'$ and unknown parameter β. If Y_i $(i = 1, 2, \ldots, n)$ is uniformly distributed over the range $\mu_i \pm \sigma$ (σ unknown), then show that the maximum likelihood estimate of β is obtained by minimizing $\max_i |y_i - \mathbf{x}_i'\beta|$. [5]

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2010-11

M. Stat. I Year (NB Stream)
Real Analysis

Date: 30.08.10 **Maximum Marks: 40** **Duration: $1\frac{1}{2}$ 1Hours**

The Paper contains questions of 46 marks. You may attempt all questions. Maximum marks you can score is 40.

1. a) Show that for $A \subseteq R$, boundary of A = boundary of A^c

   b) Show that if $A \subseteq R$ and A is open or closed, then the interior of the boundary set of A is empty.

   c) Give example of an open set (in R) whose boundary set is uncountable. Justify your answer.

   [3+4+5]

2. a) Show that the set of all algebraic numbers is countable.

   b) Show that the set of all prime numbers is infinite.

   [5+5]

3. Check if the following sequences are Cauchy or not. Give reasons.

   a) $s_n = 1 + 1/2 + 1/3 + \ldots\ldots + 1/n$

   b) $s_n = a_1 - a_2 + a_3 - \ldots\ldots + (-1)^{n-1} a_n$
   where $\{ a_n \}$ is a decreasing sequence with limit 0.

   [5+5]

4. Give examples (with justifications) :

   a) Two subsets A , B of R for which $\text{Int}(A) \cup \text{Int}(B) \neq \text{Int}(A \cup B)$

   b) $(S')' \neq S'$ where S is a subset of R and S′ denotes derived set of S.

   c) $f(A \cap B) \neq f(A) \cap f(B)$ where f is from S to T and A , B are two subsets of S.

   d) An infinite subset of R with no accumulation point.

   [2+2+2+1]

5. State and prove Bolzano- Weierstrass Theorem.

   [7]

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