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Markov Perfect Equilibria in Altruistic Growth Economies with Production Uncertainty*

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This paper concerns the existence of Markov perfect equilibria in altruistic growth economies. Previous work on deterministic models has established existence only under extremely restrictive conditions. We show that the introduction of production uncertainly yields an existence theorem for aggregative infinite horizon models with very general forms of altruism. *Journal of Economic Literature* Classification Numbers: 022, 026, 111.

1. INTRODUCTION

An altruistic growth economy consists of a sequence (possibly finite) of generations and production technologies. Each generation derives utility from its own consumption and the consumptions of some or all of its descendants.[1] As we have discussed elsewhere (Bernheim and Ray [4]), this framework is of wide applicability.

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[1] Alternatively, each generation might derive utility from its own consumption and the utilities of its descendants. This "non-paternalistic" formulation raises different issues, but we do not consider them here (see, e.g., Pearce [17], Ray [19], Streufert [20]).

A central concept describing intertemporal behaviour for such an economy is that of Markov perfect equilibrium.[2] In such an equilibrium, each generation chooses consumption optimally, given knowledge of its own endowment and the endowment-dependent behaviour of its descendants. This is true of all possible endowments, and for every generation. Since Markov equilibria are so simple, they may be more likely than complex equilibria to arise in practice, and their properties are certainly more amenable to study (see, e.g., Bernheim and Ray [6]). In addition, Markov equilibria will undoubtably turn out to be very useful in studying the properties of more complex equilibria.[3]

The natural and basic question is: do Markov equilibria exist in a reasonably wide class of altruistic growth economies? This issue remained unresolved (see, e.g., Peleg and Yaari [18]; Kohlberg [13]) until Bernheim and Ray [4] and Leininger [15] obtained independent affirmative results for an aggregative (one-commodity) model displaying limited altruism. Altruism is limited in their models in the sense that each generation derives utility only from its own consumption and the consumption of its *immediate* successor.[4]

This existence result is useful but restrictive. In particular, it is important to study whether the result can be extended to (a) a disaggregated multi-commodity model, and (b) more general and far reaching forms of altruism. Regarding (a), recent interesting work by Harris [12] employs techniques similar to that in Bernheim and Ray [4] to prove a Markov existence result in a many-commodity framework.[5] But (b) is a tougher nut to crack. In fact, Peleg and Yaari [18] construct a finite horizon counterexample, showing the difficulty of obtaining a general result.[6]

In Bernheim and Ray [6], we showed that the presence of uncertainty (embodied naturally in the production technology) paves the way for a very general Markov existence theorem in finite horizon models. Unfor-

[2] See, e.g., Dasgupta [8], Kohlberg [13], Leininger [15], and Bernheim and Ray [5], and in the non-paternalistic context, Loury [16], Streufert [20], and Ray [19].

[3] There is an analogy here with repeated games, where history dependent strategies incorporate one-shot "punishments" in order to sustain "collusive" outcomes.

[4] Such limited altruism models have been explored in a variety of contexts. See, e.g., Arrow [2], Dasgupta [8], Barro [3], Kohlberg [13], Loury [16], and Lane and Mitra [14].

[5] However, even in a stationary model, Harris [12] fails to establish the existence of a stationary equilibrium. This remains an interesting (and difficult) open question.

[6] Two points are relevant here. First, more general history dependent equilibria will still exist, as Goldman [9] shows for the finite horizon case and Harris [11] demonstrates for the infinite horizon model. But Markov equilibria still demand our attention, as we have argued elsewhere (Bernheim and Ray [5]). Second, it is of some interest that similar problems do *not* arise in a non-paternalistic framework and Markov equilibria can be shown to exist (Ray [19]).

tunately, the techniques used in that paper are not well suited for the infinite horizon problem.

The purpose of this note is to demonstrate that the introduction of uncertainty also yields an existence theorem for stationary Markov equilibria in aggregative infinite horizon models with very general forms of altruism. The uncertainty is used to show that the best response of a generation, given its descendants' strategies, exists. Our proof depends critically on the fact that each generation's equilibrium investment is a non-decreasing function of its endowment. As in Ray [19], this allows monotone savings functions to be identified wth distributions of probability measures, and endowed with the topology of weak convergence. In Bernheim and Ray [6], we have shown that this "monotonicity" property has strong implications for the positive and normative features of equilibrium programs for a related model. Unfortunately, monotonicity of policy functions depends both upon the existence of an aggregate good, and on a separability assumpton for preferences. Therefore, the infinite horizon result is more limited than its finite horizon counterpart.

We discuss the model and its assumptions in Section 2. Section 3 states and proves the main theorem.

2. The Model

Consider an infinite sequence of generations labelled $t = 0, 1, 2$, etc. There is one commodity, which may be consumed or invested. In each time period, decisions concerning production and consumption are made by a fresh generation. Thus, generation t is endowed with some initial output, $y_t \geqq 0$, which it divides between consumption, $c_t \geqq 0$, and investment, $x_t \geqq 0$ $(y_t = c_t + x_t)$. The return to this investment forms the endowment of the succeeding generation.

The well-being of each generation will be determined by the sequence of consumption choices. Specifically, we assume that generation t's preferences can be represented by a utility function $U_t\colon \mathbb{R}_+^\infty \to \mathbb{R}$, satisfying the following assumptions.

(U.1) $U_t(\langle c_s \rangle_{s=0}^\infty) = u(c_t) + v(c_{t+1}, c_{t+2}, \dots)$.

(U.2) v is continuous in the product topology on real valued sequences.

(U.3) u is strictly concave in c_t.

Remarks. (i) Note throughout that the model considered here is *stationary*. The techniques used can be adapted to demonstrate the existence of non-stationary Markov-perfect equilibrium for non-stationary environments, at the expense of additional notation.

(ii) Implicitly, we assume that each generation's well-being is independent of its ancestors' choices. Trivially, this assumption could be weakened to require separability between ancestors' choices, current choice, and descendants' choices. Further weakening of the assumption is clearly impossible: if ancestors' choices affect the current generation's *ordinal* preferences over descendants' choices, the use of Markov policy functions will, in general, be suboptimal.

The investment chosen by each generation determines the endowment of its successor up to a random disturbance, ω_t, which is realized from the state space $[0, 1]$. Specifically, the production function, $f: \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+$, and disturbances ω_t satisfy the following assumptions.

(F.1) f is strictly increasing and continuous in both x and ω_t.

(F.2) There exists $\bar{y}$ such that for all $\omega_t \in [0, 1]$ and $x > \bar{y}$, $f(x, \omega_t) < x$.

(F.3) $\omega \equiv \langle \omega_t \rangle_1^\infty$ is an i.i.d. sequence of random variables. The distribution of ω is given by an atomless probability measure η on the class of Borel sets in $[0, 1]$. Let μ denote the product measure η^∞ (see Halmos [10, p. 157]).

Remarks. (i) Under (F.2), if $y_0 \leqq \bar{y}$, then for all feasible programs $y_t \leqq \bar{y}$. On this basis, we restrict attention to endowments in $[0, \bar{y}]$.[7] It is possible to relax assumption F.2 by using a truncation argument (see Bernheim and Ray [4]).

(ii) It is relatively straightforward to relax the assumption that the ω_t are i.i.d. However, some subset of past realizations will then affect expectations concerning future realizations. Thus, one would have to allow strategies to depend on the history of past innovations, as well as current endowments. Strictly speaking, the equilibrium strategies would then not be Markov. It would not, however, be necessary to allow conditioning of strategies on past actions, independent of their effects on current endowments, in order to obtain an existence result.

A *Markov strategy* (for any generation) is a function $s: [0, \bar{y}] \to [0, \bar{y}]$ such that for all $y \in [0, \bar{y}]$, $0 \leqq s(y) \leqq y$. Let S^0 denote the set of conceivable Markov strategies.

We will focus attention on *stationary* equilibria. Thus, we wish to describe the evolution of decisions when all generations select the same Markov strategy, s. The following recursion determines the evolution

[7] Or, if $y_0 \leqslant \hat{y}$ in general, to all feasible programs with $y_t \leqslant \max\{\bar{y}, \hat{y}\}$.

of capital stocks, given ω, s, an some initial investment choice x for generation 0

$$\sigma_0(x, \omega; s) = x$$
$$\sigma_t(x, \omega; s) = s(f(\sigma_{t-1}(x, \omega; s), \omega_t)), \qquad t = 1, 2, \ldots . \tag{1}$$

This, in turn, determines the evolution of consumption decisions

$$\gamma_t(x, \omega; s) \equiv f(\sigma_{t-1}(x, \omega; s), \omega_t) - \sigma_t(x, \omega; s), \qquad t = 1, 2, \ldots .$$

Let $\sigma = (\sigma_1, \sigma_2, \ldots)$ and $\gamma = (\gamma_1, \gamma_2, \ldots)$, and define

$$V(x, \omega; s) = v(\gamma(x, \omega; s)).$$

The strategy $s \in S^0$ constitutes a (*stationary Markov perfect*) *equilibrium* if for each $y \in [0, \bar{y}]$, $s(y)$ solves

$$\max_{0 \leqq x \leqq y} u(y - x) + E_\omega V(x, \omega; s). \tag{2}$$

Remark. Note that by our continuity assumption (U.2) and the compactness of feasible programs in the product topology, the expectation in (2) is always well defined provided V is measurable.

3. Existence

We now state our central result.

THEOREM. *Under the stated assumptions, there exists a stationary Markov-perfect equilibrium. It is always the case that the equilibrium policy function, s, is non-decreasing, and may be chosen to be upper semicontinuous.*

The general line of proof used below is similar to that of Ray [19], and some specific steps are closely related to arguments therein. We have noted these steps throughout, generally leaving them to the reader, who may wish to consult Bernheim and Ray [7] for complete details.

Proof. Our first key lemma establishes that best response policy functions are always non-decreasing. The proof is identical to that of Theorem B in Ray [19]; we therefore omit it.

LEMMA 1. *Fix $s \in S^0$. Suppose that for $y \in \{y^1, y^2\}$, $y^i \in [0, \bar{y}]$ $(i = 1, 2)$, $y^1 > y^2$, problem (2) is well defined. Further, suppose x^1 and x^2 are corresponding solutions. Then $x^1 \geqq x^2$.*

Henceforth, we will restrict attention to non-decreasing, upper semicontinuous (usc) functions. Let $S \subset S^0$ denote the set of such functions. Our next two lemmas establish that when future generations select $s \in S$, then problem (2) is well defined.

LEMMA 2. *Suppose* $s \in S$. *For each* $x \in [0, \bar{y}]$, *and* $t \geqq 0$, $\sigma_t(x, \omega; s)$ *is continuous in* ω *almost everywhere.*

Proof. By induction. Suppose that σ_{t-1} does not depend upon $(\omega_t, \omega_{t+1}, ...)$, and that σ_{t-1} is continuous in ω almost everywhere (this holds for $t=1$). Not then that σ_t does not depend upon $(\omega_{t+1}, \omega_{t+2}, ...)$ (inspect (1)). Denote the set of discontinuities of σ_{t-1} by D_{t-1}. Since s is non-decreasing, it has at most a countable number of discontinuities on $[0, \bar{y}]$; call them $\langle d_1, d_2, ... \rangle$. Let $D_t^i = \{\omega | f(\sigma_{t-1}(x, \omega; s), \omega_t) = d_i\}$. σ_t is discontinuous at ω only if $\omega \in D_{t-1}$, or $\omega \in D_t^i$ for some i. Since f is strictly increasing in ω_t, and since σ_{t-1} does not depend on ω_t, every $(\omega_\tau)_{\tau \neq t}$-section of D_t^i consists of a single point, and therefore has measure zero. Thus, D_t^i has measue zero (see Halmos [10, p. 147]). Since D_t is contained in the union of a countable number of sets of measure zero, it has measure zero. This completes the induction step. Q.E.D.

Two corollaries follow immediately:

COROLLARY 2.1. *Suppose* $s \in S$. *For each* $x \in [0, \bar{y}]$, $\sigma(x, \omega; s)$, $\gamma(x, \omega; s)$, *and* $V(x, \omega; s)$ *are continuous in* ω *almost everywhere.*

COROLLARY 2.2. *Suppose* $s \in S$. *For each* $x^0 \in [0, \bar{y}]$, *define* $D(x^0) = \{\omega^0 \in [0, 1]^\infty \mid V(x, \omega^0; s)$ *is discontinuous in* x *at* $x^0\}$. *For all* $x^0 \in [0, \bar{y}]$, $D(x^0)$ *has measure zero.*

LEMMA 3. *Suppose* $s \in S$. *Then for all* x, y, $0 \leqslant x \leqslant y \leqslant \bar{y}$,

$$u(y-x) + E_\omega V(x, \omega; s)$$

is well defined, and continuous in (y, x).

Proof. The first term is continuous in (y, x). V is simply the composition of measurable functions, and is therefore measurable. Since v is bounded on the space of all feasible programs (see (U.2)), the expectation is well defined. To show continuity for the second term, take some sequence $x^n \to x$. Then

$$E_\omega V(x^n, \omega; s) - E_\omega V(x, \omega; s) = \int_{[0,1]^\infty} (V(x^n, \omega; s) - V(x, \omega; s))\, d\mu.$$

By Corollary 2.2, $V(x^n, \omega; s) \to V(x, \omega; s)$ almost everywhere. Further, since $\langle c_t \rangle_{t=1}^{\infty} \in [0, \bar{y}]^{\infty}$, which is compact in the product topology, and since v is continuous, V is bounded. Applying Lebesgue's dominated convergence theorem establishes continuity. Q.E.D.

Due to difficulties involving the behaviour of policy functions at $\bar{y}$, it is convenient to work with quasi-equilibria, defined as follows. An $s \in S$ is a *quasi-equilibrium* if for each $y \in [0, \bar{y})$, $s(y)$ solves (2). Let $\bar{S} \subset S$ consist of the functions $s \in S$ such that $s(\bar{y}) = \bar{y}$. As in Ray [19], $\bar{S}$ can be thought of as the set of distribution functions on $[0, \bar{y}]$ (where probability is rescaled). Our next lemma indicates that if we start with some element of $\bar{S}$, maximization for each $y \in [0, \bar{y}]$ generates a unique "quasi-best" response in $\bar{S}$.

LEMMA 4. *For each $s \in \bar{S}$, there is a unique function $s' \equiv H(s)$ such that $s' \in \bar{S}$, and for all $y \in [0, \bar{y})$, $s'(y)$ solves problem* (2).

Proof. Let $h(y)$ be the correspondence which maps to solutions of (2). By Lemma 4 and the maximum theorem, h is upper hemicontinuous. Let $s'(y) = \max\{h(y)\}$ for $y \in [0, \bar{y})$, and $s'(\bar{y}) = \bar{y}$. Clearly, $s' \in \bar{S}$. Now suppose there is another u.s.c. selection from $h(y)$, $s'' \in \bar{S}$. For some $\hat{y}$, $s'(\hat{y}) > s''(\hat{y})$. Since s'' is u.s.c. there is some $\tilde{y} > \hat{y}$ with $s'(\tilde{y}) > s''(\tilde{y})$. But this contradicts Lemma 1. Q.E.D.

Lemma 4 defines a mapping, $H: \bar{S} \to \bar{S}$. A fixed point of this mapping is a quasi-equilibrium. We need to establish continuity of H. The key step is to prove that $E_\omega V$ is continuous in s.

LEMMA 5. *Suppose some sequence $\langle s^n \rangle_0^{\infty}$ in $\bar{S}$ converges to $s \in \bar{S}$. Then for each $x \in [0, \bar{y}]$, $E_\omega V(x, \omega; s^n) \to E_\omega V(x, \omega; s)$.*

Proof. Choose any $\hat{\omega}$ at which σ is continuous in ω. Suppose $\sigma_{t-1}(x, \hat{\omega}; s^n) \to \sigma_{t-1}(x, \hat{\omega}; s)$ (this holds for $t = 1$). By assumption, f is continuous. Further, since $\sigma_t(x, \omega; s)$ is continuous in ω_t at $\hat{\omega}$, and since f is increasing in ω_t, s must be continuous at $f(\sigma_{t-1}(x, \hat{\omega}; s), \hat{\omega}_t)$. Thus, using (1), $\sigma_t(x, \hat{\omega}; s^n) \to \sigma_t(x, \hat{\omega}; s)$. By induction, this holds for all t. Since f is continuous, $\gamma(x, \hat{\omega}; s^n) \to \gamma(x, \hat{\omega}; s)$ in the product topology. By (U.2), $V(x, \hat{\omega}; s^n) \to V(x, \hat{\omega}; s)$.

By Corollary 2.1, σ is continuous in ω almost everywhere. Thus, by the peceding argument, $V(x, \omega; s^n) \to V(x, \omega; s)$ almost everywhere. Combining this with the boundedness of V (see the proof of Lemma 3) and Lebesgue's dominated convergence theorem yields the desired result. Q.E.D.

Given Lemma 5, one proves continuity of the mapping H in a manner completely analagous to the proof of Lemma 6 in Ray [9]. From Lemma 3 of Ray [19], if $\bar{S}$ is endowed with the topology of weak convergence, every continuous function from $\bar{S}$ to itself has a fixed point. Thus, a quasi-equilibrium exists, with some policy function $s \in \bar{S}$. Let x^0 solve (2) for $y = \bar{y}$ (by Lemma 3, x_0 exists). Define $\hat{s}(y) = s(y)$ for $y \in [0, \bar{y})$, and $\hat{s}(y) = x^0$ for $y = \bar{y}$. Since f is increasing in ω_t (so that $y_t = \bar{y}$ iff $\omega_t = 1$), and since η is atomless (so that $\mu[\{\omega \in [0, 1]^\infty \mid \omega_t = 1 \text{ for some } t\}] = 0$), it follows that $\hat{s}$ is an equilibrium. Lemma 1 assures us that $\hat{s}$ *must* be non-decreasing. By construction, $\hat{s}$ is usc. Q.E.D.

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