Effect of ion temperature on ion-acoustic solitary waves in a two-ion plasma

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The effect of ion temperature on ion-acoustic solitary waves in the case of a two-ion plasma has been investigated using the pseudopotential approach of Sagdeev. An analytical solution for relatively small amplitudes has also been obtained. Our result has been compared, whenever possible, with the experimental result obtained by Nakamura. It is found that a finite ion temperature considerably modifies the restrictions on the Mach number obtained for cold ions.

Les effets de la température des ions sur les ondes solitaires ioniques acoustiques ont été examinés, dans le cas d'un plasma à deux ions, en utilisant l'approche de pseudo-potentiel de Sagdeev. On a aussi obtenu une solution analytique pour les amplitudes relativement faibles. Notre résultat a été comparé, lorsque la chose était possible, avec le résultat expérimental obtenu par Nakamura. On trouve qu'une température ionique finie modifie les restrictions sur le nombre de Mach obtenu pour des ions froids.

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1. Introduction

The evolution of small-amplitude solitary waves studied mainly y means of the Korteweg–deVries (KdV) equation is of conderable interest in plasma dynamics (1, 2). A reductive perırbation technique has been used by many authors to show ıat ion-acoustic solitary waves propagating in a collisionless lasma of hot electrons and cold ions are governed by the KdV quation (3).

However, the reduction perturbation technique is not suitble for comparatively large amplitudes and, as shown by Sharma *t al.* (4), higher order corrections are significant. Recently, lakamura *et al.* (5) have used the pseudopotential method to xplain large-amplitude solitary waves in a two-ion plasma vith both positive and negative ions; they found experimental vidence for this.

However, no temperature effect was considered by Nakanura *et al.* in their theoretical formula for the pseudopotential. t has been shown (6) that the temperature effect is significant or the solitary-wave solution even with an ion–electron temerature ratio as small as 1/30. So far, we know of no study hat has been made using Sagdeev's method (7) to investigate he effect of ion temperature in a two-ion plasma. The reasons or taking up the present work are the following. Most of the lasma found in nature or the laboratory are composed of everal ion species. Second, the presence of another ion speies significantly modifies the characteristics of ion-acoustic waves (8). For example, Ikezi (9) has found that the addition of small quantities of hydrogen ions to an argon plasma can prevent soliton formation. Finite ion temperatures further restrict the allowed region for soliton formations. This has been shown in the present work with the help of separatrices.

The organization of the paper is as follows. In Sect. 2 we state the basic fluid equations and derive the pseudopotential. In Sect. 3, solitary-wave solutions are discussed. In Sect. 4, analytical results are given by expanding the pseudopotential in terms of ϕ and keeping terms up to third order. Also, an approximation to the pseudopotential is obtained for small ion temperatures, keeping only first-order terms for temperature but terms that are valid to all orders for ϕ, the amplitude. Section 5 is kept for results and discussions.

2. Derivation of the pseudopotential and solitary-wave solutions

The basic fluid equation in one dimension for a collisionless plasma, a mixture of two warm ions and hot isothermal electrons, is (10)

$$[1]\quad \frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_1) = 0$$

$$[2]\quad \frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + \frac{\sigma_i}{n_i}\frac{\partial p_i}{\partial x} = -\frac{Z_1}{\beta_i}\frac{\partial \phi}{\partial x}$$

$$[3]\quad \frac{\partial p_i}{\partial t} + u_i \frac{\partial p_i}{\partial x} + 3p_i \frac{\partial u_i}{\partial x} = 0$$

$$[4]\quad \frac{\partial^2 \phi}{\partial x^2} = n_e - Z_1 n_1 - Z_2 n_2$$

$$[5]\quad n_e = n_{0e} \exp(\phi)$$

where i can have values of 1 and 2, the suffixes 1 and 2 denoting, respectively, the heavier and lighter ions. μ is the ratio of the mass of the lighter ion to the mass of the heavier ion. T_i and T_e are, respectively, the ion temperature and the electron temperature. β is given by

$$[6]\quad \beta = [(1-\alpha)Z_1 + \alpha Z_2]$$

α being the measure of the light-ion concentration and is given by

$$[7]\quad \alpha = \frac{Z_2 n_{20}}{n_{0e}}$$

and

$$[8]\quad \sigma_i = T_i/T_e, \qquad i = 1, 2$$

All the quantities in the above set of equations have been normalized to make them dimensionless. The quantities involved are normalized in the following way.

n_i and n_e, u_i, ϕ, and p_i are normalized by n_{0e} (electron-number density), $(kT_e/m_i)^{1/2}$, (kT_e/e), and $(n_{0e}kT_i)$ respectively. The space and time coordinates x and t are normalized by the Debye length $(kT_e/4\pi n_{0e}^2)^{1/2}$ and the characteristic time $(4\pi n_{0e}^2/m_i)^{1/2}$. To obtain solitary-wave solutions, we make the dependent variable depend on the single independent variable $\xi = x - Vt$, where V is the velocity of the soliton. Then from the above set of equations we get the following integrals of motions:

$$[9] \quad n_i = \frac{Vn_{i0}}{(V-u_i)}$$

$$[10] \quad p_i = \frac{V^3}{(V-u_i)^3}$$

where i can take the values 1 and 2. In obtaining [9] and [10], we have used the following boundary conditions: as $x \to \infty$, $u_i \to 0$, $p_i \to 1$, $\phi \to 0$, and

$$[11] \quad n_i \to n_{i0}$$

Another set of relations can be obtained from [1] and [5] utilizing the relations [9] and [10]; it is given by

$$[12] \quad n_i = \left[\frac{F_i - (F_i^2 - 12\sigma_i n_{i0}^4 V^6)^{1/2}}{6\sigma_i V^2}\right]^{1/2}, \qquad i = 1, 2$$

where

$$[13] \quad F_i = \left(V^2 + 3\sigma_i - \frac{2Z_i\phi}{\beta_i}\right) V^2 n_{i0}^2$$

and

$$\beta_1 = \beta, \qquad \beta_2 = \mu\beta$$

Putting the values of n_1 and n_2 given by [12] and [13] into [7] we get

$$[14] \quad \frac{d^2\phi}{d\xi^2} = -\frac{\partial\psi}{\partial\phi}$$

where the pseudopotential ψ is given by

$$[15] \quad \psi(\phi) = \psi_1(\phi) + \psi_2(\phi) + 1 - e^{\phi}$$

where

$$[16] \quad \psi_i(\phi) = (3\sigma_i V^6)^{1/4}\beta_i(e^{\theta_i/2} - e^{\theta_{i0}/2}) + \tfrac{1}{3}(e^{-3\theta_i/2} - e^{-3\theta_{i0}/2})$$

where θ_i and θ_{i0} are given by

$$[17] \quad \theta_i = \cosh^{-1}\left[\frac{(3\sigma_i + V^2 - 2Z_1\phi/\beta_i)}{12\sigma_i V^2}\right]$$

$$[18] \quad \theta_{i0} = \cosh^{-1}\frac{V^2 + 3\sigma_i}{12\sigma_i V^2}$$

Equation [14] has the same form as the equation governing a particle with a potential ϕ moving in a "potential" ψ with time ξ. Its solution can be written as

$$[19] \quad \xi = \int \frac{d\phi}{\sqrt{-2\psi}}$$

If we neglect terms of order $O(\sigma^2)$, then ψ can be expressed in a simpler form; viz.,

$$[20] \quad \psi(\phi) = \Bigg\{1 - e^{\phi} - \frac{\beta V^2(1-\alpha)}{Z_1}\left[\left(1 - \frac{2Z_1\phi}{\beta V^2}\right)^{1/2}\left(1 - \frac{3\sigma_1 Z_1\phi}{\beta(V^2 - 2Z_1\phi/\beta)^2}\right) - 1\right] - \beta\frac{(1-\alpha)\sigma_1}{Z_1}\left[\left(1 - \frac{2Z_1\phi}{V^2\beta}\right)^{-3/2} - 1\right]$$
$$- \frac{\beta\mu_\alpha V^2}{Z_2}\left[\left(1 - \frac{2Z_2\phi}{\beta\mu V^2}\right)^{1/2}\left(1 - \frac{3\sigma_2 Z_2\phi}{\beta\mu(V^2 - 2Z_2\phi/(\beta\mu))}\right) - 1\right] - \frac{\beta\mu_\alpha\sigma_2}{Z_2}\left[\left(1 - \frac{2Z_2\phi}{\beta\mu V^2}\right)^{-3/2} - 1\right]\Bigg\}$$

When both σ_1 and $\sigma_2 \to 0$, our result [20] reproduces the result of Sharma *et al.* (4). Note that our definition of μ is different from that in ref. 4.

3. Solitary-wave solutions

The form of the pseudopotential determines whether soliton solutions of [14] exist or not. For simplicity we assume $\sigma_1 = \sigma_2 = \sigma$ for the following observations. An analysis similar to that done by Kuehl and Imen (11) shows that for a positive initial velocity, the conditions for existence of solitary-wave solution are

$$[21] \quad V^2 > \left(Z_1^2 + \frac{Z_2^2 r}{\mu}\right)\Big/\beta(1-r) + 3\sigma$$

where $r = n_{20}/n_{10}$ and

$$[22] \quad e^{\phi} - 1 + (3\sigma V^6)^{1/4}[\beta n_{10}(1 - e^{\theta_{10}/2}) + \tfrac{1}{3}(1 - e^{-3\theta_{10}/2}) + \beta\mu n_{20}(e^{\theta_2'/2} - e^{\theta_{20}/2}) + \tfrac{1}{3}(e^{-3\theta_2'} - e^{-3\theta_{20}/2})] < 0$$

where θ_{10} and θ_{20} are given by [18] and

$$[23] \quad \theta_2' = \cosh^{-1}\left[\frac{(3\sigma + V^2)(1 - Z_2/Z_1\mu)}{\sqrt{12\sigma V^2}}\frac{Z_2}{Z_1\mu}\right]$$

The inequality [21] is obtained from the inequality $(\partial^2\psi/\partial\phi^2)_{\phi=0} < 0$, which is the condition for a potential well, and condition [22] is obtained in the following way, assuming $Z_1 > Z_2/\mu$ and $\sigma_1 = \sigma_2 = \sigma$. When ϕ satisfies the inequality

$$[24] \quad (V^2 + 3\sigma - 2Z_1\phi/\beta)^2 < 12\sigma V^2$$

ψ becomes complex. (Note that when $Z_1 > Z_2/\mu$, [24] is approached sooner than the other possibility, $V^2 + 3\sigma - 2Z_2\phi/\beta\mu < 12\sigma V^2$.) In order that the particle moving into the region $\phi > 0$ be reflected before reaching the region of complex ψ, it is necessary for ψ $(\phi = \phi_0) > 0$ where

$$[25] \quad \phi_0 = (V^2 + 3\sigma - \sqrt{12\sigma V^2})\beta/2Z_1$$

This gives the condition [22]. To give some quantitative results, we examine the following simple case.

Consider the single-ion case with $Z_1 = 1$ and $\alpha = 0$. In the absence of the temperature term it can be shown that the solitary-wave solution in a nondrifting plasma exists only for V satisfying

$$[26] \quad 1 < V < 1.6$$

In case of $\sigma \neq 0$, this inequality gets considerably modified. For example, if we take $\sigma = 1/30$, $Z_1 = 1$, and $\alpha = 0$, then

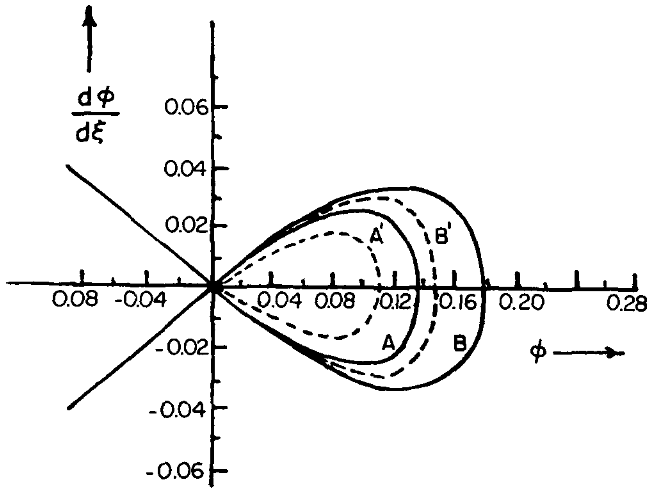

FIG. 1. Separatrices of [28] with $V = 1.1$ and $\mu = 0.4$. Solid curves, A and B, are for $\sigma_1 = \sigma_2 = 0$ and $\alpha = 0.1$ or 0.5 respectively. The broken curves, A′ and B′, are for $\sigma_1 = \sigma_2 = 0.01$ and $\alpha = 0.1$ or 0.5 respectively.

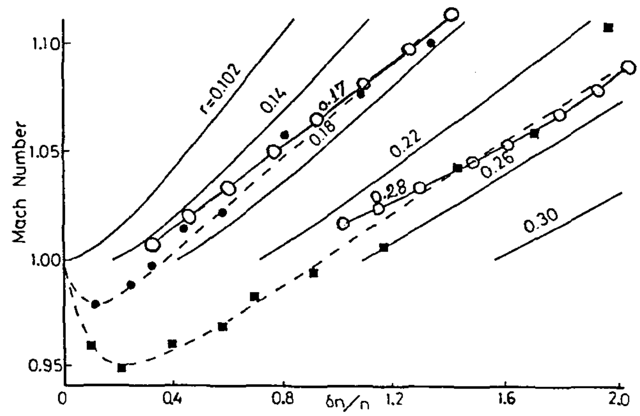

FIG. 2. The dependence of the peak velocity on the peak height. Filled circles (●): $r = 0.17$; Filled squares (■): $r = 0.28$ (experimental results).[5] The solid curves are taken from ref. 5. The solid lines with open circles (—○—) are the present solution with $\sigma_1 = \sigma_2 = \sigma = 1/20$ and the Mach number normalized by $\sqrt{1 + 3\sigma}$. All of the above curves are drawn with $\mu = 0.476$.

conditions [21] and [23] give

$$[27] \quad 1.049 < V < 1.25$$

Inequality [27] is significantly different from [26] even for a relatively small value of σ. Another factor that is very important in two-ion plasmas is the α term, i.e., the light-ion concentration. The presence of a finite temperature also puts a significant restriction on the critical values of α for soliton formation. In the presence of α, both compressional and rarefactive solitons are possible although the amplitudes of the two solitons are not equal to each other (12). To show how temperature restricts the allowed region for soliton formation, we first write [14] in the following form:

$$[28] \quad \frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 = -\psi(\phi)$$

For soliton formation $\psi(\phi)$ must be negative.

In Fig. 1, $d\phi/d\xi$ is plotted against ϕ. The solid curves are drawn with $\sigma_1 = \sigma_2 = 0$ and $\alpha = 0.1$ and 0.5 respectively. The broken curves, A′ and B′, are drawn with $\sigma_1 = \sigma_2 = 0.01$ and $\alpha = 0.1$ and 0.5 respectively. Each of the solid and broken curves represent a bounded solution, and interior and exterior regions of the separatrices correspond to periodic and aperiodic solutions respectively.

4. Analytical solution

An analytical solution of [14] for not-so-large amplitudes can be obtained by expanding $\partial\psi/\partial\phi$ in terms of ϕ and keeping terms up to second or even third order. After a lengthy but straightforward calculation we get

$$[29] \quad \frac{d^2\phi}{d\xi^2} = A\phi - B\phi^2 + C\phi^3$$

where

$$[30] \quad A = 1 - \sum_{i=1}^{2} \frac{Z_i^2 n_{i0}}{\beta_i}\left(\frac{1}{V^2} + \frac{3\sigma_i}{V^2} + \frac{9\sigma_i^2}{V^6}\right)$$

$$[31] \quad B = \frac{1}{2} - \sum_{i=1}^{2} \frac{Z_i^3 n_{i0}}{\beta_i^2}\left(\frac{3}{2V^4} + \frac{15\sigma_i}{V^6} + \frac{189\sigma_i^2}{2V^8}\right)$$

$$[32] \quad C = \frac{1}{6} - \sum_{i=1}^{2} \frac{Z_i^4 n_{i0}}{\beta_i^3}\left(\frac{5}{2V^6} + \frac{105}{2}\frac{\sigma_i}{V^8} + \frac{567\sigma^2}{V^{10}}\right)$$

where we have only kept terms up to $O(\sigma^2)$.

Equation [29] can be integrated exactly in the following way. From [29], integrating once,

$$[33] \quad \left(\frac{d\phi}{d\xi}\right)^2 = \alpha_1\phi^2 - \alpha_2\phi^3 + \alpha_3\phi^4$$

where $\alpha_1 = A$, $\alpha_2 = 2B/3$, and $\alpha_3 = C/2$. Integrating again and using the condition that at $\xi = 0$, ϕ is to be maximum, we get

$$[34] \quad \phi = \frac{2\alpha_1}{(\alpha_2^2 - 4\alpha_1\alpha_3)^{1/2}[2\cos^2 h(\xi\alpha_1/2) - 1] + \alpha_2}$$

Now if we define the width of the soliton to be the value of ξ, say $\xi = \xi_0$, such that

$$[35] \quad \phi(\xi_0) = 0.42\phi(0)$$

then it can be easily seen from [34] that the width is given by

$$[36] \quad \xi_0 = \frac{2}{\sqrt{\alpha_1}}\cosh^{-1}\left[\frac{0.6905\alpha_2}{(\alpha_2^2 - 4\alpha_1\alpha_3)^{1/2}} + 1.6905\right]^{1/2}$$

In the absence of the σ term, our result is in complete agreement with the analytical solution of Sharma *et al.* (4).

5. Results and discussions

To check our calculations we have compared our results with both previous theoretical and experimental work. On the theoretical side we have checked our result in the single-ion plasma case with that of Lai (13), and the result is in complete agreement with that of Lai's second-order calculation; the numerical results are also very close to those obtained by Lai. We have also drawn separatrices to show how a finite ion temperature reduces the allowed region for soliton formation. Because temperature plays a significant role in two-ion plasmas, we think the exact evolution of the pseudopotential will help to elucidate the criteria for soliton formation. For the two-ion plasma case, our results have been compared with the experimental results of Nakamura *et al.* (5) (taking $Z_1 = 1$, $Z_2 = -1$). It can be seen from Fig. 2 that by taking into account temperature effects, the theoretical predictions are closer to the experimental results. In conclusion, our result shows that temperature effects cannot be neglected for solitary-wave solutions. Also, in the case of multiple-ion plasmas, the present method can be applied to find the exact pseudopotential for possible solitary-wave solutions, from which second and higher order terms can easily be obtained.

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1. H. Washimi and T. Tanuiti. Phys. Rev. Lett. **17**, 966 (1966).
2. Y. Kodama and T. Tanuiti. J. Phys. Soc. Jpn. **45**, 298 (1978).
3. Y. H. Ichikawa and T. Tanuiti. J. Phys. Soc. Jpn. **34**, 513 (1973).
4. S. R. Sharma, Yashvir, and T. N. Bhatnagar. Phys. Fluids, **29**, 442 (1986).
5. Y. Nakamura, I. Tsukabayashi, G. O. Ludwig, and J. L. Ferreira. Phys. Lett. **113A**, 155 (1985).
6. R. K. Roychoudhury and Sikha Bhattacharyya. Can. J. Phys. **65**, 699 (1987).
7. R. Z. Sagdeev. *In* Reviews of plasma physics. Vol. 3. Consultants Bureau Enterprises Inc., New York, NY. 1966. p. 23.
8. R. B. White, B. D. Fried, and F. V. Coronitri. Phys. Fluids, **15**, 1484 (1972).
9. H. Ikezi. Phys. Fluids, **16**, 1668 (1973).
10. G. C. Das. Plasma Phys. **19**, 363 (1977).
11. H. H. Kuehl and K. Imen. IEEE Trans. Plasma Sci. **PS-13**, 37 (1985).
12. S. Watanabe. J. Phys. Soc. Jpn. **53**, 930 (1984).
13. C. S. Lai. Can. J. Phys. **57**, 2136 (1979).