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Midterm Examination
Large Sample Statistical Methods
First Semester
2011-2012 Academic Year
M.Stat. First Year (B-Stream Only)

Date : 02.09.11 Maximum Marks: 40 Duration :- 2 hours

Answer as many questions as you can. The maximum you can score is 40.

1. Suppose a sequence of real valued random variables $\{X_n\}$ converges in probability to both X and Y, where X and Y are real valued random variables. Show that with probability one, X equals Y. [4]

2. Suppose $X_n \sim$ Normal(n, n), for $n \geq 1$. Show that $n^{1000} P(X_n > 2n)$ tends to 0 as $n \to \infty$. [4]

3. Let $\{X_n\}$ be a sequence of iid random variables with distribution F. Suppose $w(F) = \sup\{x : F(x) < 1\}$ is finite. Show that $X_{(n)}$ converges almost surely to $w(F)$, where $X_{(n)} = \max\{X_1, \ldots, X_n\}$. [4]

4. Show that a sequence of random variables $\{X_n\}$ is stochastically bounded if and only if for every sequence of reals $\{k_n\}$ such that $k_n \to \infty$ as $n \to \infty$, $P(|X_n| > k_n) \to 0$ as $n \to \infty$. [4]

5. Suppose $X_n \sim N(0, 1 + \frac{1}{n})$ for $n \geq 1$ and $Y_n = X_n I(|X_n| \leq n)$. Does $e^{X_n} - e^{Y_n}$ converge in probability ? Prove your answer. [4]

6. State with assumptions, the weak Bahadur representation of sample quantiles due to J.K. Ghosh. Assuming that the representation is proved for the smallest sample median under the assumptions stated by you, prove it for any sample median. [1+3]

7. Suppose you have a sequence of iiid observations $(X_1, Y_1), (X_2, Y_2), \ldots$ from a bivariate distribution G supported on the unit disc $\{(x, y) : 0 \leq x^2 + y^2 \leq 1\}$. Suppose further that the distribution has a continuous density $g(x, y)$ with $g(0, 0) > 0$. Let $D_i = \sqrt{X_i^2 + Y_i^2}$, for $i = 1, 2, \ldots$. Let $D_{(1,n)} = \min\{D_1, \ldots, D_n\}$. Find real constants u_n and $v_n > 0$ such that $v_n(D_{(1,n)} - u_n)$ converges to a non-degenerate distribution. Prove your assertion and write the explicit form of the limiting distribution function. [6]

8. Suppose $X_1, \ldots, X_n$ are iid from a distribution F given by $F(x) = xI_{(0 \leq x \leq \frac{1}{2})} + (2x - \frac{1}{2})I_{(\frac{1}{2} \leq x \leq \frac{3}{4})}$. Let $\hat{\zeta}_{\frac{1}{2},n}$ denote the smallest sample median based on $X_1, \ldots, X_n$. For $n = 10000$, can you give an approximate value of the probability that the random variable $\hat{\zeta}_{\frac{1}{2},n}$ is larger than 0.51 ? Justify your answer. [6]

9. Suppose $X_n \sim \text{Beta}(\frac{2}{n}, \frac{3}{n})$ for $n \geq 1$. Does X_n converge in distribution to a non-degenerate random variable ? Prove your assertion. [8]

INDIAN STATISTICAL INSTITUTE
Midsemestral Examination
M.Stat 1st year (2011-12)

Subject: Measure Theoretic Probability | Date of Exam. : 5 September, 2011
Max. Marks : 30 | Time : 1hr 30 mins

The questions carry a total of 35 marks. Maximum marks you can score is 30.Attempt all questions.

1. Give example of a σ -field that is not countably generated. Justify your answer.

[6]

2. State Monotone class theorem. Using the theorem show that if P_1 and P_2 , two probability measures agree on a field $\mathcal{F}$ they have to agree on $\sigma(\mathcal{F})$.

[1+5]

3. Show that the collection of subsets of $(0,1]$ which are not in Lebesgue completion of $(0,1]$ has cardinality same as the cardinality of the collection of sets in Lebesgue completion of $(0,1]$.

[6]

4. Let Q be a probability measure on the Borel σ-field on $(0,1]$ and $Q(A) = 1/\sqrt{2}$ if Lebesgue measure of A is $1/\sqrt{2}$. Show that Q is identical with Lebesgue measure on $(0,1]$.

[5]

5. Let $(\Omega, \mathcal{F}, P)$ be a probability measure space (with σ-field $\mathcal{F}$) which may not be complete. Let $\mathcal{F}' = \{ A' : A' \subseteq \Omega$, there exist $A,B \in \mathcal{F}$ such that $A \Delta A' \subseteq B$ and $P(B) = 0\}$and P is extended to $\mathcal{F}'$ by identifying $P(A') = P(A)$, where A' and A are as above. Show that $\mathcal{F}'$ is a σ-field and P is well defined on $\mathcal{F}'$.
Also show that P is a complete probability measure on $\mathcal{F}'$.

[6+6]

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Indian Statistical Institute

First Midsemestral Examination, 2011-12

M. Stat I year (B-stream)
Statistical Inference I

40

Date: September 7, 2011 · Maximum Marks: [redacted] Time: 2 hours

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[redacted]

1. Let $K \subset \mathbb{R}^n$ be compact. Show that the convex hull $\mathrm{conv}(K)$, is also compact. [redacted] [10]

2. Consider a game between two players where Player 1 chooses one of the three sealed boxes at random and keeps Rs. 100 in that box without the knowledge of Player 2. Player 2 chooses one of the boxes. In the next move, Player 1 removes one of the boxe(s) not chosen by Player 2 and which does not contain the money. Finally, Player 2 can either open the box he chose earlier or swap to choose the remaining box not chosen so far. The final payoff for Player 2 is the content of the box chosen in the last move (assume the game is zero-sum). Describe the game tree indicating the information sets, if any. Write down the sets of feasible strategies. Obtain expected payoffs for all the feasible strategies of the game and hence find the optimal strategy for Player 2. [redacted] [10]

3. Let $K \subset \mathbb{R}^n$ be a compact, convex subset and $x_k \to x^*$ be a convergent sequence in $\mathbb{R}^n$ such that $x_k \notin K$ for every $k \geq 1$. Show that there exists a nonzero vector $p \in \mathbb{R}^n$ such that $p'x \leq p'x^*$ for every $x \in K$. [redacted] [20]

INDIAN STATISTICAL INSTITUTE
Mid-semester Examination: 2011-2012 (First Semester)

M. Stat I Year

Applied Stochastic Processes

Date: September 09, 2011 Maximum Marks: 50 Duration: 2 hours.

Note: Total mark is 56. Answer as many as you can.

1. (a) A radioactive source emits particle in accordance with a Poisson process with parameter λ. Each particle emitted has a probability of p of being recorded. Let $N(t)$ be the number of particles recorded in an interval of length t, derive the expression of $P[N(t) = n]$.

   (b) Suppose that customers arrive at a store in groups. Let $N(t)$ be the number of groups arrived by time t and arrival of groups is in accordance with Poisson process with parameter α. The number of customers in the ith group is given by a zero-truncated Poisson variate X_i, $i = 1, 2, \ldots$ with parameter λ. X_i's are independent and also independent of $N(t)$. Let $M(t)$ be the total number of customers arrive by time t. Find the p.g.f. of $M(t)$. Hence find the expectation of $M(t)$.

   (c) Let $X(t)$ be a Poisson process with parameter λ. Determine the covariance between $X(t)$ and $X(t+s)$ for $t > 0,\ s > 0$.

   $[4 + 8 + 3 = 15]$

2. Consider a birth and death process $X(t)$ with birth and death rates $\lambda_n = n\lambda$ and $\mu_n = n\mu$ (for $n \geq 1$), respectively and $\lambda_0 = \mu_0 = 0$.

   (a) Suppose that $\lambda = \mu$. Derive the p.g.f. of $X(t)$. Obtain the expression of probability of extinction. Find the distribution of the time to reach state 0.

   (b) Define $M_i(t) = \sum_{n=1}^{\infty} n^i p_n(t)$, for $i = 1, 2, \ldots$, where $p_n(t) = P[X(t) = n]$. Show that (using the differential equations) $M_2(t)$ satisfies the following

   $$M_2'(t) = 2(\lambda - \mu)M_2(t) + (\lambda + \mu)M_1(t).$$

   $[10 + 8 = 18]$

3. Consider a general birth and death process $X(t)$ with birth and death rates λ_n $(n \geq 0)$ and μ_n $(n \geq 1)$, respectively.

   (a) Derive the expression of equilibrium distribution. Write down the condition for existence of the equilibrium condition. Check whether equilibrium distribution exists if λ_n is 0 for some $n > 0$.

(b) Suppose customers arrive at the service station in accordance with Poisson process with parameter λ and are served by N servers. Each server work at a rate μ and the customer immediately leave the station after the service is complete. If there are more than N customers waiting for service, the excess customers form a queue. Let $X(t)$ be the number of customers in the system at time t. Write this as a suitable birth and death process, specifying the birth and death rates. Write down the corresponding equilibrium probabilities and derive the condition for existence of the same.

[7 + 6 = 13]

4. Consider a linear birth and death process with immigration, $X(t)$, with rates $\lambda_n = n\lambda + \alpha$, $n \geq 0$ and $\mu_n = n\mu$, $n \geq 1$. Show that $E[X(t)]$ satisfies the differential equation

$$\frac{d}{dt}E[X(t)] = (\lambda - \mu)E[X(t)] + \alpha.$$

Hence find $E[X(t)]$ when $X(0) = N$. [10]

INDIAN STATISTICAL INSTITUTE
MID-SEMESTRAL EXAMINATION 2011-2012

M.STAT 1st year. Design of Experiments

12.09.
[redacted], 2011, Total marks 20+3. Duration: One hour

Answer all questions.

Keep your answers brief and to the point.
Marks will be deducted for rambling answers.

1. [$3 \times 4 = 12$]

a) For a connected block design show that $(C + \frac{\mathbf{r}\mathbf{r}'}{n})^{-1}$ a generalized inverse of the C matrix, where $\mathbf{r}$ is the replication vector of the design.

b) Consider a BIB (v, b, r, k) design. Suppose each block of this design is augmented by all the v symbols applied once each, so that the resulting design still has b blocks, but now each are of size $k + v$. Will this design be balanced? Justify.

c) Will the design in (b) above be orthogonal? Justify.

d) Prove that $b \geq v$ in a BIB design with parameters v, b, r, k.

2. A factory employs 25 workers and has 5 machines; each machine needs 5 workers to operate it. Give an assignment of the workers to the machines for a week, (Monday to Saturday), so that each day every worker gets work and every pair of workers work together on a machine exactly once in the week. [5]

3. Starting from the arrangement obtained in question 2 above, can you construct a BIB design with parameters $r = 24,\ k = 20, \lambda = 19$? (**Only indicate method,** blocks need not be shown.) [3]

4. BONUS QUESTION: Starting from the arrangement in question 2 above, can you construct a BIB design with parameters $r = k = 6, \lambda = 1$? (**Only indicate method,** blocks need not be shown). [3]

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination : 2011-2012

M.Stat. I Year

GROUP –A(Sample Surveys)

12·09·
Date : [redacted]2011 | Maximum Marks : 50 | Duration : $1\frac{1}{2}$ Hours

Answer ANY TWO questions . Marks allotted to each question are given within the parentheses . Standard notations and symbols are used .

1.(a) If π_i and π_{ij} denote respectively the first order and the second order inclusion probabilities , show that

(i) $\sum_{i=1}^{N} \pi_i$ = E[$\vartheta(s)$] , where $\vartheta(s)$ is the effective sample size

(ii) $\sum_{i\neq j=1}^{N} \sum \pi_{ij}$= Var[$\vartheta(s)$] + $\vartheta(\vartheta - 1)$ where ϑ is the expected effective sample size .

(b) Show that

$$\vartheta(\vartheta - 1) + \theta(1-\theta) \leq \sum_{i\neq j=1}^{N} \sum \pi_{ij} \leq N(\vartheta - 1)$$

where $\vartheta = [\vartheta]+\theta$, $0 < \theta < 1$ and $[\vartheta]$ denotes the greatest integer contained in ϑ.

(5+10+10)=[25]

2. (a) Define the Horvitz- Thompson estimator of the population total of a finite population and derive its sampling variance .

(b) Obtain the Yates-Grundy's form of an unbiased estimator of the variance of the Horvitz-Thompson estimator . Mention a sufficient condition under which the variance estimator is non-negative . (10+10+5) =[25]

3. (a) Obtain the Des Raj's ordered estimator for a sample of size 2 drawn from a population of size N by PPSWOR sampling scheme ..

(b) Derive the sampling variance of the above estimator and also obtain an unbiased estimator of the variance . (7+15+3)=[25]

P.T.O

INDIAN STATISTICAL INSTITUTE
Semester Examination: 2011-2012 (First Semester)

M. Stat I Year

Applied Stochastic Processes

Date: 14/11/2011 Maximum Marks: 100 Duration: 3 hours.

Note: Total marks is 108. Answer as many as you can.

1. Let $\{N(t), t \geq 0\}$ be a counting process such that for given $\Lambda = \lambda$, $\{N(t), t \geq 0\}$ is a Poisson process having rate λ. Let G be the distribution function of Λ. Show that $\{N(t), t \geq 0\}$ has stationary but not independent increments. Compute $\lim_{h \to 0} P[N(h) \geq 1]/h$. Compute the conditional distribution of the time of the first event after t give $N(t) = n$. Find the variance of $N(t)$.

[6+4+4+4=18]

2. Consider a job shop consisting of M machines and a single repairman, and suppose that the amount of time a machine runs before the breaking down is exponentially distributed with mean $1/\lambda$ and the amount of time it takes the repairman to fix any broken machine is exponential with mean $1/\mu$. We say that the state is n whenever there are n machines down. Model this system as a birth and death process along with differential equations. Derive the expression of limiting probability π_n that n machines are down. Calculate the long-run proportion of time that a given machine is working.

[3+6+3=12]

3. Consider a machine that works for an exponential amount of time having mean $1/\lambda$ before breaking down; and suppose that it takes an exponential amount of time having mean $1/\mu$ to repair the machine. Let $X(t)$ denotes the state of the machine at time t. Write this as a general continuous time Markov process by specifying the matrix R of infinitesimal transition probabilities. If the machine is in working condition at time 0, then derive the probability that it will be under repair at time $t = 20$. Let $S_i(t)$ denote the time spent in state $i (= 0, 1)$ by time t. Calculate $E(S_0(t)|X(0) = 0)$.

[3+5+5=13]

4. Consider a discrete time branching process in which in each generation an individual either dies or is replaced by two offspring. Let X_n be the population size of the nth generation. Assuming $X_0 = 1$, derive the recursive relationship for the probability generating function of X_n. Hence find $P[X_2 = 2]$.

[7+3=10]

P.T.O.

5. Let $\{X_i\}$ be a renewal process with mean $\mu = E(X_1) < \infty$ and renewal function $M(t)$, then show that

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu}.$$

[12]

6. Let $\{X_i\}$ be a renewal process with mean $\mu = E(X_1) < \infty$. Let $\delta(t) = t - S_{N(t)}$ be defined as current age and $g(t) = E[\delta(t)]$, where $N(t)$ and $S_{N(t)}$ have usual meanings. Then show that

   (a) $\lim_{t\to\infty} \frac{\delta(t)}{t} = 0$, with probability 1.

   (b) $\lim_{t\to\infty} g(t) = \frac{E[X_1^2]}{2\mu}$.

[4+9=13]

7. The lifetime of a car is a random variable with distribution F. Mr. A buys a new car as soon as his old car either breaks down or reaches the age of T_0 years. A new car costs C_1 units and an additional cost of C_2 units is incurred whenever a car breaks down. Assume that a T_0-year old car in working order has resale value $R(T_0)$. There is no resale value of a failed car. Suppose that a cycle begins each time a new car is purchased. Calculate the long-run average cost per unit time.

[10]

8. Consider a s-server bank in which potential customers arrive in accordance with Poisson process with rate λ. However, an arrival can enter the bank if the number of customers in the bank is less than s. The service time distribution of each server is exponential with parameter μ. Let p_B be the probability that an arriving customer is blocked from entering the system.

   (a) Write down the steady state equations and derive the steady state distribution. Derive the expression of p_B.

   (b) Find the rate at which customers enter the bank.

   (c) Let L be the long run expected number of customers in the system. Then show that

   $$L = \frac{\lambda}{\mu}(1 - p_B)$$

   (d) Calculate the probability that a server is busy.

[10+2+5+3=20]

Indian Statistical Institute

First Semestral Examination, 2011-12

M. Stat I year (B-stream)
Statistical Inference I

Date: November 16, 2011 | Maximum Marks: 60 | Time: 3 hours

Note that the paper contains 65 marks

1. Let $K(x, y)$ be a continuous real valued function defined on $X \times Y$ where both X and Y are compact. If there exists $x_0 \in X$, $y_0 \in Y$ and a real number v such that

$$K(x_0, y) \geq v \qquad \text{for all} \quad y \in Y$$

and,

$$K(x, y_0) \leq v \qquad \text{for all} \quad x \in X$$

then show that

$$v = \min_y \max_x K(x, y) = \max_x \min_y K(x, y)$$

and conversely. [15]

2. Let $\mathscr{R} \subset \mathbb{R}^n$ be a convex risk set of a statistical decision problem with finite Θ (as discussed in class) and $\bar{\mathscr{R}}$ be its topological closure. Assume that $\mathscr{R}$ is bounded from below. Show that

   (i) the set of lower boundary points, $\lambda(\mathscr{R})$, of $\mathscr{R}$ is non-empty,

   (ii) one can find an example of a prior where Bayes rule does not exist (you have to describe $\mathscr{R}$ algebraically and prove accordingly, no pictorial arguments),

   (iii) if $\lambda(\mathscr{R}) \subset \mathscr{R}$ then minimal complete class exists.

[5 + 5 + 5 = 15]

3. State and prove Karlin's theorem on admissibility of linear estimators based on the sufficient statistic under squared error loss for one parameter exponential families. [20]

P.T.O.

4. Let X_i be independently distributed as $\mathrm{N}(\theta_i, 1)$ for $i = 1, 2, \ldots, n$, $n \geq 3$. Consider simultaneous estimation of $\theta = (\theta_1, \theta_2, \cdots, \theta_n)$ under the total mean square loss, that is, for any estimator $\delta = (\delta_1, \delta_2, \cdots, \delta_n)$ of θ based on $(X_1, X_2, \cdots, X_n)$

$$R(\delta, \theta) = E_\theta \sum_1^n (\delta_i - \theta_i)^2.$$

Let $\delta_i = (1 - \hat{B})\, X_i$, $i = 1, 2, \ldots, n$, be an estimator where $\hat{B} = B\left(\sum_1^n X_i^2\right)$ for some decreasing function B. Show that if $P_\theta(\hat{B} > 1) > 0$, for every θ then

$$R(\hat{\delta}, \theta) < R(\delta, \theta)$$

for every θ where $\hat{\delta}_i = \max(1 - \hat{B}, 0)\, X_i$, for $i = 1, 2, \ldots, n$. [15]

INDIAN STATISTICAL INSTITUTE

Firsr Semestral Examination : 2011 – 2012

M. Stat. I Year

Group A (Sample Surveys)

Date : 18.11.2011 Maximum Marks : 50 Duration : $1\frac{1}{2}$ Hours

Answer ANY TWO questions . Marks allotted to each question are given within the parentheses . Standard notations and symbols are used .

1.(a) Show that for a sampling design of fixed effective size n from a population of size N

(i) $\sum_{i=1}^{N} \pi_i = n$

(ii) $\sum_{j \neq i=1}^{N} \pi_{ij} = (n-1)\, \pi_i$

(iii) $\sum_{i\neq}^{N} \sum_{j=1}^{N} \pi_{ij} = n\,(n-1)$

where π_i and π_{ij} denote respectively the first and the second order inclusion probabilities .

(b) Show that for a fixed effective sample size design the variance of the Horvitz – Thompson estimator of the population total can be written as

$$Var(\hat{Y}) = \sum_{i<j=1}^{N} \sum_{j=1}^{N} (\pi_i \pi_j - \pi_{ij}) \left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2$$

Obtain an unbiased estimator of the above variance assuming that the second order inclusion probabilities are all positive and mention a condition under which it is non – negative .

(5+5+2+8+3+2) =[25]

2.(a) Describe a sampling scheme under which the classical ratio estimator of the population mean is unbiased . Derive the sampling variance of the ratio estimator under the same sampling scheme and also obtain an unbiased estimator of the variance of the estimator . Is it always non-negative ?

(b) What is an IPPS sampling scheme ? Describe how under some condition the sampling scheme in (a) above can be modified to an IPPS sampling scheme . Show that the Horvitz-Thompson estimator of the population total based on such a modified sampling scheme is more efficient than the Hansen-Hurwitz estimator based on PPSWR sampling scheme involving the same number of draws .

P.T.O. (5+5+4+1+2+2+6)=[25]

3.(a) What is Murthy's unordering principle ? Show that Des Raj estimator of the population total of a finite population based on a PPWWOR sample of size $n(\geq 2)$ can be improved upon by Murthy's unordering principle .

(b) Derive an expression for the sampling variance of the improved unbiased estimator for n=2 and obtain an unbiased estimator of the variance .

(3+7+10+5) =[25]

GROUP-B

INDIAN STATISTICAL INSTITUTE

First Semester Examination : 2011-12

Course Name: M.Stat. 1st Year (B-Stream)

Subject Name: Design of Experiments

Date: Total Marks: 33. The maximum you can score is 30.

Duration: $1\frac{1}{2}$ hours Note: Answer **all** Questions.

Answers should be to the point. Marks may be deducted for unduly lengthy answers.

1. (a) A factorial experiment is to be conducted with 3 factors F_1, F_2 and F_3 at levels 2, 3 and 4, respectively. Using suitable notation, **write down** the expression for a full set of orthonormal treatment contrasts belonging to interaction F_1F_2.

   (b) Prove that contrasts belonging to any two distinct interactions are mutually orthogonal.

   (c) A factorial experiment is to be designed with 4 factors, A, B, C, D, each at 2 levels. All main effects and all interactions are to be estimated. Furthermore, the main effects and 2-factor interactions have to be estimated with more precision than the other interactions. **Indicate** how you can construct a design for this in 4 replicates, each replicate consisting of 2 blocks of size 8 each. Show **any one** replicate in full.

   (d) **Write down** the variances of the estimators of main effect A, interaction ABC and interaction $ABCD$. [2+2+3+3=10]

2. (a) Define an orthogonal array with s symbols and strength t, $OA(N, k, s, t)$.

   (b) For each of the following statements **write** whether it is true or false.

   (i) An $OA(N, k, s, t)$ is also an $OA(N, k, s, t')$, where $0 < t' < t$.

   (ii) An $OA(s^2, k, s, 2)$ always exists if $k \leq s - 1$.

   (iii) Not all $N \times k'$ subarrays of an $OA(N, k, s, t)$ are orthogonal arrays of strength t.

   (iv) Suppose $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ is an $OA(N, k, s, t)$. If A_1 is itself an $OA(N_1, k, s, t_1)$, then A_2 is an $OA(N - N_1, k, s, t_2)$ with $t_2 \geq t_1$.

   (c) Prove that for $N \geq 4$, the existence of a Hadamard matrix of order N implies the existence of an $OA(N, N-1, 2, 2)$. [2+(2× 4) + 2=12]

3. (a) A factorial experiment is to be conducted with 5 factors, each at 2 levels. No blocks are to be used. Construct a fractional factorial design with only 8 treatment combinations so that all main effects are estimable when all interactions are absent.

   (b) Prove that for a symmetric BIB design, all pairs of distinct blocks intersect in the same number of treatments.

   (c) Given a symmetric BIB design with $v = 11, r = 6, \lambda = 3$, indicate how you can construct a BIB design with 6 treatments in 10 blocks of size 3 each. [5+3+3=11]

INDIAN STATISTICAL INSTITUTE
First Semester Examination: 2011-12

M. Stat. I Year (B Stream)
Measure Theoretic Probability

Date: 22.11.11 **Maximum Marks: 50** **Duration: 3 Hours**

The Paper contains questions of 56 marks.
Attempt all question. Maximum marks you can score is 50.

1. a) Let $\{X_n\}$ be a sequence of i.i.d. random variables each with mean 0 and variance 1.

   Let $Y_n = (X_1 + X_2 + \ldots + X_n)/\sqrt{n}$

   Show that there does not exist a random variable U such that $Y_n \to U$ (almost surely).

   b) Give example of a continuous distribution function F of some random variable X such that $\exists$ E with Lebesgue measure zero, but $P(X \in E) = 1$

   c) Derive characteristic functions of Standard Normal random variable and also of Cauchy random variable.

   d) State Inversion Theorem for characteristic functions. Use it to show that characteristic functions uniquely determine the distribution of the random variable.

[8+6+4+4]

2. a) Let $\Omega = (0,1]$. A is a Borel set in Ω, such that Lebesgue Measure of A is positive. Show that A has a subset B which is not Lebesgue measurable.

   b) Prove that the set of points at which a sequence of measurable real functions converges is a measurable set.

   c) Let $f:[0,1] \to R$ be a Riemann integrable function. Show by an example that f may not be Borel measurable.

[6+5+4]

3 a) Let $\Omega = (0,1]$ with Borel σ – field $\boldsymbol{B}$. P_1 and P_2 are two probability measures defined on it. Show that there exists an $A_0 \in B$ such that $\sup_{A \in B} (P_1(A) - P_2(A))$ is attained at $A = A_0$ s.t.

$$P_1(A) \geq P_2(A) \quad \text{if} \quad A \subset A_0$$

And $$P_1(A) \leq P_2(A) \quad \text{if} \quad A \subset A_0^c$$

P.T.O.

b) Let P be a π - system on Ω and μ_1, μ_2 are two measures defined on $(\Omega, \sigma(P))$ such that μ_1 , μ_2 agree on P and are also σ - finite with $P-$ sets $A_1, A_2, A_3, \ldots$

i.e. $\bigcup_{i=1}^{\infty} A_i = \Omega$ and $\mu_j(A_i) < \infty$ for all i, j.

Show that

$$\mu_1(A \cap (\bigcup_{i=1}^{n} A_i)) = \mu_2(A \cap (\bigcup_{i=1}^{n} A_i))$$

for all $A \in \sigma(P)$ and n =1,2,3,.....

Hence show that μ_1 , μ_2 agree on $\sigma(P)$.

C) Let $f : R \to [o, \infty)$ be a Bored measurable function with $\int_R f \, d\lambda < \infty$.

Show that to each $\in > o$, $\exists$ $\delta > o$ such that $\int_E f \, d\lambda < \in$ whenever $\lambda(E) < \delta$.

Here λ denotes Lebesgue measure on R.

[7 + 6 + 6]

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Indian Statistical Institute
Semestral Examination First Semester (2011-2012)
M.Stat. First Year
Large Sample Statistical Methods

Maximum Marks: 60 Date : 26.11.2011 Duration :- 3 hours

This question paper carries 66 marks. Answer as many questions as you can. The maximum you can score in this exam is 60.

1. (a) Let $X_1, \ldots, X_n$ be iid from a distribution F, given by $F(x) = (1 - \frac{1}{\ln x}) I_{x \geq e}$. Let $X_{(n)}$ denote the sample maximum. Can you find two sequences of real constants $a_n > 0$ and b_n such that $\frac{X_{(n)} - b_n}{a_n}$ converges to a non-degenerate limit distribution? Prove your assertion. [7]

   (b) Let $X_1, \ldots, X_n$ be iid with an exponential distribution with density $f(x) = e^{-x} I_{x \geq 0}$. Let $\bar{X}_n$ denote the sample mean and $X_{(1)}$ the sample minimum. Does there exist any sequence a_n of positive reals such that $a_n((\bar{X}_n)^{100} - 1 - X_{(1)})$ converges to a non-degenerate limit distribution? Prove your assertion. [6]

2. Let $\{X_n\}$ be iid $N(\mu, \sigma^2)$, where both μ and σ are unknown. Let $S_n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$. Find a variance stabilizing transformation for S_n and hence get a confidence interval for σ^2 with asymptotic coverage probability 0.95. Prove your answers. (You may assume the asymptotic distribution of (suitably centred and scaled) S_n.) [4+2=6]

3. (a) Let $X_1, \ldots, X_n$ be iid normal with unknown mean μ and variance 1. Suppose you are also told that the unknown mean μ is an integer. Find maximum likelihood estimator $\hat{\mu}_n$ of μ. Prove that it is not possible to find a sequence of real constants a_n such that $a_n(\hat{\mu}_n - \mu)$ converges to a non-degenerate limit distribution. [6]

   (b) Suppose now that $X_1, \ldots, X_n$ are iid with common density $f(x, \theta)$, where $\theta \in \Theta$, where Θ consists of **only finitely many real numbers**. Assume also that density under each θ has the same support and that the distributions under different θ's are diferent. If θ_0 is the true value of θ, prove that with probability tending to 1 (under θ_0) as $n \to \infty$, the likelihood function will be maximized at the value $\theta = \theta_0$. [6]

   (c) Let $X_1, \ldots, X_n$ be iid with common density $f(x, \theta)$ given by

$$f(x, \theta) = \frac{e^{-(x-\theta)}}{[1 + e^{-(x-\theta)}]^2}, -\infty < x < \infty,$$

   where $\theta \in \mathbf{R}$ is unknown. Does a consistent maximum likelihood estimator for θ exist in this problem? Justify your answer. Can you suggest any sequence of functions $g_n(.)$ (dependent on n) of $\bar{X}_n$ such that $g_n(\bar{X}_n)$ (after suitable centering and scaling) converges in distribution to $N(0, 1/I(\theta))$, where $I(\theta)$ is the Fisher Information? Justify your answer. [4+4=8]

P.T.O

(d) Let $X_1, \ldots, X_n$ be iid $N(\theta, 1)$ where $\theta \in \mathbf{R}$. Let T_n denote the Hodges' estimator that estimates θ by 0 if the sample mean lies in $[-n^{-1/4}, n^{-1/4}]$ and by the sample mean otherwise. Find the asymptotic distributions of $\sqrt{n}(T_n - \theta)$ and $\sqrt{n}(\bar{X}_n - \theta)$ for each $\theta \in \mathbf{R}$. Can you propose a criterion with respect to which the asymptotic performance of the sample mean will be better than that of the Hodges' estimator? Justify your answer. [6+2=8]

(e) Suppose $X_1, \ldots, X_n$ are iid having density $f(x, \theta)$, where $\theta \in \mathbf{R}$. Invoking appropriate regularity assumptions, derive the asymptotic null distribution of the likelihood ratio test statistic for testing $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$. [10]

4. (a) What is the Jackknife estimator of bias of an estimator ? What is the Jackknife bias-adjusted estimator? Can you give an example of an estimator whose bias is totally removed by the Jackknife bias-adjusted estimator? Justify your answer. [1+1+3=5]

(b) What is the asymptotic justification behind using the Jackknife bias-adjusted estimator? [4]

Indian Statistical Institute
Backpaper
M.Stat. First Year
First Semester, 2011-12 Academic Year
Large Sample Statistical Methods

Date : 30.12.2011 Maximum Marks: 100 Duration :- $3\frac{1}{2}$ hours

Answer all questions

1. Prove that a sequence of random variables $\{X_n\}$ converges in probability to zero if and only if $E\left(\frac{X_n}{1+X_n}\right) \to 0$ as $n \to \infty$. [10]

2. Suppose $X_i, i \geq 1$ are independent random variables such that $P(X_i = i) = \frac{1}{2} = P(X_i = -i)$, $\forall i \geq 1$. Does there exist a sequence $\{a_n\}$ of positive constants such that $a_n S_n$ converges in distribution to a non-degenerate distribution, where $S_n = \sum_{i=1}^{n} X_i$? Prove your answer. [10]

3. Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} F$ and F has a unique median denoted by $\zeta_{\frac{1}{2}}$. Show that $F_n^{-1}(\frac{1}{2})$ converges to $\zeta_{\frac{1}{2}}$ almost surely, where $F_n^{-1}(\frac{1}{2})$ denotes the smallest sample median. [10]

4. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ Poisson(λ), $\lambda > 0$. Can you find two sequences $a_n > 0$ and b_n of real constants (possibly dependent on λ) such that $a_n\left((1 - \frac{1}{n})^{n\bar{X}_n} - b_n\right)$ converges in distribution to a non-degenerate random variable? Prove your assertion. [10]

5. Stating appropriate assumptions, prove asymptotic normality of a consistent sequence of roots of the likelihood equation. [20]

6. Give an example where the maximum likelihood estimator is inconsistent. Prove your answer. [10]

7. Suppose $X_1, \ldots, X_n$ are iid Bernoulli($\frac{1}{2}$) random variables. Do there exist sequences of real constants a_n and b_n such that $a_n(s_n^2 - b_n)$ converges in law to a non-degenerate limit distribution? Here $s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. Prove your claim. [10]

8. Suppose you have an iid sample of size n from a multinomial population with k classes. Let π_i, $i = 1, 2, \ldots, k$ denote the probabilities of the classes. Suppose you want to test $H_0 : \pi_i = \pi_i^0$, $i = 1, \ldots, k$ vs $H_1 : \pi_i \neq \pi_i^0$ for at least one $i \in \{1, 2, \ldots, k\}$, where $\pi_i^0 > 0, \forall i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} \pi_i^0 = 1$ with π_i^0's as known constants.
Find the asymptotic distribution of $T_n = \sum_{i=1}^{k} \frac{(n_i - n\pi_i^0)^2}{n\pi_i^0}$ under H_0, where $n_i, i = 1, 2, \ldots, k$ denote the number of members in the sample falling in the i-th class. [10]

9. State and prove the Glivenko-Cantelli Theorem. [10]

Indian Statistical Institute
Multivariate Statistical Analysis
M-I, Midsem

Date: Feb 20, 2012 Duration: 2hrs.

Attempt any 5 questions. The maximum you can score is 40. Justify all your steps. This is a closed book, closed notes examination. You may use your own calculator.

If copying is detected in the solution for any problem, all the students involved in the copying will get 0 for that problem. Also an additional penalty of 5 will be subtracted from the overall aggregate of each of these students.

1. Let $M(x, y)$ denote the minimum copula defined as follows:

   $$\forall x, y \in [0, 1] \;\; M(x, y) = \min\{x, y\}.$$

   Show that it has the following property:

   If $C(x, y)$ is any copula such that $\forall x \in [0, 1] \;\; C(x, x) = M(x, x)$ then $C \equiv M$.

   Is there any other copula with the same property? Prove or provide a counterexample as appropriate. [10]

2. We want to distinguish between two types of mangoes based on length X_1, girth X_2 and weight X_3. These measurements have been made for 50 mangoes of type *Langra* and 50 mangoes of type *Himsagar*. Assuming multivariate normality (with common unknown, p.d. covariance matrix) obtain the LRT to test if these measurements have any discriminatory power. Also suggest how you can construct a good discriminant rule to classify any mango of these two types based on these measurements. [10]

3. Pillai's test for MANOVA rejects the null hypothesis (of equality of means) for large values of $tr(B(B + W)^{-1})$, where B is the "between" sum-of-squares matrix, and W is the "within" sum-of-squares matrix. Show that this test statistic may be expressed in terms of the eigenvalues of BW^{-1}. [10]

4. Consider the following bivariate data.

i	1	2	3	4	5
X_1	0	0	2	5	6
X_2	0	1	0	0	0

   Construct the single linkage and complete linkage dendrograms where the vertical axes show the Euclidean distances. [10]

5. Let $X_1, ..., X_n$ be iid $N_p(\mu, \Sigma)$ where μ is known, but Σ (pd) is unknown. Derive the union-intersection test for

$$H_0 : \Sigma = I \text{ vs. } H_1 : \Sigma \neq I.$$

[10]

6. Suggest how you may construct two random variables X, Y such that all the following conditions are met:

   (a) $X \sim N(0, 1)$

   (b) $Y \sim Cauchy(1, 1)$.

   (c) $X = f(Y)$ for some nondecreasing (deterministic) function f.

[10]

7. You have a discriminant analysis problem that you can solve using either C&RT or LDA. Suggest how you can use the available data set to decide which one is more suitable for it. [10]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination : 2010-11(First Semester)

M. Stat. I year

Discrete Maths.

Date: 23.12.10   Maximum Marks : 80   Duration : 2 1/2 hrs

Note: Answer as many as you can. The maximum you can score is 80. Notation is as used in the class.

1. (a) Let $G(x)$ denote the generating function for the Fibonacci sequence $\{F_n\}$. Show that $G(x) = \frac{1}{1-x-x^2}$.
   Suppose $G(x) = \frac{1}{(1-\lambda x)(1-\mu x)}$. What are the values of λ, μ? Hence, derive a formula for F_n.

   (b) Let $S_{n,t}$ denote the number of ways to partition an n-element set into t non-empty subsets. Show that

   $$S_{n,t} = tS_{n-1,t} + S_{n-1,t-1}, 1 \leq t < n.$$

   Compute $S_{5,3}$

   [10+8]

2. Suppose $\{a_n\}$ satisfy the following recurrence, for $n \geq 2$

   $$na_n = 2(a_{n-1} + a_{n-2}),$$

   with $a_0 = e$ and $a_1 = 2e$. Let $A(x)$ denote the ordinary generating function for $\{a_n\}$. Show that

   $$A'(x) = 2(1+x)A(x).$$

   Find $A(x)$ [8]

3. (a) Consider the recurrence

   $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_p a_{n-p}, \quad n \geq p,$$

   where $a_i \in I\!R$ and $a_p \neq 0$. Show that the set of solutions of the recurrence relation forms a vector space over $I\!R$

(b) A codeword over the alphabet is said to be legitimate if no two 0's appear consecutively. Let b_n denote the number of legitimate codewords of length n. Find a recurrence for the number b_n.

[6+4]

4. State the Inclusion-Exclusion Principle.
Hence, find $\phi(n)$, where $n = pqr$ is a product of three distinct primes and ϕ is the Euler's function. [4+6]

5. Define the Stirling's numbers $S(n, k)$ and $s(n, k)$.
Derive the generating function for the (unsigned) Stirling numbers $|s(n, k)|$.

[4+7]

6. Let $p(n, k)$ denote the number of partitions of n into (exactly) k parts. Derive the following recurrence relation

$$p(n, k) = p(n-1, k-1) + p(n-k, k).$$

[6]

7. (a) Let G be a group of permutations of a finite set A and let S be the equivalence relation induced on A by G. Let $C(a)$ denote the equivalence class containing $a \in A$. Let

$$St(a) = \{\pi \in G : \pi(a) = a\}$$

. Show that

$$|St(a)||C(a)| = |G|.$$

(b) Let G be a (finite) group of permutations. Define its cycle index $P_G(x_1, \ldots, x_k)$.

(c) Let G be a group of permutations of a (finite) set D and let $C(D, R)$ be the set of all colourings of D using colours in R. Let w be a weight assignment on R. What do you mean by the pattern inventory of colourings in $C(D, R)$? Express the pattern inventory of $C(D, R)$ in terms of the cycle index.(No proof required)

(d) Consider carbon molecules modelled as regular tetrahedrons with the carbon atom located at the centre and the vertices occupied by one of CH_3, C_2H_5, Cl, H. A typical such molecule is CH_2Cl_2. Two molecules x and y are considered the same (i.e. xSy) if if y can be obtained from x by one of the following 12 symmetries.

- no change
- a rotation by 120° or 240° around a line connecting a vertex and the centre of the opposite face
- rotation by 180° around a line joining the midpoints of two opposite edges.
- Identify the vertices with a, b, c, d and hence express the group of symmetries with a group G of permutations of $\{a, b, c, d\}$ Show that the cycle index of G is $\frac{1}{12}\{x_1^4 + 8x_1x_3 + 3x_2^2\}$
- Colour the set $D = \{a, b, c, d\}$ with the 4 colours $R = \{CH_3, C_2H_5, H, Cl\}$. Find the number of distinct molecules(colourings) with no CH_3

[8+3+4+ 12]

INDIAN STATISTICAL INSTITUTE

M.Stat I year, 2011-12

Second Semester

Mid-Semestral examination

Subject: Time Series Analysis — Date: 22-Feb-2012
Full Marks: 30 — Duration: 1 hr 30 min.

The paper contains questions of 35 marks. Maximum marks you can score is 30.

1. a) Define the components of a time series with examples.
   b) Discuss different techniques to estimate or eliminate the seasonal component of a time series.
   c) The filter $a_{-2}B^{-2} + a_{-1}B^{-1} + a_0 + a_1B^1 + a_2B^2$
   passes third-degree polynomial trends without distortion and eliminates arbitrary seasonal components with period 3. Derive the values of a_i's.

[5+5+6]

2. a) Let $\{X_t\}$ be stationary solution to the equation
   $$X_t - \emptyset X_{t-1} = Z_t + Z_{t-5}$$
   where $\{Z_t\}$ is $WN(0, \sigma^2)$

   Show that if $|\emptyset| < 1$, the equation admits unique stationary solution.

   b) In question 2 a) above, show that no stationary solution exists if $\emptyset = -1$.

[6+6]

3. a) Let $\{X_t\}$ be a stationary time series with mean μ and auto-covariance function $\gamma(t)$. On the basis of first n-observations $x_1, x_2, \dots, x_n$ derive the best linear unbiased estimate of μ when $\gamma(t)$ is known.
   b) Compare the asymptotic behavior of the above estimate in 3 a) with that of the sample mean.

[5+2]

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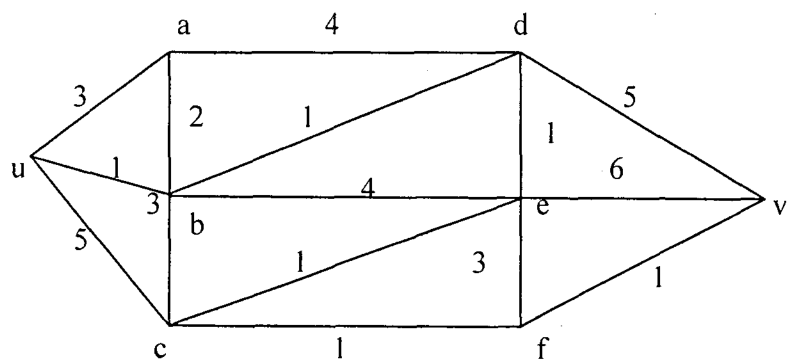
INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination: 2011-12
M. Stat. I Year
Optimization Techniques

Date: 24.02.2012 **Full Marks: 60** **Duration: 2 Hours**

Attempt all questions.

1.(a) Using Dijkstra's algorithm find a shortest path from u to v in the following graph.



[10]

2. Solve the following LPP using Simplex algorithm.

min $2x_1 + x_2$

subject to
$$x_1 - x_2 \geq 4$$
$$3x_1 + 2x_2 \leq 20$$
$$x_1, x_2 \geq 0$$

[15]

3. Find a basic feasible solution for the Transportation problem given below. Find another bfs which has less cost than this one.

8	1	2	5	4	30
4	2	1	3	2	60
5	3	2	5	4	40
20	10	40	30	30	

[10]

P.T.O.

4.(a) Can two different column basis give the same bfs? Justify.

(b) How does one go from phase-I to phase-II in the Simplex algorithm? Mention the cases, interpret them and explain the consequences.

[7+8=15]

5. If in a graph with 2n vertices, every vertex has degree at least n, then prove that it has a perfect matching.

[10]

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INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination: (2011–2012)

M. Stat First Year

Regression Techniques

Date: 27/02/2012 Marks: 30 Duration: 2 hours

Attempt all questions

1. (a) Is the sum of the residuals always zero? Justify.

   (b) Show that, for any linear model,

$$\sum_{i=1}^{n} Var(\hat{Y}_i)/n = p\sigma^2/n,$$

   where n is the sample size, p is the number of predictors, σ^2 is the error variance, and $\hat{Y}_i$ is the fitted i-th response value Y_i.

   (c) Suppose that $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \epsilon$ is a regression model containing a β_0 term in the first position. Then show that $\mathbf{1}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{1} = n$, where n is the sample size and $\mathbf{1} = (1, 1, \ldots, 1)'$.

   (d) Consider the following data set: $(X_1, Y_1) = (2.5, 6)$, $(X_2, Y_2) = (4, 9)$. If we wish to fit one of the two models given by (i) $Y = \beta_0 + \beta_1 X + \epsilon$ and (ii) $Y = \beta_1 X + \beta_2 X^2 + \epsilon$, which of the two models is expected to perform better? Justify your answer.

   (e) Show that in the presence of pure error the square of the multiple correlation coefficient $R^2 < 1$.

Marks: 2+2+2+2+2=10

2. (a) In the ridge regression set up with $\boldsymbol{\mu}_\lambda = \mathbf{S}_\lambda \mathbf{y}$ $(\lambda \geq 0)$, $\mathbf{y} = (y_1, \ldots, y_n)'$ the vector of observations, and $\mathbf{S}_\lambda = \mathbf{X}\left(\mathbf{X}'\mathbf{X} + \lambda\mathbf{I}\right)^{-1}\mathbf{X}'$, where $\mathbf{X}$ is the design matrix and $\mathbf{I}$ is the identity matrix, prove that

$$|E\{GCV(\lambda)\} - P(\lambda)|/R(\lambda) \leq 3\frac{p}{n} + O\left(\left(\frac{p}{n}\right)^2\right)$$

   In the above, p denotes the number of predictors, n is the number of observations, $GCV(\lambda) = n^{-1}RSS(\lambda)/\left(n^{-1}tr[I - \mathrm{S}_\lambda]\right)^2$, with $RSS(\lambda) = \sum_{i=1}^{n}(y_i - \mu_{\lambda i})^2$, $P(\lambda) = n^{-1}\sum_{i=1}^{n} E(y_i^* - \mu_{\lambda i})^2$, with y_i^* standing for the i-th future observation, and with $\mu_i = E(y_i)$, $R(\lambda) = E\left\{n^{-1}\sum_{i=1}^{n}(\mu_i - \mu_{\lambda i})^2\right\}$.

   (b) Show that in the regression model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \epsilon$, where the response and the predictor variables are standardized, and $\boldsymbol{\beta}'\boldsymbol{\beta}$ is bounded above, the ridge regression estimator of $\boldsymbol{\beta}$ is nothing but the least squares estimator.

Marks: 6+4=10

3. (a) Suppose that the data $\{y_1, \ldots, y_n\}$ come from the model

$$y_i = \mu(t_i) + \epsilon_i,$$

where $\mu(t_i) = \beta_0 + \beta_1(t_i - 0.5)$ is the true regression function, ϵ_i are zero-mean, uncorrelated random variables with common variance σ^2, and $t_i = (2i - 1)/2n$; $i = 1, \ldots, n$, n being the sample size. If we fit the data correctly guessing the form of the true model, then calculate $R(\hat{\mu})$, the risk of $\hat{\mu}$ (mean squared error). Calculate the asymptotic order of $R(\hat{\mu})$ as $n \to \infty$.

(b) Now suppose that the true regression function is of the form $\mu(t_i) = \beta_0 + \beta_1(t_i - 0.5) + f(t)$, where $f \neq 0$ is a continuously differentiable function satisfying $\int_0^1 f(t)dt = 0 = \int_0^1 (t - 0.5)f(t)dt$. If we fit the data "assuming" the model $y_i = \beta_0 + \beta_1(t_i - 0.5) + \epsilon_i$, what is $R(\hat{\mu})$ in this case? Calculate the asymptotic order of $R(\hat{\mu})$ as $n \to \infty$.

Marks: 4+6=10

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2011-2012, Second Semester
M-Stat I
Metric Topology and Complex Analysis

Date: 02-03-2012 Max. Marks 30 Duration: 2 Hours

Note: Answer all questions.
You must state clearly any result that you may be using.

Throughout X is assumed to be a metric space unless stated otherwise.

1. Check the following statements, prove if true and give counter-example if false:
   i) Let $A \subset X$. A is totally bounded implies $\bar{A}$ is totally bounded.
   ii) Any connected component of X is closed.
   iii) A sequence of real-valued continuous functions $\{f_n\}$ defined on X converges to f pointwise. Then f is continuous.
   iv) Let X be compact and $\mathcal{C}(\mathcal{X}, \mathbf{R})$ denote the space of real-valued continuous functions defined on X, equipped with sup-norm. Then $\mathcal{C}(\mathcal{X}, \mathbf{R})$ is complete.
   v) I, the set of irrationals is a Baire space.
   vi) Countable intersection of connected sets is connected.

   [2 × 6]

2. Let X be a compact metric space with metric d and $f : X \to X$ be a map such that $d(x, y) = d(f(x), f(y))$. Prove that f is surjective.

   [6]

3. Let $A \subset \mathbf{R}^2$ such that A is the set of all those points which have at least one co-ordinate irrational. Prove that A is connected.

   [6]

4. Prove that if $f : \mathbf{C} \to \mathbf{C}$ is holomorphic and of the form $f(x + iy) = u(x) + iv(y)$, where u and v are real functions, then $f(z) = \lambda z + c$ for all $z \in \mathbf{C}$ where λ is a real constant and c complex. [6]

[P.T.O]

5. Prove that

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \exp(z).$$

(Hint: Show that

$$\exp(z) - \left(1+\frac{z}{n}\right)^n = \sum_{k=2}^{n} \frac{z^k}{k!}\left(1-\prod_{j=1}^{k-1}(1-j/n)\right) + \sum_{k=n+1}^{\infty} \frac{z^k}{k!}.$$

[6]

INDIAN STATISTICAL INSTITUTE
Semester Examination: 2011-2012, Second Semester
M-Stat I
Metric Topology and Complex Analysis

Date: 23.4.12 Max. Marks 70 Duration: 3 Hours

Note: Answer all questions.
$\mathcal{H}(G)$ denotes the space of holomorphic functions on G

1. a) Prove that $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ for all z_1, z_2 in $\mathcal{C}$.

   b) Let $G \subset \mathcal{C}$ be a domain and $f \in \mathcal{H}(G)$. If $|f|$ is constant on G show that f is constant on G.

   [5+8]

2. a) Compute the integral $\int_i^{2i} cos z dz$.

   b) Let $G \subset \mathcal{C}$ be a domain. If a sequence $\{f_n\} \subset \mathcal{H}(G)$ converges to f locally uniformly (i.e. uniformly over all compact subsets of G) then show that $f \in \mathcal{H}(G)$ and derivatives of f_n converges to derivative of f locally uniformly on G.

   [3+10]

3. a) Let f be an entire function and assume that $|f(z)| \leq A + B|z|^n$ where A, B are positive reals and n is a positive integer. Show that f is a polynomial of degree less than or equal to n.

   b) Suppose f and g are entire functions and $|f(z)| \leq |g(z)|$ for all $z \in \mathcal{C}$. What conclusion can you draw?

   [8+8]

4. Use residue theorem to prove that

$$\int_{-\pi}^{\pi} \frac{1}{1+\sin^2 x}\, dx = \pi\sqrt{2}.$$

[8]

5. Let C be a positively oriented simple closed curve. Let f be analytic inside and on C and nonzero on C. Suppose that f has n zeros z_k $(k = 1, 2, \ldots, n)$ inside C, where each z_k is of multiplicity m_k. Prove that

$$\int_C \frac{zf'(z)}{f(z)}\, dz = 2\pi i \sum_{k=1}^{n} m_k z_k.$$

[8]

6. Determine the number of roots, counting multiplicities, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \leq |z| < 2$. [7]

7. Find the linear fractional transformation S mapping 0 to -1, i to 0, and ∞ to 1. Determine the images of the following sets under S:

(a) the imaginary axis; (b) the real axis; (c) the horizontal line through i.

[10]

INDIAN STATISTICAL INSTITUTE
Second Semester Examination: 2011-12
M. Stat. I Year
Time Series Analysis

Date: 25.04.12 **Maximum Marks: 60** **Duration: 3 Hours**

Note : The paper contains questions of 70 marks.
Attempt all questions. Maximum marks you can score is 60.

1 Describe all possible solutions in stationary time series $\{X_t\}$ to the equation

$X_t + X_{t-1} = Z_t + Z_{t-3}$

Where $\{Z_t\} \sim$ White Noise $(0, \sigma^2)$

[5]

2 a) State Wold decomposition theorem for a stationary time series $\{X_t\}$.

Use it to show that a stationary time series is q-correlated iff it is some MA(q) process.

b) What are deterministic and non-deterministic components of a stationary time series. Derive the deterministic part of the stationary time series $\{X_t\}$ given by the equation

$X_t = U + \frac{1}{2} Z_t + \frac{1}{3} Z_{t-1}$

Where U ~ $N(0,1)$

$\{Z_t\} \sim$ IID $(0, \sigma^2)$

and $\{Z_t\}$ is independent to U.

[9+8]

3 a) Let $\{X_t\}$ be a stationary time series with mean μ and autocovariance function $\gamma(h)$. For h>0 , let $P_n X_{n+h}$ be best linear predictor of X_{n+h} on the basis of $\{1, X_n, X_{n-1}, \dots, X_1, X_0\}$. Show that $P_n X_{n+h}$ is unique even if $X_n, X_{n-1}, \dots, X_1, X_0$ are linearly dependent.

b) Let $P_n X_{n+1} = \mu$, where $P_n X_{n+1}$ and μ are as described in part a). What can you say about the autocovariance function?

[5+2]

4. a) For a stationary time series X_t , define spectral density $f(\lambda)$, when it exists.

Give example of some stationary time series $\{X_t\}$, for which spectral density does not exist. Justify your answer.

b) For what values of ρ in R, the following is an ACVF of some stationary time series.

$$\gamma(h) = \begin{cases} 1 & \text{if } h = 0 \\ \rho & \text{if } |h| = 1 \\ -\rho^2 & \text{if } |h| = 2 \\ 0 & \text{otherwise.} \end{cases}$$

P.T.O

c) Let $\{X_t\}$ be an AR (1) time series given by $X_t = \phi X_{t-1} + Z_t$ where $\{Z_t\}$~ WN $(0,\sigma^2)$ and $|\phi| < 1$.

Derive the expression for spectral density of it for different values of ϕ. Give rough sketches of the graphs and interpret.

[5+5+6]

5 a) Let $\overline{X_n}$ be the sample mean of the first n observations from an AR (p) process defined from an IID$(0,\sigma^2)$ process $\{Z_t\}$. Show that $\overline{X_n}$ is asymptotically normal as $n \to \infty$.

b) Define sample autocovariance function $\hat{\gamma}(h)$ based on first n observations from a stationary time series $\{X_t\}$. Show that the variance covariance matrix based on $\hat{\gamma}(h)$ is positive definite if the observations are not constant.

[7+6]

6 The following gives first 16 observations from a stationary time series $\{X_t\}$. Calculate $\hat{\rho}(1)$, $\hat{\rho}(2)$ and $\hat{\rho}(3)$. On the basis of them test the hypothesis $H_0: \{X_t\}$ is IID.

0.4	0.7	0.4	0.4	0.8	0.1	0.6	0.0
0.7	0.2	0.8	0.9	0.3	0.3	0.4	0.2

It is given that $\chi^2_{3,0.05} = 7.81$

[12]

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INDIAN STATISTICAL INSTITUTE
Second Semestral Examination : 2011-12
M. STAT. I YEAR
Optimization Techniques

Date: 26 April 2012 Maximum Marks: 100 Duration: 3 hours

Note: There are two groups - A and B. *Use separate answer booklet for each group.* Group A has questions of 40 marks and Group B of 70 marks. *Maximum you can score 40 and 60 in groups A and B respectively.*

Group A

1 Prove that if an LPP has a feasible solution, it also has a basic feasible solution. [15]

2 Formulate the shortest path problem as a linear programming problem. Write down its dual. Can you interpret the complementary slackness conditions in this case? [15]

3 There are six committees of a state legislature: Finance, Environment, Health, Transportation, Education, and Housing. Suppose that there are 10 legislators who need to be assigned to committees, each to one committee. The following matrix has its (i, j)-th entry equal to 1 iff the i-th legislator would like to serve on the j-th committee.

	Finance	Environment	Health	Transportation	Education	Housing
Allen	1	1	1	0	0	0
Barnes	1	1	0	1	1	0
Cash	1	1	1	0	0	0
Dunn	1	0	0	1	1	1
Ecker	0	1	1	0	0	0
Frank	1	1	0	0	0	0
Graham	1	1	1	0	0	0
Hall	1	0	0	0	0	0
Inman	1	1	1	0	0	0
Johnson	1	1	0	0	0	0

Suppose that we want to choose exactly one new member for each committee, choosing only a legislator who would like to serve. Can you do so? (Not every legislator needs to be assigned to a committee, and no legislator can be assigned to more than one committee.) [10]

Group B

1 (a) Consider the following problem. Find a sharp upper bound for the objective function and solve the problem.

$$\max\ z = 2x_1 + x_2 + 4x_3$$
$$\text{subject to}$$
$$x_1 + 2x_2 + x_3 \le 16,$$
$$x_1 - x_2 + 3x_3 \le 12,$$
$$x_1, x_2, x_3 \ge 0, \text{ are integers.}$$

(b) Formulate the problem suitably and obtain a solution by Balas' algorithm:

$$\max\ z = x_1 + 2x_2 - 3x_3$$
$$\text{subject to}$$
$$20x_1 + 15x_2 - x_3 \le 10$$
$$\text{or}\quad 12x_1 - 3x_2 + 4x_3 \le 20,$$
$$x_1, x_2, x_3 \in \{0, 1\}.$$

[10+15=25]

2 (a) Consider the problem with $c_j > c_0 > 0$ for $j = 1, 2, \ldots, N$:

$$\max\ z = \prod_{j=1}^{N} p_j$$
$$\text{subject to}$$
$$\sum_{j=1}^{N} c_j p_j \le c_0, \qquad 0 \le p_j \le 1\ \forall j.$$

(i) Formulate recursive relations of dynamic programming for this problem.

(ii) Solve the problem using the relations of (i).

(b) For $k \ge 1$ and a given function $g(x)$, write down a dynamic programming formulation of the problem:

$$\min\ z = g(x_1) + \frac{g(x_2)}{x_1} + \frac{g(x_3)}{x_1 x_2} + \cdots + \frac{g(x_N)}{x_1 x_2 \ldots x_{N-1}}$$
subject to
$$x_1 x_2 \ldots x_N = k,$$
$$x_j \ge 1 \text{ for } j = 1, 2, \ldots, N.$$

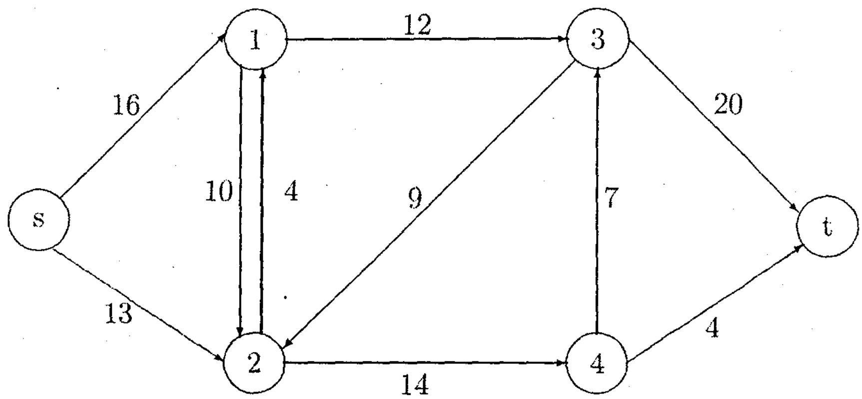
[(7+8)+10=25]

[P.T.O.]

3 (a) State the Max-flow Min-cut theorem.

(b) Write the definition of flow augmenting path.

(c) Solve the Max-flow problem for the following network (the number adjacent to an arc is its capacity).



[3+2+15=20]

———*** xXx ***———

INDIAN STATISTICAL INSTITUTE

Semester Examination : 2011-12(Second Semester)

M. Stat. I year
Discrete Maths.

Date:27.04.12 Maximum Marks : 90 Duration : 3 hrs

Note: Answer as many as you can. The maximum you can score is 90. Notation is as used in the class.

1. Let $V = \{1; 2, \ldots, p\}$ and D denote the set of all 2-element subsets of V. Consider a $1-1$ correspondence between graphs $G = (V, E)$ and 0-1 functions f defined on D as follows:

$$f(\{i, j\}) = 1 \longleftrightarrow \{i, j\} \in E.$$

   (a) Let G, G' be two graphs on V and let f, f' be their corresponding functions. Show that G and G' are isomorphic iff there is a permutation π on D such that for all $\{i, j\} \in D$,

$$f(\{i, j\}) = f'(\pi(\{i, j\})).$$

   (b) Use Burnside's Lemma to determine the number of distinct (non-isomorphic) graphs of 3 vertices.

[5+7]

2. (a) When is an integer N said to have the (p, q) Ramsey property? (Here $p, q \geq 2$ are integers.)
   Show that $R(3, 3) = 6$.

   (b) Show that for every graph G with 9 vertices, either G contains a triangle or its complement $\bar{G}$ contains K_4 [2+5+6]

3. Write down an algorithm for finding a spanning tree of a connected graph. Discuss its correctness and the computational complexity of this algorithm [8]

4. Describe Dijkstra's shortest path algorithm.

   Apply the algorithm to the following problem.
   A manufacturing process starts with a **raw wood**. The wood must be **cut** into shape, **stripped**, have **holes punched** and be **painted**. The cutting must precede hole punching and stripping must precede painting. The cutting/stripping takes 1 unit of time, painting takes 2 units of time for uncut wood and 4 units of time for cut wood: and punching takes 3 units of time for unstripped wood, 5 units for stripped but unpainted and 7 units for painted wood. Find the sequence of activities that will allow completion of the process in the shortest possible time. [5+8]

5. Show that for $n \geq 2$, there are n^{n-2} labelled trees with n vertices. How many rooted labelled trees with n vertices are there? Verify your claim for $n = 3$. [10]

6. (a) Write an algorithm to compute the Jacobi symbol $\left(\frac{m}{n}\right)$. Your algorithm should not use factoring apart from dividing out powers of 2. Use your algorithm to compute $\left(\frac{7411}{9283}\right)$

   (b) Describe the Solovay-Strassen algorithm for primality testing.

   (c) Show that for $n = pq$, where p, q are distinct primes, the set

   $$\{a \in Z_n^* : \left(\frac{a}{n}\right) \equiv a^{(n-1)/2} \bmod n\}$$

   has at most $(n-1)/2$ elements. [8+5+6]

7. Define a strongly regular graph with parameters (n, k, a, c).

   Consider the Paley graph $P(13)$. What are its parameters? Find the common neighbours of 4 and 10. Find its eigenvalues. [8]

8. Show that a connected regular graph with exactly 3 eigenvalues is strongly regular.

   Show that if p is prime, all strongly regular graphs with $p+1$ vertices is imprimitive. [6+6]

9. Describe the graph G defined by an orthogonal array $OA(k, n)$. Show that it is strongly regular with parameters
   $(n^2, (n-1)k, (n-2) + (k-1)(k-2), k(k-1))$. [8]

INDIAN STATISTICAL INSTITUTE

Semestral Examination: (2011–2012)

M. Stat First Year

Regression Techniques

Date:. 04.05.12 Full Marks: ..100.. Duration: .3 hours.

Attempt all questions

[You may use the following result if necessary, without proof: If $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$, then $\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} \\ \mathbf{A}^{21} & \mathbf{A}^{22} \end{pmatrix}$ where $\mathbf{A}^{11} = \left(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right)^{-1} = \mathbf{A}_{11}^{-1} - \mathbf{A}^{12}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}$, $\mathbf{A}^{12} = -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{A}^{22}$, $\mathbf{A}^{22} = \left(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right)^{-1}$, $\mathbf{A}^{21} = -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}^{11}$.]

1. (a) Consider fitting a straight line $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$; $i = 1, \ldots, n$. Suppose that $2\beta_0 - \beta_1 = 4$. Show the unrestricted and the restricted estimation spaces diagrammatically.

(b) Provide vectors that span the restricted estimation space and its complement.

(c) Consider the regression model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \epsilon$ (assume that it contains an intercept), where $\boldsymbol{\beta}$ is a $p \times 1$ vector and $\mathbf{X}$ is $n \times p$ and has full rank. Consider testing $\mathbf{A}\boldsymbol{\beta} = \mathbf{0}$, where $\mathbf{A}$ is $q \times p$ $(q < p)$, has full rank and does not involve the intercept β_0. Show geometrically that the F-statistic for testing this hypothesis can be written in terms of the difference between the R^2-statistics, where R^2 stands for the coefficient of determination.

Marks: 8+7+10=25

2. (a) Using the ridge regression set up establish the existence of biased estimators which have smaller risk than unbiased ones.

(b) Assume that there are infinitely many non-zero β_j's in the model $y_i = \sum_{j=1}^{\infty} \beta_j x_{ij} + \epsilon_i$, $i = 1, \ldots, n$, where $\epsilon_i; i = 1, \ldots, n$, are normal random errors, $\sum_{j=1}^{\infty} \beta_j^2 < \infty$ and, for each n, for $j, k = 1, \ldots, n$, $\sum_{i=1}^{n} x_{ij}x_{ik} = n\delta_{jk}$, where $\delta_{jk} = 1$ if $j = k$ and 0 otherwise. Let $\hat{\lambda}$ be the minimizer of $\hat{P}(\lambda) = n^{-1}RSS(\lambda) + 2n^{-1}\sigma^2 tr(S_\lambda)$ over $\Lambda_n = \{1, \ldots, a_n\}$ with $a_n \to \infty$ as $n \to \infty$. Show that $P(\hat{\lambda} \leq x) \to 0$ as $n \to \infty$ for any finite x.

Marks: 10+15=25

3. (a) Suppose that data $\{y_1, \ldots, y_n\}$ are available where, for $i = 1, \ldots, n$, $E(y_i) = \mu(t_i)$; $t_i = (2i-1)/2n$. Assuming that $\mu(\cdot)$ is unknown, consider its regressogram estimator, given by $\mu_\lambda(t) = \sum_{i=1}^{n} K(t, t_i; \lambda)y_i$, where $K(t, t_i; \lambda) = \frac{\sum_{r=1}^{\lambda} I_{P_r}(t) I_{P_r}(t_i)}{\sum_{j=1}^{n} \sum_{r=1}^{\lambda} I_{P_r}(t) I_{P_r}(t_j)}$. Here $I_{P_r}(\cdot)$ denotes the indicator function for the interval $P_r = [\frac{r-1}{\lambda}, \frac{r}{\lambda})$, $r = 1, \ldots, \lambda - 1$, and $P_\lambda = [\frac{\lambda-1}{\lambda}, 1]$. Show that, under suitable assumptions, the risk of the regressogram estimator converges to zero at the rate $n^{-2/3}$.

(b) Consider the proportional hazards model $h(t; x) = \lambda(t) \exp\{\beta' x\}$. Let $\Lambda(t) = \int_{-\infty}^{t} \lambda(u) du = \exp\{\alpha t\}$. Suppose there are n uncensored and m censored observations. Discuss a methodology for computing the maximum likelihood estimates of the parameters *(Complete mathematical details not required)*.

Marks: 15+10=25

4. (a) Show that the ordinary residuals e_i and the leverage values p_{ii} satisfy the inequality

$$p_{ii} + \frac{e_i^2}{SSE} \leq 1$$

where SSE is the residual sum of squares.

(b) Show that the variance of the regression coefficient β_j $(j = 1, \ldots, p)$ in the usual linear regression set up can be written in terms of the variance inflation factor VIF_j.

(c) Suppose that case i is an outlier, in the sense that an amount δ is added to its expected value. Let $\mathbf{u}_i$ be the unit vector with one in the i-th position and zeros elsewhere and write the revised model as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \delta \mathbf{u}_i + \epsilon$$

Develop the test statistic for the hypothesis $H_0 : \delta = 0$, and show that the resulting t-statistic is equivalent to the externally Studentized residual.

Marks: 8+7+10=25

INDIAN STATISTICAL INSTITUTE
Semester Examination (back): 2011-2012, Second Semester
M-Stat I
Metric Topology and Complex Analysis

Date: 25.06.12 **Max. Marks 100** **Duration: 3 Hours**

Note: Answer all questions.

1. a) Let $X \subset \mathbf{R}$. Show that X is compact if and only if it is closed and bounded.

   b) Let X be a metric space. Show that X is compact if and only if all real-valued continuous functions on X are bounded.

   [7+8]

2. a) Let (X, d) be a complete metric space. Let $A \subset X$. Show that A is complete under metric d if and only if A is closed.

   b) Give complete metrics on $(0, 1)$ and on the set of irrationals inducing the usual topology.

   [7+8]

3. a) Let G be a bounded domain in C and $f : cl(G) \to C$ be continuous and f be holomorphic in G. Then show that $\sup\{|f(z)|\}$ is attained at a point on the boundary of G and not at any point in G unless f is constant.

   b) Show that an analytic function on a domain is completely determined by its values on any set of points containing a limit point.

   [10+10]

4. a) Let γ be a piecewise smooth closed curve in C and $z \in C - \{\gamma\}$. Show that $(1/2i\pi) \int_\gamma \frac{1}{\zeta - z} d\zeta$ is an integer-valued function and is constant on each component of $C - \{\gamma\}$.

   b) Calculate the integral $\int_0^\infty sin(x^2)dx$ using the fact that $\exp(iz^2)$ is an entire function. [10+10]

INDIAN STATISTICAL INSTITUTE

Semester Examination : 2011-12(Back Paper)

M. Stat. I year

Discrete Maths.

Date 28.06.12 : Duration : 3 hrs

Note: Answer as many as you can.

1. State and prove Burnside's lemma. [12]

2. Define Ramsey number $R(p, q)$.

   Show that for $p, q \geq 3$, $R(p, q)$ exists. Hence, or otherwise, show that $R(3, 4) \leq 10$. [2+7+4]

3. Give an example of a graph with 8 vertices that neither contains a triangle nor its complement has K_4.

   Show that the generalized Ramsey number $R(p, q : 1) = p + q - 1$.[4+6]

4. Show that in a complete graph with 9 vertices there are exactly 4 edge-disjoint Hamiltonian circuits. [8]

5. Describe the Depth-First-Search algorithm. Show that the algorithm also outputs a spanning tree, if it exists. [10]

6. Show that if a graph G contains exactly two vertices u, v with odd degrees, then there must be a path from u to v. [5]

7. Show that if G is k-regular and connected, then k is an eigenvalue with multiplicity 1. [10]

8. Show that if G is strongly regular, then its complement is also strongly regular. Find the parameters of the complement in terms of the parameters of G. [8]

9. Show that if G is strongly regular, then it has exactly 3 eigenvalues. Under what conditions is the converse true? [8]

P.T.O

10. Let G be strongly regular with parameters (n, k, a, c). Show that if $gcd(k, n-k-1) = 1$, then either G is disconnected or its complement is disconnected. [8]

11. Define an orthogonal array $OA(n, k)$ When is it said to be extendible?

Show that the graph associated with an $OA(k, n)$ has Chromatic number n iff it is extendible. [12]